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Garside monoids *vs* divisibility monoids

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Divisibility monoids are a natural algebraic generalisation of Mazurkiewicz trace monoids, namely monoids in which the distributivity of the underlying lattices is kept as a hypothesis, but the relations between the generators are not necessarily supposed to be commutations. Garside monoids are a natural algebraic generalisation of spherical Artin monoids, namely monoids in which the existence of common multiples is kept as a hypothesis, but the relations between the generators are not necessarily supposed to be of Coxeter type.

Here, we show that the quasi-centre of these monoids can be studied and described in a similar way, and then we exhibit the intersection between the two classes of monoids.

1. Introduction

The purpose of this paper is to study some possible connections between the classes of Garside monoids and divisibility monoids, which are both natural algebraic generalisations of monoids involved in two mathematical areas, namely braid theory and trace theory, respectively.

Garside monoids can be used as a powerful tool in the study of automatic structures for groups arising in topological or geometric contexts. Generalising Mazurkiewicz trace monoids, divisibility monoids have been introduced as a mathematical model for the sequential behavior of some concurrent systems considered in several areas in computer science and in which two sequential transformations of the form ab and cd, with a, b, c, d viewed as atomic transitions, can give rise to the same effect.

Here we show that Garside monoids and divisibility monoids are both strangely similar (although they come from apparently unrelated mathematical theories) and genuinely different (even if they share some essential algebraic properties). Their similarity will be illustrated by the fact that their quasi-centre – roughly speaking, some supmonoid of the centre – can be studied using very similar techniques and described through the same statement. Their particularity will be emphasised by the fact that the intersection between the two classes, which we exhibit by using the result about the quasi-centre, is reduced to a somewhat confined subclass.

Main Theorem

(i) A divisibility monoid is a Garside monoid if and only if every pair of its irreducible elements admits common multiples.

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(ii) A Garside monoid is a divisibility monoid if and only if the lattice of its simple elements is a hypercube.

The paper is organised as follows. In Section 2, we gather the needed basic properties of Garside monoids and divisibility monoids. In Section 3, we introduce a specific tool, which we call *local delta*, that allows us to compute a minimal generating set for the quasi-centre of every divisibility monoid (Propositions 3.12 and 3.14). We compare these results with those obtained for Garside monoids in Picantin (2001b). In Section 4, we finally prove the main theorem of this paper (Theorem 4.1) and illustrate it.

2. Background for Garside and divisibility monoids

In this section, we list some basic properties of Garside monoids and divisibility monoids, and summarise some results by Dehornoy and Paris for Garside monoids and by Droste and Kuske for divisibility monoids. For all the results quoted here, refer to Dehornoy and Paris (1999), Picantin (2000) and Dehornoy (2002), and to Droste and Kuske (2001) and Kuske (2001).

2.1. *Divisors and multiples in a monoid*

Assume that M is a monoid. We say that M is *conical* if 1 is the only invertible element in M. For a, b in M, we say that b is a left divisor of a, or that a is a right multiple of b, if $a = bd$ holds for some d in M. The set of the left divisors of b is denoted by $\downarrow(b)$. An element c is a right lower common multiple, or a right lcm, of a and b if it is a right multiple of both a and b, and every right common multiple of a and b is a right multiple of c. Right divisor, left multiple and left lcm are defined symmetrically. For a, b in M, we say that b *divides* a, or that b is a divisor of a, if $a = cbd$ holds for some c, d in M.

If c, c' are two right lcm's of a and b, then c necessarily is a left divisor of c', and c' is a left divisor of c. If we assume M to be conical and cancellative, we have $c = c'$: the unique right lcm of a and b is then denoted by $a \vee b$. If $a \vee b$ exists, and M is left cancellative, there exists a unique element c satisfying $a \vee b = ac$: this element is denoted by $a\backslash b$. In particular, we obtain the identities

$$a \vee b = a \cdot (a\backslash b) = b \cdot (b\backslash a).$$

Cancellativity and conicity imply that left and right divisibility are order relations. Now, the additional assumption that any two elements admitting a common multiple admit an lcm, which we will see is satisfied by both Garside monoids and divisibility monoids, allows us to obtain the following algebraic properties for the operation $\backslash$.

