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On the Independent Domination Number of Random Regular Graphs

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A dominating set $\mathscr{D}$ of a graph G is a subset of $V(G)$ such that, for every vertex $v \in V(G)$, either in $v \in \mathscr{D}$ or there exists a vertex $u \in \mathscr{D}$ that is adjacent to v. We are interested in finding dominating sets of small cardinality. A dominating set $\mathscr{I}$ of a graph G is said to be *independent* if no two vertices of $\mathscr{I}$ are connected by an edge of G. The size of a smallest independent dominating set of a graph G is the *independent domination number* of G. In this paper we present upper bounds on the independent domination number of random regular graphs. This is achieved by analysing the performance of a randomized greedy algorithm on random regular graphs using differential equations.

1. Introduction

Throughout this paper we consider simple graphs that are undirected, unweighted and contain no loops or multiple edges. A graph G is said to be d-regular if every vertex in $V(G)$ has degree d (*i.e.*, each vertex is adjacent to precisely d other vertices in G). When discussing any graph G, we let n denote the cardinality of $V(G)$ and for d-regular graphs on n vertices, we assume dn to be even to avoid parity problems. For other basic graph-theoretical definitions we refer the reader to, for example, Diestel [3].

A *dominating set* $\mathscr{D}$ of a graph G is a subset of the vertices of G such that, for every vertex $v \in V(G)$, either in $v \in \mathscr{D}$ or there exists an edge $uv \in E(G)$ with $u \in \mathscr{D}$. We are interested in finding dominating sets of small cardinality. The *domination number* of a graph G, which we denote by $\gamma(G)$, is the size of a smallest dominating set of G.

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For an arbitrary graph G the problem of determining $\gamma(G)$ is one of the core NP-hard optimization problems in graph theory and this problem remains NP-hard even for planar graphs of maximum degree 3 [5]. Johnson [7] showed that for general graphs on n vertices, $\gamma(G)$ is approximable within $1 + \log n$. Raz and Safra [11] showed that $\gamma(G)$ is not approximable within $c \log n$ for some $c > 0$. When restricted to graphs of bounded degree $d \geqslant 3$, Papadimitriou and Yannakakis [10] showed that the problem of determining $\gamma(G)$ is APX-complete and is approximable within $-\frac{1}{2} + \sum_{i=1}^{d+1} i^{-1}$.

A dominating set $\mathscr{I}$ of a graph G is said to be *independent* if no two vertices of $\mathscr{I}$ are connected by an edge of G. The *independent domination number* of a graph G, denoted by $\gamma_i(G)$, is the size of a smallest independent dominating set of G. Halldórsson [6] showed that for an arbitrary graph G, $\gamma_i(G)$ is not approximable within $n^{1-\epsilon}$ for any $\epsilon > 0$. Note that for a d-regular graph G it is simple to verify that determining $\gamma_i(G)$ is approximable within $(d+1)/2$. For graphs with bounded degree d, Alimonti and Calamoneri [1] improved upon this trivial bound when d is small and Kann [8] showed that this problem is APX-complete for bounded degree graphs.

We consider random d-regular graphs that are generated *uniformly at random* (u.a.r.), and need some associated notation. We say that a property $\mathscr{B} = \mathscr{B}_n$ of a random graph holds asymptotically almost surely (a.a.s.) if the probability that $\mathscr{B}$ holds tends to 1 as n tends to infinity. When d-regular graphs are the objects of consideration, this is modified so that n is restricted to even numbers if d is odd. For other basic random graph theory definitions we refer the reader to Bollobás [2].

We [4] improved upon an earlier result of Molloy and Reed [9] by showing that for a random cubic (*i.e.*, 3-regular) graph G, $\gamma_i(G)$ a.a.s. satisfies $0.2641n \leqslant \gamma_i(G) \leqslant 0.27942n$. The upper bound was achieved by using differential equations to analyse the performance of a randomized greedy algorithm that is based on repeatedly choosing vertices of current minimum degree and deleting edges. The lower bound was calculated by means of a direct expectation argument.

Zito [16] presented upper and lower bounds on $\gamma_i(G)$ when G is a random d-regular graph and gave explicit values for these bounds when $3 \leqslant d \leqslant 7$. The lower bounds were, again, calculated by means of a direct expectation argument whilst the upper bounds were calculated by using differential equations to analyse the performance of a randomized algorithm that is based on repeatedly choosing vertices of a particular degree and deleting edges. Note that for $d = 3$, the upper bound in [16] is larger than the upper bound result presented in [4].

