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AN EFFECTIVE BENFORD'S LAW FOR STRICTLY INCREASING AND LOG-CONCAVE SEQUENCES

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Abstract

We establish an effective form of Benford's law for a class of strictly increasing and log-concave sequences $(f_n)_{n\geq 0}$. We apply our result to the partition and plane partition functions, obtaining improvements on bounds due to Iyer for the first occurrence of prescribed leading digit strings.

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1. Introduction

1.1. Main results. Let $b \geq 2$ be an integer and let $w \geq 1$ be an integer with base-b expansion

$$w = \sum_{0\leq i<t} w_i b^i,$$

where $t \geq 1$, $w_i \in \{0, \ldots, b-1\}$ and $w_{t-1} \neq 0$. Thus, w represents a base-b string of length t and

$$b^{t-1} \leq w < b^t.$$

Throughout this paper, let $f = (f_k)_{k\geq 0}$ be a sequence of positive real numbers. We say that f_k begins with the base-b string w if there exists an integer h such that

$$wb^h \leq f_k < (w+1)b^h.$$

For $x \geq 0$, define

$$S_{w,b,f}(x) := \#\{k : 0 \leq k < x, f_k \text{ begins with the base-}b \text{ string } w\}.$$

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We say that f satisfies Benford's law in base b if, for every base-b string w,

$$\lim_{x\to\infty} \frac{S_{w,b,f}(x)}{x} = \rho_{w,b},$$

where $\log_b$ denotes the logarithm to base b and

$$\rho_{w,b} := \log_b(w+1) - \log_b w = \log_b\left(1+\frac{1}{w}\right).$$

Let n_0 and n be nonnegative integers with $n \geq n_0 + 2$. We say that a sequence $(f_k)_{n_0\leq k\leq n}$ of positive real numbers is *strictly increasing and log-concave* if

$$f_k > f_{k-1} \quad (n_0 < k \leq n),$$

and

$$f_k^2 \geq f_{k-1}f_{k+1} \quad (n_0 < k < n).$$

We use the same terminology for an infinite sequence $(f_k)_{k\geq n_0}$ when every finite subsequence $(f_k)_{n_0\leq k\leq n}$, with $n \geq n_0 + 2$, has these properties.

The main result of this paper is the following quantitative discrepancy estimate for the counting function $S_{w,b,f}(x)$.

THEOREM 1.1. *Let $n_0 \geq 0$ be an integer. Suppose that $(f_n)_{n\geq n_0}$ is strictly increasing and log-concave. Then, for every integer $n > n_0$,*

$$|S_{w,b,f}(n) - \rho_{w,b}n| \leq 2\log_b f_n + \frac{\rho_{w,b}(1-\rho_{w,b})}{\log_b f_n - \log_b f_{n-1}} + \lambda_{w,b,f}(n_0),$$

where

$$\lambda_{w,b,f}(n_0) := 2 + n_0 + 2\rho_{w,b} - 2\log_b f_{n_0}.$$

Theorem 1.1 yields the following Benford criterion, since

$$S_{w,b,f}(x) = S_{w,b,f}(\lceil x\rceil),$$

where $\lceil x\rceil$ denotes the least integer $\geq x$.

COROLLARY 1.2. *Under the hypotheses of Theorem 1.1, suppose further that*

$$\frac{\log f_n}{n} \to 0 \quad \textit{and} \quad n(\log f_n - \log f_{n-1}) \to \infty$$

as $n \to \infty$. Then, f satisfies Benford's law in every base $b \geq 2$. More precisely,

$$S_{w,b,f}(x) = \rho_{w,b}x + o_{w,b}(x) \quad (x \to \infty).$$

General sufficient criteria for Benford's law have previously been established (see, for example, Anderson *et al*. [1]). The criterion in Corollary 1.2 is instead derived from the structural assumptions of strict increase and log-concavity. Moreover, Theorem 1.1 gives an explicit discrepancy estimate for the occurrence of each prescribed initial string.

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For a sequence $f = (f_n)_{n\geq 0}$ of positive real numbers, let $N_f(w, b)$ denote the least integer $n \geq 0$ such that f_n begins with the string w in base b, whenever such an integer exists. Equivalently,

$$N_f(w, b) = \min\{n \geq 1 : S_{w,b,f}(n) > 0\} - 1. \tag{1.1}$$

Theorem 1.1 immediately yields the following upper bound.