Lemma 2.1. Assume that M is a cancellative conical monoid in which any two elements admitting a common right multiple admit a right lcm. Then the following identities hold in M:

$$(ab) \vee (ac) = a(b \vee c), \qquad c\backslash(ab) = (c\backslash a)((a\backslash c)\backslash b), \qquad (ab)\backslash c = b\backslash(a\backslash c),$$
$$(a \vee b)\backslash c = (a\backslash b)\backslash(a\backslash c) = (b\backslash a)\backslash(b\backslash c), \qquad c\backslash(a \vee b) = (c\backslash a) \vee (c\backslash b).$$

For each identity, this means that both sides exist and are equal, or that neither exists.

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Notation 2.2. For some subsets A, B of elements, we denote by $A\backslash B$ the set of elements $a\backslash b$ provided that all of them exist.

2.2. *The quasi-centre of a monoid*

The quasi-centre of a monoid turns out to be a useful supmonoid of its centre.

Definition 2.3. Let M be a monoid. An *irreducible element* of M is defined to be a non-trivial element a such that $a = bc$ implies either $b = 1$ or $c = 1$. The set of the irreducible elements in M can be written as $(M\backslash\{1\})\backslash(M\backslash\{1\})^2$.

Definition 2.4. Assume that M is a monoid with set of irreducible elements Σ. The *quasi-centre* of M is defined to be its submonoid $\{a \in M;\ a\Sigma = \Sigma a\}$.

Although straightforward, the two following lemmas capture key properties of quasi-central elements. They will be used frequently in the remaining sections.

Lemma 2.5. Assume that M is a cancellative monoid. Then, for every element a in M and every quasi-central element b in M, the following are equivalent:

(i) a divides b;
(ii) a divides b on the left;
(iii) a divides b on the right.

Proof. By definition, both (ii) and (iii) imply (i).

Now, assume (i). Then there exist elements c, d in M satisfying $b = cad$. Since b is quasi-central, we have $cb = bc'$ for some c' in M. We find $cb = bc' = cadc'$, hence, by left cancellation, $b = adc'$, which implies (ii). Symmetrically, (i) implies (iii). □

Lemma 2.6. Assume that M is a cancellative monoid and a is a quasi-central element in M. Then, for any two elements b, c satisfying $a = bc$, b is quasi-central if and only if c is also.

Proof. Assume c to be quasi-central. Let x be an irreducible element in M. Since a is quasi-central, there exists an irreducible element x' in M satisfying $xa = ax'$, hence $xbc = bcx'$. Now, since c is quasi-central, there exists an irreducible element x'' in M satisfying $cx' = x''c$. We find $xbc = bx''c$, hence, by right cancellation, $xb = bx''$, which implies that b is quasi-central. Symmetrically, if b is quasi-central, then c is also. □

2.3. *Main definitions and properties for Garside monoids*

Definition 2.7. A monoid M is said to be *Garside*[†] if M is conical and cancellative, every pair of elements in M admits a left lcm and a right lcm, and M admits a *Garside element*, defined to be an element whose left and right divisors coincide, are finite in number and generate M.

[†] Garside monoids as defined above are called Garside monoids in Charney *et al.* (2002), Dehornoy (2002), Picantin (2002) and Picantin (2003a), but they were called either 'small Gaussian' or 'thin Gaussian' in earlier papers (Dehornoy and Paris 1999; Picantin 2000; Picantin 2001a; Picantin 2001b), where a more restricted notion of Garside monoid was also considered.

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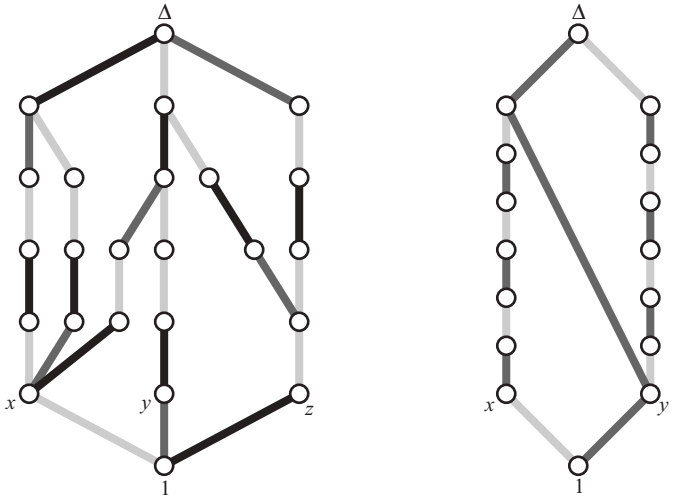


Fig. 1. The lattice of simple elements of the Garside monoids M_χ (left) and M_κ (right).