In this paper we analyse the average-case performance of a simple heuristic, which is a random greedy algorithm that gives upper bounds on $\gamma_i(G)$ when G is a random d-regular graph. This algorithm is an extension of that for $d = 3$ presented in [4] and improves all upper bounds presented in [16]. Using the direct expectation argument presented in [16], we also evaluate corresponding lower bounds on $\gamma_i(G)$ for several small values of d.

In the following section we give a description of our algorithm and in section 3 we outline the method used for its analysis. Our analysis uses a theorem of the second author [15], which we restate in section 3. The results of this paper are encompassed by the following theorem, the proof of which is given in section 4.

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Table 1. Bounds on $\gamma_i(G)$ when G is a random d-regular graph on n vertices.

d	$\mathscr{C}_\ell(d)n$	$\mathscr{C}_u(d)n$
03	0.2641n	0.27942n
04	0.2236n	0.24399n
05	0.1959n	0.21852n
06	0.1755n	0.19895n
07	0.1596n	0.18329n
08	0.1468n	0.17037n
09	0.1362n	0.15948n
10	0.1273n	0.15015n
15	0.0976n	0.11783n
20	0.0803n	0.09830n
30	0.0606n	0.07526n
40	0.0494n	0.06181n
50	0.0420n	0.05285n

Theorem 1.1. *Let $d \geqslant 3$ be fixed. Then, for a random d-regular graph G on n vertices, the size of a minimum independent dominating set is asymptotically almost surely less than $\mathscr{C}_u(d)n$, where the constant $\mathscr{C}_u(d)$ is given in Table* 1.

The constant $\mathscr{C}_u(d)$ referred to in Theorem 1.1 arises from the solution of particular sets of differential equations. In Table 1, corresponding lower bounds are also given by evaluating constants $\mathscr{C}_\ell(d)$ from the argument in [16] (the details of which are restated in the final section).

2. Prioritizing choices

Consider the following algorithm that greedily finds a dominating set of a graph G. Repeatedly choose a vertex u randomly to add to a set $\mathscr{I}$. After each vertex is chosen, remove u and its neighbours from G along with all their incident edges. Once no vertices remain, the set $\mathscr{I}$ is a dominating set in G. It is not difficult to see that the set $\mathscr{I}$ is also a maximal independent set and therefore an independent dominating set.

We modify this algorithm slightly by giving priority, in the choice of u, to the vertices of current minimum degree in G, and add to $\mathscr{I}$, not always u, but sometimes one of its neighbours. To be precise, the new algorithm is the following. Repeatedly choose a vertex u randomly from those vertices of current minimum degree in G. If u has a neighbour of degree strictly larger than that of u, select v randomly from the set of neighbours of u with degree maximum among all neighbours of u. Otherwise, select v to be u. Add v to $\mathscr{I}$ and remove v from G along with its neighbours and all their incident edges.

The deletion of the edges incident with the neighbours of v may cause the current degree of a vertex in G (that is not a vertex in $\mathscr{I}$ or adjacent to a vertex of $\mathscr{I}$ in G) to be decreased to zero. We refer to such vertices as *accidental isolates*. Owing to the priority assigned to vertices of current minimum degree, accidental isolates that are created are immediately selected in the subsequent steps to be part of $\mathscr{I}$.

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Note that no two accidental isolates may be connected by an edge of G. This, and the fact that all edges incident with neighbours of vertices of $\mathscr{I}$ in G are deleted, ensures that $\mathscr{I}$ is an independent dominating set in G at the end of the algorithm.

Define an *operation* to be the process of selecting a vertex, u, of current minimum (nonzero) degree, the selection of a vertex, v, to add to $\mathscr{I}$, the deletion of all edges incident with neighbours of v and adding all accidental isolates that are created to $\mathscr{I}$. In particular, an operation of type r is an operation in which u has degree r.

