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Presentations for Conway's group Co_1

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1. *Introduction*

In [2], chapter S, and also in the ATLAS ([1], p. 131), we give two presentations for $3 \cdot Suz:2$, the normalizer of a certain 3-element in Conway's largest sporadic simple group Co_1. These presentations extend naturally to presentations for groups having $Co_1 \times 2$ as image, leading to conjectured presentations for $Co_1 \times 2$ and Co_1 (see [1], p. 183, although the form of one of the presentations there is slightly different from what we give here).

In this paper we prove the correctness of our conjectured presentations. We make use of one enumeration of 1545600 cosets performed by D. Ž. Djoković's coset enumerator in eight hours of CPU time on a University of Waterloo VAX-11/785, as well as some much smaller enumerations performed by the author's coset enumerator on a Concordia University VAX-11/780. We close the paper by discussing some natural extensions of our presentations.

Remarks

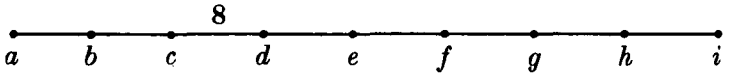
In [3] we give a different presentation for Co_1, via a presentation for $2 \cdot Co_1$. In these presentations, generators and relations for many (Leech) sublattice stabilizers are visible, including Co_2, Co_3, McL, HS and M_{24}.

Our typical presentation consists of a Coxeter graph together with further relations, where the Coxeter graph denotes certain generators and relations in the usual way: each node represents a generator x and the relation $1 = x^2$; if two such generators x, y are unjoined then $xy = yx$; if they are joined by an unlabelled edge then $1 = (xy)^3$; and if joined by an edge labelled n then $1 = (xy)^n$. We use ATLAS notation [1] for group structures.

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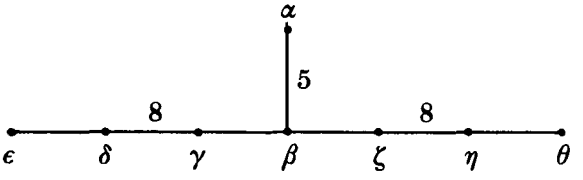
2. *The presentations*

Let G be the group presented by



$$a = (cd)^4, \quad 1 = (bcde)^8 \tag{1}$$

and let Γ be the group presented by



$$\alpha = (\gamma\delta)^4 = (\zeta\eta)^4, \quad 1 = ((\beta\gamma\delta\gamma\delta)^5\epsilon)^3 = (\alpha\beta\gamma\zeta)^5 = (\beta\zeta\eta\theta)^{10}. \tag{2}$$

The following facts follow immediately from [2], chapter S:

$Co_1 \times 2$ is a quotient of G and of Γ.

The relations $1 = (\alpha\beta\gamma)^5 = (\alpha\beta\zeta)^5 = (\beta\gamma\delta\epsilon)^{10}$ hold in Γ.

In G, $\langle a,b,c,d,e,f\rangle \simeq J_2{:}2$, $\langle a,b,c,d,e,f,g\rangle \simeq G_2(4){:}2$, and $\langle a,b,c,d,e,f,g,h\rangle \simeq 3\cdot Suz{:}2$, in which the O_3-subgroup is generated by $(bcdcdefgh)^{13}$.

In Γ, $\langle \alpha,\beta,\gamma,\delta,\epsilon\rangle \simeq J_2$, $\langle \alpha,\beta,\gamma,\delta,\epsilon,\zeta\rangle \simeq G_2(4)$ and $\langle \alpha,\beta,\gamma,\delta,\epsilon,\zeta,\eta\rangle \simeq 3\cdot Suz{:}2$, in which the O_3-subgroup is generated by $(\beta\zeta\eta\zeta\eta)^5 = \omega$ (say). Also, $\langle \alpha,\beta,\gamma,\delta,\zeta,\eta,\theta\rangle \simeq 3\cdot Suz{:}2\times 2$, whose central involution $(\omega\theta)^3 = (\omega^{-1}\theta)^3$ is central in Γ.

3. $\Gamma \simeq G \simeq Co_1 \times 2$

We now show that $\Gamma \simeq G \simeq Co_1 \times 2$, and by giving words for the central involutions of G and Γ, we implicitly give presentations for Co_1.

THEOREM 1. $\Gamma \simeq G$.

Proof. Let $\Sigma = \langle \alpha,\beta,\gamma,\delta,\epsilon,\zeta,\eta\rangle$, and $S = \langle a,b,c,d,e,f,g,h\rangle$. Then $\Sigma \simeq S \simeq 3\cdot Suz{:}2$, and both η and h are 'J_2-type' outer involutions in their respective copies of $3\cdot Suz{:}2$. Thus, there is an isomorphism Φ from Σ onto S such that $\Phi(\eta) = h$. We will extend a suitably chosen Φ to an isomorphism of Γ onto G, such that $\Phi(\theta) = i$.