Example 2.8. All spherical Artin monoids are Garside monoids. Braid monoids of complex reflection groups (Broué *et al.* 1998; Picantin 2000), Garside's hypercube monoids (Garside 1969; Picantin 2000), Birman–Ko–Lee monoids for spherical Artin groups (Birman *et al.* 1998; Picantin 2002) and monoids for the torus link groups in Picantin (2003a) are also Garside monoids.

The monoid M_χ with presentation

$$\langle x, y, z : xzxy = yzx^2,\ yzx^2z = zxyzx,\ zxyzx = xzxyz \rangle$$

is a typical example of a Garside monoid, which has the distinguishing feature of not being antiautomorphic.

The monoid M_κ defined by the presentation $\langle x, y : xyxyxyx = yy \rangle$ is another example of a Garside monoid, which, in this case, admits no additive norm, that is, no norm ν satisfying $\nu(ab) = \nu(a) + \nu(b)$ (a *norm* for a monoid M is a mapping μ from M to $\mathbb{N}$ satisfying $\mu(a) > 0$ for every $a \neq 1$ in M, and satisfying $\mu(ab) \geqslant \mu(a) + \mu(b)$ for every a, b in M).

Every element in a Garside monoid has finitely many left divisors, and, for any two elements a, b, the left common divisors of a and b admit a right lcm, which is therefore the left gcd of a and b. This left gcd is denoted by $a \wedge b$.

Every Garside monoid admits a minimal Garside element, denoted, in general, by Δ. The set of the divisors of Δ, called the *simple elements*, endowed with the operations $\vee$ and $\wedge$ is a finite lattice.

Example 2.9. The lattices of simple elements of monoids M_χ and M_κ of Example 2.8 are displayed in Figure 1, using Hasse diagrams, where light, mid and dark grey edges represent the irreducible elements x, y and z, respectively.

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It is easy to check that every Garside element is a quasi-central element. Now, this is far from being sufficient to describe what the quasi-centre of every Garside monoid looks like. This was done in Picantin (2001b), where the following structural result was established.

Theorem 2.10. Every Garside monoid is an iterated crossed product of Garside monoids with an infinite cyclic quasi-centre.

2.4. *Main definitions and properties for divisibility monoids*

Definition 2.11. A monoid M is called a *left divisibility monoid*, or simply a *divisibility monoid* if M is cancellative and finitely generated by its irreducible elements, if any two elements admit a left gcd and if every element a dominates a finite[†] distributive lattice $\downarrow(a)$.

Note that cancellativity and the lattice condition imply conicity. As in the Garside case, the left gcd of two elements a, b will be denoted by $a \wedge b$. The *length* $|a|$ of an element a is defined to be the height of the lattice $\downarrow(a)$.

Example 2.12. Every finitely generated trace monoid is a divisibility monoid. The monoids $\langle x, y, z : xy = yz\rangle$, $\langle x, y, z : x^2 = yz\rangle$ and $\langle x, y, z : x^2 = yz,\ yx = z^2\rangle$ are not trace, but they are divisibility monoids. The monoid $\langle x, y, z : x^2 = yz,\ xy = z^2\rangle$ is not a divisibility monoid (but it is a Garside monoid!); indeed, the lattice $\downarrow(x^3)$ is not distributive.

An easy but crucial fact about divisibility monoids is the following.

Lemma 2.13. Assume that M is a divisibility monoid. Then finitely many elements in M admitting at least a right common multiple admit a unique right lcm.

The following result states that there exists a decidable class of presentations that gives rise precisely to all divisibility monoids.

Theorem 2.14. Assume that M is a monoid finitely generated by the set Σ of its irreducible elements. Then M is a left divisibility monoid if and only if:

(i) $\downarrow(xyz)$ is a distributive lattice;
(ii) $xyz = xy'z'$ or $yzx = y'z'x$ implies $yz = y'z'$;
(iii) $xy = x'y'$, $xz = x'z'$ and $y \neq z$ imply $x = x'$, for any x, y, z, x', y', z' in Σ; and if
(iv) we have $M \cong \Sigma^*/\sim$, where $\sim$ is the congruence on Σ^* generated by the pairs (xy, zt) with x, y, z, t in Σ satisfying $xy = zt$.