The algorithm above for finding a small independent dominating set of d-regular graphs is a direct extension of the algorithm in [4] that finds a small independent dominating set of cubic graphs. The algorithm in [4] is analysed as follows. Letting variables Y_i $(i = 0, \ldots, 3)$ denote the number of vertices of current degree i, the expected values of Y_i are estimated throughout the algorithm for each i using differential equations. It is shown that with high probability the variables are concentrated near their expected values. The analysis in [4] has major complications arising from the fact that priority is given to vertices currently of minimum degree. We call such an algorithm a *prioritized algorithm*.

We do not analyse exactly the algorithm presented above, but a similar one, using a technique introduced by the second author [15]. This approach approximates the performance of a prioritized algorithm by analysing associated *deprioritized* algorithms. These algorithms entirely avoid prioritizing by using a randomized mixture of operations. The particular mixture used for any step is prescribed in advance but changes over the course of the algorithm in order to approximate the prioritized algorithm.

One of the main objectives of using this new technique is to reduce the number of conditions that are required to be checked. Arguments in [4] require steps involving branching processes and large deviation inequalities. They also require checking complex conditions regarding derivatives, especially at the transition between phases, which are those points at which the smallest 'commonly occurring' degree undergoes a 'sudden' shift.

3. Use of deprioritized algorithms

The operations and priorities described in the prioritized algorithm given in section 2 can be analysed using [15, Theorem 1]. This provides us with a set of differential equations whose solution describes the state of a deprioritized version of the algorithm during its execution. From this, we deduce asymptotically almost sure bounds on the size of the independent dominating set at the end of the algorithm. (We claim, but do not prove, that, in addition, the deprioritized algorithm produces the same asymptotic result as the original prioritized algorithm.)

The standard model for random d-regular graphs is as follows. Take a set of dn points in n buckets labelled $1, 2, \ldots, n$, with d points in each bucket, and choose u.a.r. a pairing $P = p_1, \ldots, p_{dn/2}$ of the points such that each p_i is an unordered pair of points and each point is in precisely one pair p_i. The resulting probability space of pairings is denoted by $\mathscr{P}_{n,d}$. Form a d-regular pseudograph on n vertices by placing an edge between vertices i and j for each pair in P having one point in bucket i and one in bucket j.

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In order to prove that a property is a.a.s. true of a uniformly distributed random d-regular (simple) graph, it is enough to prove that it is a.a.s. true of the pseudograph corresponding to a random pairing (see Bollobás [2] and Wormald [13]).

As in [14], we redefine this model slightly by specifying that the pairs are chosen sequentially. The first point in a random pair may be selected using any rule whatsoever, as long as the second point in that random pair is chosen u.a.r. from all the remaining free (unpaired) points. This preserves the uniform distribution of the final pairing.

When a pair has been determined in the sequential process, we say that it has been *exposed*. By exposing pairs in the order which an algorithm requests their existence, the generation of the random pairing may be combined with the algorithm (as in [4, 12, 14]). In this way, the algorithm such as the one in the previous section, which deletes edges, may be described in terms of operations incorporated into the pairing generation. The definition of the operations may be extended to do whatever other tasks the algorithm needs to carry out.

The algorithm proper acts upon the final (pseudo)graph of the generation process, but the set of exposed pairs builds up this final graph during the course of the generation process which incorporates the algorithm. The order in which the edges are deleted corresponds to the order in which the pairs were exposed.

The setting of [15, Theorem 1] requires a number of definitions, and may be described as follows. It concerns a class of processes applied to the random pairing. As described above, this may be defined in terms of the generation algorithm which exposes pairs. The beginning of the generation algorithm is the empty pairing G_0. The pairing G_{t+1} is obtained from G_t by applying an operation which may expose some of the pairs; the *degree* of a bucket is the number of points it contains in exposed pairs. The operation, op_t, which is applied to G_t must be one of some prespecified set of operations, Op_i, $i = 1, \ldots, d$, where Op_i consists of selecting a bucket u of degree $d - i$ (vertex of degree i) in G_t u.a.r., and then applying some specified set of tasks involving some random choices, resulting in G_{t+1}. A subset I of $V(G) \cup E(G)$ is selected during the operations, with $I_0 = \emptyset$ initially, and $I = I_t$ for the pairing G_t.

For $1 \leqslant i \leqslant d$, let $Y_i = Y_i(t)$ denote the number of buckets of degree $d - i$ in G_t, and let $Y_{d+1} = Y_{d+1}(t)$ denote cardinality of the set I_t. Put $\mathbf{Y}(\mathbf{t}) = (Y_1(t), \ldots, Y_{d+1}(t))$.