COROLLARY 1.3. *Under the hypotheses of Theorem 1.1, suppose that the set of integers $n > n_0$ satisfying*

$$n > \frac{2}{\rho_{w,b} \log b} \log f_n + \frac{(1 - \rho_{w,b}) \log b}{\log f_n - \log f_{n-1}} + \frac{\lambda_{w,b,f}(n_0)}{\rho_{w,b}}$$

is nonempty. Let $N_{w,b,f}$ denote its least element. Then,

$$N_f(w, b) \leq N_{w,b,f}.$$

1.2. Applications to partitions and plane partitions. For $n \geq 0$, let $p(n)$ denote the number of partitions of n and let $\mathrm{PL}(n)$ denote the number of plane partitions of n. Recall that a partition of n is a nonincreasing sequence of positive integers with sum n. A plane partition of n is an array of nonnegative integers $(\pi_{i,j})_{i,j\geq 1}$ whose rows and columns are weakly decreasing, and such that $\sum_{i,j\geq 1} \pi_{i,j} = n$. We refer the reader to [2] for background on partitions and plane partitions.

The leading digits of both $p(n)$ and $\mathrm{PL}(n)$ obey Benford's law (see [1, 4]). Motivated by this phenomenon, Douglass and Ono [4] asked for nontrivial upper bounds for $N_p(w, b)$ and $N_{\mathrm{PL}}(w, b)$. Here, w is a base-b digit string, and $N_p(w, b)$, respectively $N_{\mathrm{PL}}(w, b)$, is the least index n for which $p(n)$, respectively $\mathrm{PL}(n)$, begins with the digit string w in base b.

Let t be the length of w. Using exponential sums and transcendence theory, Luca [6, 7] proved that

$$N_p(w, b) \leq \exp(2 \cdot 10^{25}(t + 12)(\log b)^2),$$

and, using results concerning prime numbers, that

$$N_{\mathrm{PL}}(w, b) \leq b^{51t+688}.$$

More recently, Iyer [5] developed an elementary discrepancy method and obtained the substantially sharper bounds

$$N_p(w, b) \leq 288b^{2t} + 2$$

and

$$N_{\mathrm{PL}}(w, b) \leq 130b^{3t} + 29400b^{3t/2}.$$

We now apply Theorem 1.1 to obtain explicit discrepancy estimates for $p(n)$ and $\mathrm{PL}(n)$. These estimates yield improved bounds for the first occurrence of a prescribed leading digit string.

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The sequences $(p(n))_{n\geq 2}$ and $(\mathrm{PL}(n))_{n\geq 2}$ are strictly increasing. Moreover, Nicolas [8] and DeSalvo and Pak [3, Theorem 1.1] proved that $(p(n))_{n\geq 25}$ is log-concave, whereas Ono *et al.* [9] proved that $(\mathrm{PL}(n))_{n\geq 12}$ is log-concave. Consequently, Theorem 1.1 applies to $p(n)$ with $n_0 = 25$ and to $\mathrm{PL}(n)$ with $n_0 = 12$.

COROLLARY 1.4. *For every integer $n \geq 100$,*

$$|S_{w,b,p}(n) - \rho_{w,b}n| < \frac{4\pi}{\sqrt{6}\log b}\sqrt{n} + 1.1\rho_{w,b}\sqrt{n}\log b + 27.$$

For every integer $n \geq 2829$,

$$|S_{w,b,\mathrm{PL}}(n) - \rho_{w,b}n| < \frac{6\zeta(3)^{1/3}}{2^{2/3}\log b}n^{2/3} + 0.9\rho_{w,b}n^{1/3}\log b + 14.$$

Here, $\zeta(s)$ denotes the Riemann zeta function.

We derive the following upper bounds from Corollary 1.4.