3. The quasi-centre of a divisibility monoid

This section deals only with divisibility monoids, and no knowledge of Garside monoids is needed, except perhaps for the final remark. After defining the *local delta*, we use it as a fundamental tool for giving a generating set for the quasi-centre of every divisibility

[†] This finiteness requirement is in fact unnecessary since it follows from the other stipulations (see, for example, Kuske (2001)).

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monoid and showing that it is minimal. We establish then that the quasi-centre of every divisibility monoid is a free Abelian submonoid.

3.1. *A generating set for the quasi-centre*

We first introduce a partial version of what the author called *local delta* in Picantin (2001b).

Definition 3.1. Assume that M is a divisibility monoid. Let a be an element in M. If the set $\{b\backslash a;\ b \in M\}$ is well-defined[†] and admits a right lcm, the latter is called the *right local delta of* a or simply the *local delta of* a and is denoted by

$$\Delta_a = \bigvee\{b\backslash a;\ b \in M\}.$$

Otherwise, we say that a does not admit a right local delta. Note that the equality $1\backslash a = a$ implies that a is a left divisor of Δ_a, whenever it exists, and that, having $b\backslash 1 = 1$ for every b in M, we obtain $\Delta_1 = 1$.

Lemma 3.2. Assume that M is a divisibility monoid. Then an element a in M admits a local delta if and only if $a \vee b$ exists for every element b in M.

Proof. Assume that M is a divisibility monoid with Σ the set of its irreducible elements and $\Sigma_1 = \Sigma \cup \{1\}$. For every a in M, we define $\Upsilon_i(a)$ by $\Upsilon_0(a) = \{a\}$ and $\Upsilon_i(a) = \Sigma_1 \backslash \Upsilon_{i-1}(a)$ for $i > 0$ whenever both $\Upsilon_{i-1}(a)$ exists and $c\backslash b$ exists for every c in Σ_1 and every b in $\Upsilon_{i-1}(a)$.

Assume that a is an element such that $a \vee b$ exists for every b in M. From the distributivity of $\downarrow(a \vee b)$, we can deduce $|b\backslash a| \leqslant |a|$. The set $\{b\backslash a;\ b \in M\}$ is then finite, and there exists a positive integer i_a satisfying

$$\{a\} = \Upsilon_0(a) \subsetneq \Upsilon_1(a) \subsetneq \ldots \subsetneq \Upsilon_{i_a}(a) = \Upsilon_{i_a+1}(a) = \ldots = \{b\backslash a;\ b \in M\}.$$

Indeed, by Lemma 2.1, we have $M\backslash a = \Sigma_1^{i_a}\backslash a = \Sigma_1 \backslash (\Sigma_1^{i_a-1}\backslash a)$. Now, assume c, d belonging to $M\backslash a$ and h satisfying $d = h\backslash a$. By hypothesis, the element $(hc)\backslash a$ exists, hence, by Lemma 2.1, the element $c\backslash d$, which is $c\backslash(h\backslash a)$, exists, and so does $c \vee d$. Therefore, the finite set $\{b\backslash a;\ b \in M\}$ admits a right lcm, namely Δ_a.

The converse implication is straightforward. □

Remark 3.3. Following the previous proof, we can compute the right local delta of any element a by recursively computing the sets $\Upsilon_i(a)$ whenever they exist and then by computing the right lcm of $\Upsilon_{i_a}(a)$, which must exist by Lemma 3.2. See Examples 3.10 and 4.2 below.

We are going to prove that the quasi-centre of any divisibility monoid is generated by the (possibly empty) set of the local delta of its irreducible elements (Proposition 3.9). The proof of this result relies on several preliminary statements.

Lemma 3.4. Assume that M is a divisibility monoid. Then, for every element a and every quasi-central element b in M, a dividing b implies Δ_a exists and divides b.

[†] That is, the element $b\backslash a$ exists, or, equivalently, the element $a \vee b$ exists, for every b in M.

