109.05 A fruitful blunder

Recently while evaluating the scripts of a math contest I came across some scripts bearing the following claim:

> "Since all the roots of each of the two real polynomials $P(x)$ and $Q(x)$ are real, the sum-polynomial $P(x) + Q(x)$ must have at least one real root if it happens to be a non-constant polynomial."

While the claim is patently true if $P(x)$ and $Q(x)$ are polynomials of degree one, it fails to hold in general if at least one of the real polynomials is of degree at least two. One may consider $P(x) = x^2$ and $Q(x) = x + 1$ to obtain $P(x) + Q(x) = x^2 + x + 1$ which does not have a real root. If one insists on counterexamples with polynomials of same degree then one may work with $P(x) = x(x - 1)$ and $Q(x) = (x - 2)(x - 3)$ to the same effect. But note that if $P(x)$ and $Q(x)$ are real polynomials of degree two and have a common root then the sum-polynomial $P(x) + Q(x)$ is a real polynomial with at most two real roots. Can this be extended to three or more real polynomials of degree two? We consider monic real polynomials of degree two to keep matters simple. Let $P_i(x), i = 1, 2, 3$ be three monic real polynomials of degree two and suppose that any two of them have exactly one real root in common. If a root is common to all the polynomials then trivially the sum-polynomial has a real root. Hence assume that, that is not the case and that the common root of one pair of polynomials is distinct from the common root of another pair. So,

$$P_1(x) = (x - a)(x - b);$$

$$P_2(x) = (x - b)(x - c);$$

$$P_3(x) = (x - c)(x - a);$$

for three distinct reals a, b, c, and

$$S(x) := P_1(x) + P_2(x) + P_3(x) = 3x^2 - 2(a + b + c)x + (ab + bc + ca).$$

The discriminant of $S(x)$ is

$$4\left(a^2 + b^2 + c^2 - ab - bc - ca\right) = 2\left\{(a - b)^2 + (b - c)^2 + (c - a)^2\right\} > 0$$

since x, b, c are distinct. Thus both the roots of the sum-polynomial $S(x)$ are real. How about four monic real polynomials of degree two with $P_i(x)$ and $P_{i+1}(x)$ having exactly one real root in common and distinct pair of polynomials having distinct common roots? Here $i = 1, 2, 3, 4$ and $P_5(x) = P_1(x)$. Note that we are not considering every possible pair of polynomials but the pairs are chosen cyclically. For $i = 1, 2, 3, 4$ let

$$P_i(x) = \left(x - a_1\right)\left(x - a_{1+1}\right)$$

where a_1, a_2, a_3, a_4 are distinct reals and $a_5 = a_1$. Then

$$S(x) = \left(2x - a_1 - a_3\right)\left(2x - a_2 - a_4\right)$$

and it has two real roots.

These two cases may prompt one to ask the following question.

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What is the largest value of a positive integer n such that for every set $\{a_1, \ldots, a_n\}$ *of real numbers which are necessarily distinct, the polynomial*

$$P(x) = \sum_{i=1}^{n} (x - a_i)(x - a_{i+1})$$

with $a_{n+1} = a_1$ *has a real root*?

From what we have seen so far, $n \geqslant 4$.

Note that the discriminant Δ of $P(x)$ is

$$\Delta = 4\left\{\left(\sum_{i=1}^{n} a_i\right)^2 - n\sum_{i=1}^{n} a_i a_{i+1}\right\}.$$

For $P(x)$ to have a real root it is necessary and sufficient to have $\Delta \geqslant 0$. For $m \geqslant 3$ let

$$O_{2m+1} = \{-m, -m+1, -m+2, \ldots, -2, -1, 0, 1, 2, \ldots, m-2, m-1, m\}$$

and

$$O_{2m+1} = \{-m, -m+1, -m+2, \ldots, -2, -1, 1, 2, \ldots, m-2, m-1, m\}.$$

Then

$$\Delta_{O_{2m+1}} = -\frac{4m(m-2)(2m+1)^2}{3} < 0$$

and

$$\Delta_{E_{2m}} = \frac{8m(2m^3 - 3m^2 - 2m - 3)}{3} < 0$$

because

$$2m^3 - 3m^2 - 2m - 3 = 2\{m^2(m-2) - 2\} + (m-1)^2 > 0.$$

Thus we have exhibited sets of real numbers of size at least six for which $P(x)$ does not have a real root. Note that, if we consider

$$O_5 = \{-2, -1, 0, 1, 2\}$$

then

$$P(x) = 5x^2$$

which has a double root at $x = 0$ implying $\Delta_{O_5} = 0$. This shows that the construction for odd integers greater than 5 cannot be mimicked for the case $n = 5$ to produce a counterexample. But a little playing around throws up a handful of them. For instance, if one considers the set $\{1, 2, 3, 5, 6\}$, then

$$P(x) = 5x^2 - 34x + 59$$

and its discriminant is negative.

Hence the largest value of n such that for all n-element sets of real

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numbers $\{a_1, \ldots, a_n\}$ the polynomial

$$P(x) = \sum_{i=1}^{n} (x - a_i)(x - a_{i+1})$$

with $a_{n+1} = a_1$ has a real root is 4.

The problem analysed here could not have occurred to me had not some of the students committed the blunder in the contest. This explains the choice of the title.

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109.06 With or without L'Hospital's rule

The purpose of this Note is to prove the following:

Proposition: Consider a real-valued continuous function f defined in a neighborhood of $+\infty$ such that $\lim_{x \to \infty} f(x)$ exists and is finite. Then, for any antiderivative F of f, we have

$$\lim_{x \to \infty} \frac{F(x)}{x} = \lim_{x \to \infty} f(x).$$

Proof:

Case 1. Assume that $\lim_{x \to \infty} f(x) = L \neq 0$. Write $F(x) = c + \int_A^x f(t)\,dt$, where $A > 0$ is chosen so that, for $t > A$, $f(t) > \alpha > 0$ if $L > 0$ or $f(t) < \beta < 0$ if $L < 0$. It

follows that lim F (x) = + if L > 0 and lim F (x) = − if L < 0. L'Hospital's rule* (see [1]) says that, if a function F is defined and differentiable in a neighborhood of $+\infty$, $\lim_{x \to +\infty} F(x) = \pm\infty$, and $\lim_{x \to +\infty} F'(x)$ exists, then

$$\lim_{x \to +\infty} \frac{F(x)}{x} = \lim_{x \to +\infty} \frac{F'(x)}{1}.$$

In our case, the conditions of L'Hospital's rule are satisfied, and the conclusion of the Proposition readily follows:

$$\lim_{x \to +\infty} \frac{F(x)}{x} = \lim_{x \to +\infty} f(x).$$

* This is the original 17th-18th century French spelling; in time, the silent 's' has been removed and replaced with the circumflex over the preceding vowel. The former spelling is still used in English where there is no circumflex. Published in 1696, the rule is believed to be, in fact, the work of Johann Bernoulli.

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