COROLLARY 1.5. *We have*

$$N_p(w,b) \leq \frac{8\pi^2/3}{\rho_{w,b}^2(\log b)^2} + \frac{66}{\rho_{w,b}} + 29(\log b)^2$$

and

$$N_{\mathrm{PL}}(w,b) \leq \frac{54\zeta(3)}{\rho_{w,b}^3(\log b)^3} + \frac{65}{\rho_{w,b}} + 15\rho_{w,b}(\log b)^3 + 2\rho_{w,b}^3(\log b)^6 + 2697.$$

REMARK 1.6. The bounds in Corollary 1.5 depend on w and b through $\rho_{w,b}$, and therefore retain more information than bounds depending only on the length t of w. Consequently, for a fixed digit string w, they can be substantially sharper than the corresponding bounds obtained by replacing $\rho_{w,b}$ with a length-based estimate. For comparison, since $b^{t-1} \leq w < b^t$, the bounds in Corollary 1.5 yield

$$N_p(w,b) < 27b^{2t} + 66b^t\log b + 29(\log b)^2$$

and

$$N_{\mathrm{PL}}(w,b) < 65b^{3t} + 2697.$$

These estimates improve the corresponding bounds of Iyer [5].

2. Proof of Theorem 1.1

We begin by translating the leading-string condition into a condition on logarithms.

LEMMA 2.1. *Let $y > 0$. Then, y begins with the base-b string w if and only if*

$$\log_b y \in \mathcal{E}_{w,b},$$

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where

$$\mathcal{E}_{w,b} := \bigsqcup_{h\in\mathbb{Z}} [h + \log_b w,\ h + \log_b w + \rho_{w,b}).$$

PROOF. By definition, y begins with the string w if and only if there exists $h \in \mathbb{Z}$ such that $wb^h \le y < (w+1)b^h$. Taking logarithms to base b, this is equivalent to

$$h + \log_b w \le \log_b y < h + \log_b(w+1).$$

Since $\log_b(w+1) = \log_b w + \rho_{w,b}$, the result follows. □

We now consider a general 1-periodic set of the form

$$\mathcal{E} = \bigsqcup_{h\in\mathbb{Z}} [h+\alpha, h+\alpha+\rho), \quad \alpha \in \mathbb{R},\ 0 < \rho \le 1.$$

LEMMA 2.2. *Let $a < b$ and let $g : [a,b] \to [0,\infty)$ be nondecreasing. Then,*

$$\left| \int_a^b (\mathbf{1}_{\mathcal{E}}(u) - \rho) g(u)\, du \right| \le \rho(1-\rho) g(b^-).$$

Here, $\mathbf{1}_{\mathcal{E}}(u)$ equals 1 if $u \in \mathcal{E}$ and 0 otherwise.

PROOF. Put $q(u) := \mathbf{1}_{\mathcal{E}}(u) - \rho$. Then, q is 1-periodic and has mean zero on every period. Let $t := \{u - \alpha\}$ and define

$$Q(u) := \begin{cases} (1-\rho)t - \frac{1}{2}\rho(1-\rho), & 0 \le t < \rho, \\ \rho(1-t) - \frac{1}{2}\rho(1-\rho), & \rho \le t < 1. \end{cases}$$

Then, Q is continuous and 1-periodic, with

$$Q'(u) = q(u) \text{ for almost every } u, \quad |Q(u)| \le \tfrac{1}{2}\rho(1-\rho).$$

By Lebesgue–Stieltjes integration by parts,

$$\int_a^b q(u) g(u)\, du = Q(b) g(b^-) - Q(a) g(a^+) - \int_{(a,b)} Q(u)\, dg(u).$$

Since g is nondecreasing, dg is a positive measure. Hence,

$$\left| \int_a^b q(u) g(u)\, du \right| \le \frac{1}{2}\rho(1-\rho)\left(g(b^-) + g(a^+) + \int_{(a,b)} dg \right) = \rho(1-\rho) g(b^-).$$

This completes the proof. □

PROPOSITION 2.3. *Let $(c_k)_{0\le k\le n}$ be a strictly increasing concave sequence of real numbers. Then,*

$$|\#\{k : 0 \le k < n, c_k \in \mathcal{E}\} - \rho n| \le 2(c_n - c_0 + 1 + \rho) + \frac{\rho(1-\rho)}{c_n - c_{n-1}}.$$