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Proof. By hypothesis and Lemma 2.5, there exists an element d in M satisfying $b = ad$. As b is quasi-central, for every c in M, there exists an element c' in M satisfying $cb = adc'$. By Lemma 2.13, for every c in M, $c \vee a$, which exists and is $c(c\backslash a)$, divides cb on the left, and, by left cancellation, $c\backslash a$ exists and divides b on the left. Therefore, Δ_a exists (by Lemma 3.2) and divides b on the left. □

A weaker but convenient version of Lemma 3.4 is given by the following.

Lemma 3.5. Assume that M is a divisibility monoid. Then every quasi-central element a in M satisfies $\Delta_a = a$.

The converse assertion is as follows.

Lemma 3.6. Assume that M is a divisibility monoid. Then every element a in M satisfying $\Delta_a = a$ is quasi-central.

Proof. Let x be an irreducible element of M. From $\bigvee(M\backslash a) = a$, we can deduce that $x\backslash a$ is a left divisor of a. Therefore, $x(x\backslash a)$ is a left divisor of xa. Now, by definition, $x(x\backslash a)$ is $a(a\backslash x)$, so there exists d in M satisfying $xa = ad$. We obtain $|xa| = |ad|$, hence $|x| + |a| = |a| + |d|$. We can deduce $|d| = |x| = 1$, thus d is an irreducible element of M. So, there exists a mapping f_a from the irreducible elements of M into themselves such that $xa = af_a(x)$ holds for every irreducible element x. By cancellativity, f_a is injective, and hence surjective, so a is quasi-central by definition. □

Proposition 3.7. Assume that M is a divisibility monoid. Then, for every a in M, the element Δ_a is quasi-central, whenever it exists.

Proof. Let a be an element in M such that Δ_a exists. We claim that Δ_{Δ_a} exists and is Δ_a. By hypothesis, the set $\{b\backslash a;\ b \in M\}$ is well defined and admits a right lcm, namely Δ_a. As $\downarrow(\Delta_a)$ is finite (by definition of a divisibility monoid), the set $\{b\backslash a;\ b \in M\}$ is also. Therefore, M admits a finite subset T satisfying $M\backslash a = T\backslash a$. Let $T = \{c_1, \ldots, c_r\}$. For every element b in M, the element $((c_1 b)\backslash a) \vee \cdots \vee ((c_r b)\backslash a)$ exists and divides Δ_a on the left. Using Lemma 2.1, we find

$$((c_1 b)\backslash a) \vee \cdots \vee ((c_r b)\backslash a) = b\backslash((c_1\backslash a) \vee \cdots \vee (c_r\backslash a)) = b\backslash\Delta_a,$$

which implies that $b\backslash\Delta_a$ exists and divides Δ_a (on the left) for every b in M. We deduce that Δ_{Δ_a} exists (by Lemma 3.2) and divides Δ_a. Now, Δ_a dividing Δ_{Δ_a}, cancellativity and conicity imply $\Delta_{\Delta_a} = \Delta_a$. Therefore, by Lemma 3.6, Δ_a is quasi-central. □

Corollary 3.8. Assume that M is a divisibility monoid. Then the partial application $a \mapsto \Delta_a$ is a surjection from M onto the quasi-centre of M.

Proposition 3.9. Assume that M is a divisibility monoid with Σ the set of its irreducible elements. Then $\{\Delta_x;\ x \in \Sigma\}$ is a generating set of the quasi-centre of M.

Proof. Let b be a quasi-central element in M. We show using induction on the length $|b|$ of b that there exist an integer n and irreducible elements $x_1, \ldots, x_n$ satisfying $b = \Delta_{x_1} \cdots \Delta_{x_n}$. For $|b| = 0$, n is 0. Now assume $|b| > 0$. Then there exist an irreducible element x and an element b' in M satisfying $b = xb'$. By Lemma 3.4, Δ_x exists and we

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have $b = \Delta_x b''$ for some b'' in M with $|b''| < |b|$. By Proposition 3.7, the element Δ_x is quasi-central, and hence, by Lemma 2.6, b'' is also. By the induction hypothesis, there exist an integer m and irreducible elements $y_1, \ldots, y_m$ admitting local delta and satisfying $b'' = \Delta_{y_1} \cdots \Delta_{y_m}$. We thus obtain $b = \Delta_x \Delta_{y_1} \cdots \Delta_{y_m}$. □