We refer the reader to [15, Theorem 1] for motivation for the following definitions, and provide a little explanation below. Let $\mathbf{y}$ denote $(y_1(x), \ldots, y_{d+1}(x))$. Given functions $f_{i,r}(x, \mathbf{y})$, define

$$\begin{aligned} \alpha_k(x, \mathbf{y}) &= f_{d-k-1,d-k}(x, \mathbf{y}), \\ \tau_k(x, \mathbf{y}) &= -f_{d-k-1,d-k-1}(x, \mathbf{y}), \end{aligned} \tag{3.1}$$

where

$$x = \frac{t}{n}, \qquad \mathbf{y}(x) = \frac{\mathbf{Y}(t)}{n}. \tag{3.2}$$

We will consider the equations

$$\frac{\mathrm{d}y_i}{\mathrm{d}x} = F(x, \mathbf{y}, i, k) \tag{3.3}$$

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where

$$F(x,\mathbf{y},i,k)=\begin{cases}\frac{\tau_k}{\tau_k+\alpha_k}f_{i,d-k}(x,\mathbf{y})+\frac{\alpha_k}{\tau_k+\alpha_k}f_{i,d-k-1}(x,\mathbf{y}) & \text{if } k\leqslant d-2,\\ f_{i,1}(x,\mathbf{y}) & \text{if } k=d-1,\end{cases} \tag{3.4}$$

and work with the parameters of $f_{i,\ell}$ in the domain

$$\mathscr{D}_\epsilon=\{(x,\mathbf{y}):0\leqslant x\leqslant d,\ 0\leqslant y_i\leqslant d \text{ for } 1\leqslant i\leqslant d+1,\ y_d\geqslant\epsilon\} \tag{3.5}$$

for some pre-chosen value of $\epsilon>0$. The behaviour of the process will be described in terms of the function $\tilde{\mathbf{y}}=\tilde{\mathbf{y}}(x)=(\tilde{y}_1(x),\dots,\tilde{y}_{d+1}(x))$ defined as follows, with reference to an initial value $x=x_0=t_0/n$ of interest:

$$\begin{aligned}&\tilde{y}_i(x_0)=Y_i(t_0)/n,\ i=1,\dots,d+1, \text{ and inductively for } k\geqslant 1,\ \tilde{\mathbf{y}} \text{ is the solution of (3.3) with initial conditions } \mathbf{y}(x_{k-1})=\tilde{\mathbf{y}}(x_{k-1}),\\ &\text{extending to all } x\in[x_{k-1},x_k], \text{ where } x_k \text{ is defined as the infimum of those } x>x_{k-1} \text{ for which at least one of the following holds: (i) } \tau_k\leqslant 0\\ &\text{and } k<d-1;\ \text{(ii) } \tau_k+\alpha_k\leqslant\epsilon \text{ and } k<d-1;\ \text{(iii) } \tilde{y}_{d-k}\leqslant 0;\ \text{or}\\ &\text{(iv) the solution is outside } \mathscr{D}_\epsilon \text{ or ceases to exist.}\end{aligned} \tag{3.6}$$

The interval $[x_{k-1},x_k]$ is called phase k. This inductive definition of $\tilde{\mathbf{y}}$ continues for phases $k=1,2,\dots,m$, where

$$\begin{aligned}&m \text{ denotes the smallest } k \text{ for which either } k=d-1, \text{ or any of the}\\ &\text{termination conditions (ii), (iii) or (iv) for phase } k \text{ in (3.6) come}\\ &\text{into effect at } x_k.\end{aligned} \tag{3.7}$$