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PROOF. Define $F : [0, n] \to \mathbb{R}$ by $F(k) = c_k$ for $0 \le k \le n$ and by requiring F to be affine on each interval, $[k, k+1]$. Thus,

$$F(x) = c_k + (x-k)(c_{k+1} - c_k) \quad (k \le x \le k+1).$$

The slopes of F are positive and nonincreasing, so F is strictly increasing and concave. Hence, $G := F^{-1} : [c_0, c_n] \to [0, n]$ is convex; let G' be its nondecreasing piecewise-constant derivative. Write

$$\mathcal{E} \cap [c_0, c_n) = I_1 \sqcup \cdots \sqcup I_m$$

as a disjoint union of intervals. Each I_j lies in some $[\ell + \alpha, \ell + \alpha + \rho)$ meeting $[c_0, c_n)$, whence $m < c_n - c_0 + 1 + \rho$. Since $F([0, n)) = [c_0, c_n)$,

$$\#\{k : 0 \le k < n, c_k \in \mathcal{E}\} = \sum_{j=1}^{m} \#(\mathbb{Z} \cap G(I_j)).$$

Moreover,

$$|G(I_j)| = \int_{I_j} G'(u)\,du, \quad \rho n = \int_{c_0}^{c_n} \rho G'(u)\,du.$$

Therefore,

$$\#\{k : 0 \le k < n, c_k \in \mathcal{E}\} - \rho n = \int_{c_0}^{c_n} (\mathbf{1}_{\mathcal{E}}(u) - \rho) G'(u)\,du + \sum_{j=1}^{m} (\#(\mathbb{Z} \cap G(I_j)) - |G(I_j)|).$$

For every interval J, we have $|\#(\mathbb{Z} \cap J) - |J|| \le 2$. Since G' is nondecreasing, Lemma 2.2 gives

$$\left| \int_{c_0}^{c_n} (\mathbf{1}_{\mathcal{E}}(u) - \rho) G'(u)\,du \right| \le \rho(1-\rho) G'(c_n^-).$$

Finally, since F has slope $c_n - c_{n-1}$ on $[n-1, n)$,

$$G'(c_n^-) = \frac{1}{c_n - c_{n-1}}.$$

The result follows. □

PROOF OF THEOREM 1.1. Put $c_k := \log_b f_{k+n_0}$ for all $k \ge 0$. Then, $(c_k)_{k \ge 0}$ is strictly increasing and concave. By Lemma 2.1, f_{k+n_0} begins with the base-b string w if and only if $c_k \in \mathcal{E}_{w,b}$, where $\mathcal{E}_{w,b}$ is of the form considered in Proposition 2.3, with $\alpha = \log_b w$ and $\rho = \rho_{w,b}$. Applying Proposition 2.3 to $(c_k)_{0 \le k \le n - n_0}$ and noting that $c_k = \log_b f_{k+n_0}$ for $k \ge 0$, we obtain

$$\begin{aligned} &|\#\{0 \le k < n - n_0 : c_k \in \mathcal{E}_{w,b}\} - \rho_{w,b}(n - n_0)| \\ &\quad \le 2(\log_b f_n - \log_b f_{n_0} + 1 + \rho_{w,b}) + \frac{\rho_{w,b}(1 - \rho_{w,b})}{\log_b f_n - \log_b f_{n-1}}. \end{aligned}$$

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Moreover,

$$\#\{k : 0 \le k < n - n_0, c_k \in \mathcal{E}_{w,b}\} = S_{w,b,f}(n) - S_{w,b,f}(n_0)$$

and

$$|S_{w,b,f}(n_0) - \rho_{w,b} n_0| \le n_0.$$

The triangle inequality therefore gives

$$|S_{w,b,f}(n) - \rho_{w,b} n| \le 2(\log_b f_n - \log_b f_{n_0}) + 2(1 + \rho_{w,b}) + \frac{\rho_{w,b}(1 - \rho_{w,b})}{\log_b f_n - \log_b f_{n-1}} + n_0,$$

which simplifies to the asserted bound. □

3. Proofs of the applications

We use the following explicit estimate of Iyer [5, Lemma 2.2].