Example 3.10. Let us consider the following divisibility monoids $M_1 = \langle x, y, z : xy = yz,\ yx = zy \rangle$ and $M_2 = \langle x, y, z : xz = yx,\ yz = zx \rangle$. In both monoids, by Lemma 3.2, the irreducible elements x and z clearly do not admit right local delta. On the one hand, the quasi-centre of M_1 is thus generated by the single element $\Delta_y = \bigvee\{y\} = y$. On the other hand, let us try to compute the local delta of y. According to Remark 3.3, we compute $\Upsilon_0(y) = \{y\}$, $\Upsilon_1(y) = \{1, x, y, z\} \backslash \{y\} = \{1, x, y, z\}$. Now, since $x \vee z$ does not exist, $\Upsilon_2(y)$ cannot exist either. Therefore, no irreducible element admits local delta: $\{\Delta_s;\ s \in \{x, y, z\}\}$ is empty and the quasi-centre of M_2 is then trivial.

3.2. *Minimality of the generating set*

We now prove that the generating set given in Proposition 3.9 is minimal.

Lemma 3.11. Assume that M is a divisibility monoid. Then any two irreducible elements x, y in M admitting local delta satisfy either $\Delta_x = \Delta_y$ or $\Delta_x \wedge \Delta_y = 1$.

Proof. We first prove that, for any two irreducible elements x, y admitting local delta and every b in M, that $\Delta_x = b\Delta_y$ implies $b = 1$. According to Proposition 3.7, Δ_x and Δ_y are quasi-central, and, by Lemma 2.6, b is also quasi-central. Now, as $1 \backslash x = x$ holds, x divides Δ_x, and, by Lemma 2.5, we have $\Delta_x = dx$ for some d in M. We find

$$\Delta_x = dx = b\Delta_y,$$

and, by left cancellation, the element $d \backslash b$, which exists by Lemma 2.13, divides x: since x is irreducible, $d \backslash b$ is either x or 1. Assume $d \backslash b = x$. Then, b being quasi-central, Lemma 3.5 implies $\Delta_b = b = \bigvee(M \backslash b)$, and, therefore, x divides b. By Lemma 3.4, Δ_x divides b, which, by cancellativity and conicity, implies $\Delta_y = 1$, a contradiction. We deduce $d \backslash b = 1$. We then find $\Delta_y = (b \backslash d)x$, and, by Lemmas 2.5 and 3.4, Δ_x divides Δ_y, which, by cancellativity and conicity, implies $b = 1$.

Finally, let x, y be irreducible elements in M admitting local delta. Assume $\Delta_x \wedge \Delta_y \neq 1$. Then there exists an irreducible element z in M dividing both Δ_x and Δ_y. By Lemma 3.4, Δ_z divides both Δ_x and Δ_y, which, by the result above, implies $\Delta_x = \Delta_z = \Delta_y$. □

Proposition 3.12. Assume that M is a divisibility monoid with Σ the set of its irreducible elements. Then $\{\Delta_x;\ x \in \Sigma\}$ is a minimal generating set of the quasi-centre of M.

Proof. By Proposition 3.9, the set $\{\Delta_x;\ x \in \Sigma\}$ generates the quasi-centre of M. Let x be an irreducible element such that Δ_x exists, and let a, b be quasi-central elements in M. It suffices to show that $\Delta_x = ab$ implies either $a = 1$ or $b = 1$. Assume $a \neq 1$. Then we have $a = ya'$ for some irreducible element y and some a' in M. As a is quasi-central, by Lemma 3.4, Δ_y exists and is a left divisor of a, and, therefore, Δ_y is a left divisor of Δ_x.

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We have $\Delta_y \neq 1$, and hence, by Lemma 3.11, $\Delta_y = \Delta_x$. Cancellativity and conicity then imply $b = 1$. □

3.3. *A free Abelian submonoid*

We conclude with the observation that the quasi-centre of every divisibility monoid is a free Abelian submonoid (that is, a monoid isomorphic to a direct product of copies of the monoid $\mathbb{N}$).

Lemma 3.13. Assume that M is a divisibility monoid. Let a, b be elements in M admitting local delta. Then $a \vee b$ exists and admits a local delta, namely $\Delta_a \vee \Delta_b$.