We first alter the presentation for Γ. Let $\tau = (\delta^{\gamma\beta\zeta}\eta)^3$. Then τ is an involution in Γ centralizing η and θ, and a new presentation $(\hat{2})$ for Γ is obtained if we substitute $1 = [(\delta^{\gamma\beta\zeta}\eta)^3, \theta]$ for $1 = (\beta\zeta\eta\theta)^{10}$ in (2). (These facts are established by using coset enumeration calculations in $\langle \alpha,\beta,\gamma,\delta,\zeta,\eta,\theta\rangle \simeq 3\cdot Suz{:}2\times 2$.) In Σ, we can show that $\langle \alpha,\beta,\gamma,\delta,\epsilon,\zeta,\tau\rangle \simeq G_2(4){:}2$, and $\langle \alpha,\beta,\gamma,\delta,\epsilon,\tau\rangle \simeq J_2{:}2$.

Let Φ be an isomorphism from Σ onto S such that $\Phi(\eta) = h$. Now $C_\Sigma(\eta) \simeq J_2{:}2\times 2$, and τ is an outer involution of Σ. Thus Φ takes $\langle \alpha,\beta,\gamma,\delta,\epsilon,\tau\rangle$ to the unique copy of $J_2{:}2$ in S which centralizes h and contains outer involutions of S, and so $\Phi(\langle \alpha,\beta,\gamma,\delta,\epsilon,\tau\rangle) = \langle a,b,c,d,e,f\rangle$. Any $J_2{:}2$ which extends to a $G_2(4){:}2$ in $3\cdot Suz{:}2$ is contained in precisely two copies of $G_2(4){:}2$. These two copies of $G_2(4){:}2$ are

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interchanged by the outer involution in $3 \cdot Suz:2$ which centralizes the $J_2:2$. Now $\langle a,b,c,d,e,f,g\rangle \simeq G_2(4):2$, and so either Φ, or Φ followed by conjugation by h, takes $\langle \alpha,\beta,\gamma,\delta,\epsilon,\zeta,\tau\rangle$ to $\langle a,b,c,d,e,f,g\rangle$.

Let Φ now denote the appropriate choice, so that Φ is an isomorphism from Σ onto S such that $\Phi(\eta) = h$, and $\Phi(\langle \alpha,\beta,\gamma,\delta,\epsilon,\zeta,\tau\rangle) = \langle a,b,c,d,e,f,g\rangle$. Defining $\Phi(\theta) = i$, we can extend Φ to an isomorphism of Γ onto G, because θ and i satisfy Φ-corresponding relations with respect to Σ and S, in Γ and G, respectively. ∎

THEOREM 2. $(\Gamma \simeq) G \simeq Co_1 \times 2$. *The central involution of* Γ *is* $((\beta\zeta\eta\zeta\eta)^5\theta)^3$, *and the central involution of* G *is* $((bcdcdefgh)^{13}i)^3$.

Proof. Let $\omega = (\beta\zeta\eta\zeta\eta)^5$. Then $(\omega\theta)^3 = (\omega^{-1}\theta)^3$ is a central involution in Γ, and so $\Phi((\omega\theta)^3) = ((bcdcdefgh)^{13}i)^3 = z$ (say) is a central involution in G.

Djoković's enumeration of 1545600 cosets of $\langle a,b,c,d,e,f,g,h,z\rangle \simeq 3 \cdot Suz:2 \times 2$ in G shows that $G \simeq Co_1 \times 2$ (since G is already known to have $Co_1 \times 2$ as an image). ∎

Another presentation for Γ may be obtained by substituting $1 = (\alpha\beta\gamma\delta\epsilon\gamma\beta\delta\gamma)^7$ for $1 = ((\beta\gamma\delta\gamma\delta)^5\epsilon)^3$ in (2). It follows that the 'Presentations' for $Co_1 \times 2$ and Co_1 given in ([1], p. 183) are correct, in that the qualifiers 'more relations?' may be removed.

4. *Extending* Γ *and* G

Let (2)* be the presentation obtained from (2) by adjoining the generator ι, and the relations $1 = \iota^2$, ι commutes with α, β, γ, δ, ϵ, ζ, η, and $1 = (\theta\iota)^3$. Let Γ^* be the group presented by (2)*, and define $\Gamma^*(x,y,\ldots)$ to be the group presented by generators $x, y, \ldots$, and the relations of (2)* involving $x, y, \ldots$ only. Using techniques similar to those in [4], we can show that:

$$\Gamma^*(\alpha,\beta,\zeta,\eta,\theta,\iota) \simeq 2 \cdot (J_2 \operatorname{wr} 2)$$

$$\Gamma^*(\alpha,\beta,\gamma,\zeta,\eta,\theta,\iota) \simeq 2 \cdot (G_2(4) \operatorname{wr} 2)$$

$$\Gamma^*(\alpha,\beta,\gamma,\delta,\zeta,\eta,\theta,\iota) \simeq 2 \cdot ((3 \cdot Suz) \operatorname{wr} 2):2$$

$$\Gamma^* \simeq 2 \cdot (Co_1 \operatorname{wr} 2).$$

In each case the central involution is $(((\beta\zeta\eta\zeta\eta)^5\theta)^3\iota)^3$.

Now let (1)* be the presentation obtained from (1) by adjoining the generator j, and the relations $1 = j^2$, j centralizes a, b, c, d, e, f, g, h, and $1 = (ij)^3$. It follows immediately that the group G^* presented by (1)* is isomorphic to $2 \cdot (Co_1 \operatorname{wr} 2)$, with central involution $(((bcdcdefgh)^{13}i)^3j)^3$.

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