LEMMA 3.1. *For every integer $n \ge 4$,*

$$\left| \log p(n) - \frac{2\pi}{\sqrt{6}}\sqrt{n} + \log n - \log\left(\frac{\sqrt{3}}{12}\right) \right| \le \frac{4}{\sqrt{n}}.$$

Set

$$C := \pi\sqrt{\frac{2}{3}}, \quad \beta := C(\sqrt{2} - 1) - \frac{\log 2}{\sqrt{100}} - \frac{4 + 2\sqrt{2}}{100} > 0.9249.$$

LEMMA 3.2. *For every integer $n \ge 100$,*

$$\log p(n) \le C\sqrt{n}, \quad \log p(n) - \log p(n-1) \ge \beta n^{-1/2}.$$

PROOF. The first assertion follows from Lemma 3.1, since

$$-\log n + \log\left(\frac{\sqrt{3}}{12}\right) + \frac{4}{\sqrt{n}} < 0 \quad (n \ge 100).$$

For the second, put $d_n := \log p(n) - \log p(n-1)$. Since $p(n)$ is log-concave for $n \ge 25$, the sequence d_n is nonincreasing. Consequently,

$$n d_n \ge \log p(2n) - \log p(n) \ge C(\sqrt{2} - 1)\sqrt{n} - \log 2 - \frac{4 + 2\sqrt{2}}{\sqrt{n}}.$$

Hence,

$$d_n \ge n^{-1/2}\left(C(\sqrt{2} - 1) - \frac{\log 2}{\sqrt{n}} - \frac{4 + 2\sqrt{2}}{n}\right) \ge \beta n^{-1/2},$$

because the expression in parentheses is increasing in n. □

Set

$$\alpha := 3 \cdot 2^{-2/3}\zeta(3)^{1/3}, \quad B := \frac{2^{25/36} e^{\zeta'(-1)} \zeta(3)^{7/36}}{\sqrt{12\pi}}$$

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and

$$\gamma := \alpha(2^{2/3}-1) - \frac{25\log 2}{36\cdot 2829^{2/3}} - \frac{200(1+2^{-2/3})}{2829^{4/3}} > 1.1697.$$

We also use the following estimate of Iyer [5, Lemma 2.3].

LEMMA 3.3. *For every integer $n \geq 2829$,*

$$\left|\log \mathrm{PL}(n) - \alpha n^{2/3} + \frac{25}{36}\log n - \log B\right| \leq \frac{200}{n^{2/3}}.$$

LEMMA 3.4. *For every integer $n \geq 2829$,*

$$\log \mathrm{PL}(n) \leq \alpha n^{2/3}, \quad \log \mathrm{PL}(n) - \log \mathrm{PL}(n-1) \geq \gamma n^{-1/3}.$$

PROOF. The first assertion follows from Lemma 3.3, since

$$-\frac{25}{36}\log n + \log B + \frac{200}{n^{2/3}} < 0 \quad (n \geq 2829).$$

For the second, set

$$d_n := \log \mathrm{PL}(n) - \log \mathrm{PL}(n-1).$$

Log-concavity of $\mathrm{PL}(n)$ for $n \geq 12$ gives $nd_n \geq \log \mathrm{PL}(2n) - \log \mathrm{PL}(n)$ and Lemma 3.3 gives

$$\log \mathrm{PL}(2n) - \log \mathrm{PL}(n) \geq \alpha(2^{2/3}-1)n^{2/3} - \frac{25}{36}\log 2 - \frac{200(1+2^{-2/3})}{n^{2/3}}.$$

Therefore,

$$d_n \geq n^{-1/3}\left(\alpha(2^{2/3}-1) - \frac{25\log 2}{36n^{2/3}} - \frac{200(1+2^{-2/3})}{n^{4/3}}\right) \geq \gamma n^{-1/3},$$

as the expression in parentheses is increasing in n. □

PROOF OF COROLLARY 1.4. For every integer $n \geq 100$, apply Theorem 1.1 with $(f_n, n_0) = (p(n), 25)$. Lemma 3.2 gives

$$2\log_b p(n) \leq \frac{4\pi}{\sqrt{6}\log b}\sqrt{n}$$

and

$$\frac{\rho_{w,b}(1-\rho_{w,b})}{\log_b p(n) - \log_b p(n-1)} \leq \frac{\rho_{w,b}\log b}{\beta}\sqrt{n} < 1.1\rho_{w,b}\sqrt{n}\log b.$$

Since $p(25) > 2$ and $\rho_{w,b} \leq \log_b 2$,

$$\lambda_{w,b,p}(25) = 27 + 2\rho_{w,b} - 2\log_b p(25) \leq 27.$$

This proves the asserted estimate for $p(n)$.