Proof. Let a, b be elements in M admitting local delta. First, by Lemma 3.2, $a \vee b$ exists. Next, by Lemma 3.2 again, the element $b \vee c$ exists for every element c in M, and so does $a \vee (b \vee c)$. Associativity of the operation $\vee$ implies that $(a \vee b) \vee c$ exists for every element c in M. Therefore, by Lemma 3.2, $\Delta_{a\vee b}$ exists. Now, by Lemma 2.1, $(c\backslash a) \vee (c\backslash b) = c\backslash(a \vee b)$ holds for every c in M. We then obtain $\Delta_a \vee \Delta_b = \Delta_{a\vee b}$. □

Proposition 3.14. Assume that M is a divisibility monoid. Let QZ be its quasi-centre. Then QZ is a free Abelian submonoid of M, and the partial function $a \mapsto \Delta_a$ is a partial surjective semilattice homomorphism from $(M, \vee)$ onto $(QZ, \vee)$.

Proof. Let Σ be the set of the irreducible elements in M. By Proposition 3.12, $\{\Delta_x; x \in \Sigma\}$ is the minimal generating set of QZ. So, in order to prove that QZ is free Abelian, it suffices to show that, for any two x, y in Σ admitting local delta with $\Delta_x \neq \Delta_y$, the element $\Delta_x\backslash\Delta_y$, which exists by Lemma 3.13, is Δ_y. Let x, y be irreducible elements admitting local delta with $\Delta_x \neq \Delta_y$. Then, by Lemma 3.11, we have $\Delta_x\backslash\Delta_y \neq 1$. Now, $\Delta_x\backslash\Delta_y$ divides Δ_y, and, by Lemma 3.13, the element $\Delta_x\backslash\Delta_y$ is quasi-central, which implies $\Delta_x\backslash\Delta_y = \Delta_y$ by Proposition 3.12. The second part of the assertion then follows from Lemma 3.13. □

Propositions 3.12 and 3.14 both hold for every Garside monoid. The corresponding proofs are different, however, even though the current approach follows Picantin (2001b) closely. The point is that, in the Garside case, the function $a \mapsto \Delta_a$ is total and coincides with its left counterpart $a \mapsto \widetilde{\Delta}_a$, but there need not exist an additive length (see, for instance, the monoid M_κ in Example 2.8), while, in the divisibility case, there exists an additive length, but the function $a \mapsto \Delta_a$ is only partial and does not admit a left counterpart in general.

4. The intersection

Now we can finally prove our main theorem.

Theorem 4.1.

(i) A divisibility monoid is a Garside monoid if and only if every pair of its irreducible elements admits common multiples.

(ii) A Garside monoid is a divisibility monoid if and only if the lattice of its simple elements is a hypercube.

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Proof.

(i) The condition is necessary by the definition of a Garside monoid. We have to show that it is also sufficient. Let M be a divisibility monoid with Σ the finite set of its irreducible elements. Assume that every pair of irreducible elements in M admits right multiples. Then, by Lemma 2.13, every pair of irreducible elements in M admits a unique right lcm, and, by Theorem 2.14 (iv) or simply by distributivity, such an lcm is of length 2. From this, a straightforward induction shows that any two elements a, b in M admit a unique right lcm. Therefore, by Lemma 3.2, Δ_a exists for every a in M. In particular, Δ_x exists for every irreducible element x of M. Finally, by Proposition 3.14, $\bigvee_{x\in\Sigma}\Delta_x$ is a minimal Garside element for M, thus M is a Garside monoid.

(ii) Let M be a Garside monoid with Σ the finite set of its irreducible elements and Δ its minimal Garside element. First assume that M is a divisibility monoid. Then the finite lattice of its simple elements $\downarrow(\Delta)$ is distributive. By Proposition 3.14, we have $\Delta = \bigvee_{x\in\Sigma}\Delta_x$. Now, by Theorem 2.14 (iv), for every x in Σ, $M\backslash x$ is a subset of $\Sigma_1 = \Sigma \cup \{1\}$ including x, say $M\backslash x = \Sigma_x \cup \{x\}$, so we have $\Delta_x = \bigvee_{y\in\Sigma_x\cup\{x\}\subset\Sigma_1} y$, and hence $\Delta = \bigvee_{x\in\Sigma}\Delta_x = \bigvee_{x\in\Sigma} x$. Therefore, the lattice $\downarrow(\Delta)$ is a finite distributive lattice whose upper bound is the join of its atoms, so $\downarrow(\Delta)$ is a hypercube (see Birkhoff (1967) or, for instance, Stanley (1998, page 107).