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Likewise, for every integer $n \geq 2829$, apply Theorem 1.1 with $(f_n, n_0) = (\mathrm{PL}(n), 12)$. Lemma 3.4 gives

$$2\log_b \mathrm{PL}(n) \leq \frac{2\alpha}{\log b} n^{2/3} = \frac{6\zeta(3)^{1/3}}{2^{2/3}\log b} n^{2/3}$$

and

$$\frac{\rho_{w,b}(1-\rho_{w,b})}{\log_b \mathrm{PL}(n) - \log_b \mathrm{PL}(n-1)} \leq \frac{\rho_{w,b}\log b}{\gamma} n^{1/3} < 0.9\rho_{w,b} n^{1/3}\log b.$$

Since $\mathrm{PL}(12) > 2$ and $\rho_{w,b} \leq \log_b 2$,

$$\lambda_{w,b,\mathrm{PL}}(12) = 14 + 2\rho_{w,b} - 2\log_b \mathrm{PL}(12) \leq 14.$$

This proves the plane-partition estimate. □

Before giving the proof of Corollary 1.5, we record the following elementary lemma.

LEMMA 3.5. *Let s be a positive integer, $a > 0$, and $a_1, \ldots, a_s \geq 0$. If a real number x satisfies*

$$x > a + \sum_{1 \leq \nu \leq s} \frac{a_\nu}{a^\nu},$$

then

$$x > a + \sum_{1 \leq \nu \leq s} \frac{a_\nu}{x^\nu}.$$

PROOF. The hypothesis implies $x > a > 0$. Hence, $a_\nu/x^\nu \leq a_\nu/a^\nu$ for each ν and summing these inequalities gives the result. □

PROOF OF COROLLARY 1.5. Put $\rho = \rho_{w,b}$ and $L = \log b$. For partitions, Corollary 1.4 implies that

$$\frac{S_{w,b,p}(n)}{\rho\sqrt{n}} > \sqrt{n} - \left(\frac{4\pi}{\rho\sqrt{6}L} + 1.1L\right) - \frac{27}{\rho\sqrt{n}}.$$

Lemma 3.5, applied with

$$a = \frac{4\pi}{\rho\sqrt{6}L} + 1.1L, \quad a_1 = \frac{27}{\rho},$$

therefore shows that $S_{w,b,p}(n) > 0$ for $n \geq 100$, as soon as

$$\sqrt{n} > \frac{4\pi}{\rho\sqrt{6}L} + 1.1L + \frac{27}{\rho}\left(\frac{4\pi}{\rho\sqrt{6}L} + 1.1L\right)^{-1}.$$

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Squaring the right-hand side and bounding its coefficients gives the sufficient condition

$$n \geq \frac{8\pi^2/3}{\rho^2 L^2} + \frac{66}{\rho} + 29L^2.$$

The right-hand side is greater than 100. Thus, every integer n satisfying the displayed sufficient condition lies in the range of Corollary 1.4. It follows that $S_{w,b,p}(n) > 0$ and (1.1) yields the asserted bound for $N_p(w, b)$.

For plane partitions, write $x = n^{1/3}$. Corollary 1.4 gives

$$\frac{S_{w,b,\mathrm{PL}}(n)}{\rho n^{2/3}} > x - \frac{3 \cdot 2^{1/3}\zeta(3)^{1/3}}{\rho L} - \frac{0.9L}{x} - \frac{14}{\rho x^2}.$$

Set

$$A := \frac{3 \cdot 2^{1/3}\zeta(3)^{1/3}}{\rho L}.$$

Since

$$\frac{0.9L}{A} + \frac{14}{\rho A^2} = \left(\frac{0.9}{3 \cdot 2^{1/3}\zeta(3)^{1/3}} + \frac{14}{9 \cdot 2^{2/3}\zeta(3)^{2/3}}\right)\rho L^2 < 1.1\rho L^2,$$

Lemma 3.5 shows that it suffices to require $x \geq A + 1.1\rho L^2$. Cubing and using

$$(3 \cdot 2^{1/3}\zeta(3)^{1/3})^3 = 54\zeta(3)$$

shows that this is implied by

$$n \geq \frac{54\zeta(3)}{\rho^3 L^3} + \frac{65}{\rho} + 15\rho L^3 + 2\rho^3 L^6.$$

The right-hand side is greater than 132. Thus, after adding 2697, every integer n satisfying the preceding sufficient condition is at least 2829. Corollary 1.4 therefore applies and (1.1) yields the second assertion. □

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