Conversely, assume that the lattice $\downarrow(\Delta)$ is a hypercube, that is, $\downarrow(\Delta)$ is isomorphic to $\mathbf{1}^{|\Sigma|}$ with $\mathbf{1}$ the rank 1 chain. According to Picantin (2001a, Proposition 3.12), for every positive integer k, every element a dividing Δ^k admits a unique decomposition as $a = b_1 \vee \ldots \vee b_k$ with b_i $\vee$-indecomposable for $1 \leqslant i \leqslant k$, that is, $b_i = b_i' \vee b_i''$, which implies that either $b_i = b_i'$ or $b_i = b_i''$. Then, for every positive integer k, $\downarrow(\Delta^k)$ is isomorphic to $\mathbf{k}^{|\Sigma|}$ with $\mathbf{k}$ the rank k chain, which is a distributive lattice. Now, for every element a in M, the lattice $\downarrow(a)$ is a sublattice of a lattice $\downarrow(\Delta^k)$ for some positive integer k. Every sublattice of a distributive lattice being distributive, the lattice $\downarrow(a)$ is thus distributive for every element a in M. Therefore, all the requirements in the definition of a divisibility monoid are satisfied. □

Example 4.2. Up to isomorphism, there are 2, 5 and 23 Garside divisibility monoids with ranks 2, 3 and 4, respectively. Those with rank 2 are $\mathbb{N}^2 = \langle x, y : xy = yx\rangle$ and $\mathbb{K} = \langle x, y : x^2 = y^2\rangle^{\dagger}$. Figure 2 (to be compared with Figure 1) shows the lattice of simple elements of each of the 5 Garside divisibility monoids with rank 3. Note that, by using Theorem 2.10, we can recognise (from left to right) the monoids $\mathbb{N}^3$, $\mathbb{N} \bowtie \mathbb{N}^2$, $\mathbb{N} \times \mathbb{K}$, $\mathbb{N} \bowtie \mathbb{K}$ and some indecomposable (as a crossed product) monoid. By using Proposition 3.14, we compute that their quasi-centre is isomorphic to $\mathbb{N}^3$, $\mathbb{N}^2$, $\mathbb{N}^2$, $\mathbb{N}^2$ and $\mathbb{N}$, respectively.

In Remark 3.3 and Example 3.10, we have seen how local delta in a divisibility monoid is effectively computable. For instance, let us compute the quasi-centre of the fifth rank 3 Garside divisibility monoid, namely $M_{3,5} = \langle x, y, z : x^2 = yz, y^2 = zx, z^2 = xy\rangle$. We have

[†] $\mathbb{K}$ is for Klein Bottle.

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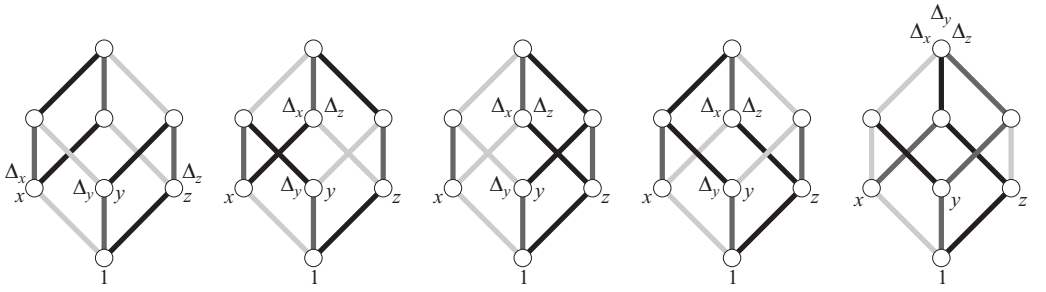


Fig. 2. The lattices of simple elements of the rank 3 Garside divisibility monoids.

$\Upsilon_1(x) = \{1, x, y, z\}\backslash\{x\} = \{1, x, z\}$, $\Upsilon_2(x) = \{1, x, y, z\}\backslash\{1, x, z\} = \{1, x, y, z\} = \Sigma \cup \{1\} = \Upsilon_3(x) = M\backslash x$. Then x admits a local delta, namely $\Delta_x = \bigvee\{1, x, y, z\} = x^3$. Symmetrically, we have $\Delta_y = \Delta_z = x^3$. Therefore, the quasi-centre of $M_{3,5}$ is quite isomorphic to $\mathbb{N}$.

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