It turns out that the intervals called phases all have nonempty interior provided

$$\tau_k>0 \ \text{ and } \ \tau_k+\alpha_k>\epsilon \ \text{ at } (x_{k-1},\tilde{\mathbf{y}}(x_{k-1})) \quad (1\leqslant k\leqslant\min\{d-2,m\}), \tag{3.8}$$

$$\begin{aligned}f_{d-1,d-1}>0 \ &\text{at } (x_0,\tilde{\mathbf{y}}(x_0)),\\ f'_{d-k,d-k}\tau_k+f_{d-k,d-k-1}f'_{d-k-1,d-k}>0 \ &\text{at } (x_{k-1},\tilde{\mathbf{y}}(x_{k-1}))^+ \quad (1<k\leqslant\min\{d-2,m\}),\\ f'_{d-k,d-k}>0 \ &\text{at } (x_{k-1},\tilde{\mathbf{y}}(x_{k-1}))^- \quad (1<k\leqslant m),\\ f'_{1,1}>0 \ &\text{at } (x_{d-2},\tilde{\mathbf{y}}(x_{d-2}))^+ \quad \text{if } m=d-1,\end{aligned} \tag{3.9}$$

with f' denoting $\frac{\mathrm{d}f(x,\tilde{\mathbf{y}}(x))}{\mathrm{d}x}$ and $(x,\tilde{\mathbf{y}}(x))^+$ and $(x,\tilde{\mathbf{y}}(x))^-$ referring to the right-hand and left-hand limits as functions of x.

We may now restate [15, Theorem 1], which we will use in the following section in connection with the independent dominating set algorithm.

Theorem 3.1. ([15]) *Let $d\geqslant 3$. Assume that for some fixed $\epsilon>0$ the operations Op_r satisfy*

$$\mathbf{E}\big(Y_i(t+1)-Y_i(t)\mid G_t\wedge\{\mathrm{op}_t=\mathrm{Op}_r\}\big)=f_{i,r}(t/n,Y_1/n,\dots,Y_{d+1}/n)+o(1) \tag{3.10}$$

for some fixed functions $f_{i,r}(x,y_1(x),\dots,y_{d+1}(x))$ and for $i=1,\dots,d+1$, $r=1,\dots,d$, with the convergence in $o(1)$ uniform over all t and G_t for which $Y_r(t)>0$ and $Y_d(t)>\epsilon n$. Assume furthermore that:

(i) *there is an upper bound, depending only upon d, on the number of pairs exposed, and on the number of elements added to I (i.e., $|I_{t+1}|-|I_t|$), during any one operation;*

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(ii) *the functions* $f_{i,r}$ *are rational functions of* $x, y_1, \ldots, y_{d+1}$ *with no pole in* $\mathscr{D}_\epsilon$ *defined in* (3.5);

(iii) *there exist positive constants* C_1, C_2 *and* C_3 *such that for* $1 \leqslant i < d$, *everywhere on* $\mathscr{D}_\epsilon$, $f_{i,r} \geqslant C_1 y_{i+1} - C_2 y_i$ *when* $r \neq i$, *and* $f_{i,r} \leqslant C_3 y_{i+1}$ *for all* r.

Define $\tilde{\mathbf{y}}$ *as in* (3.6), *set* $x_0 = 0$, *define* m *as in* (3.7), *and assume that* (3.8) *and* (3.9) *both hold. Then there is a randomized algorithm on* $\mathscr{P}_{n,d}$ *for which a.a.s. there exists* t *such that* $|I_t| = n\tilde{y}_{d+1}(x_m) + o(n)$ *and* $Y_i(t) = n\tilde{y}_i(x_m) + o(n)$ *for* $1 \leqslant i \leqslant d$. *Also* $\tilde{y}_i(x) \equiv 0$ *for* $x_{k-1} \leqslant x \leqslant x_k$, $1 \leqslant i \leqslant d-k-1$ $(1 \leqslant k \leqslant m)$.

Some of these definitions can be explained easily. The algorithm in section 2 works by deleting edges; the edges deleted correspond to pairs exposed in the corresponding pairing generation algorithm as described above. In particular, a vertex of degree i in the original algorithm corresponds to a bucket of degree $d-i$ in the pairing version; we use *vertex degree* and *bucket degree* to distinguish these complementary measures. The algorithm gives higher priority to the buckets of highest degree (vertices of lowest degree). The phase is determined by the set of bucket degrees which are reasonably common (meaning, roughly, more than cn buckets have that degree for some $c > 0$). Phase k corresponds to a period in which the smallest such common vertex degree is $d-k$ (*i.e.*, largest common bucket degree is k). At such a time, vertices of degree $d-k-1$, when created, will immediately be used up, by being chosen for u in the subsequent steps, until the minimum positive vertex degree returns to $d-k$. So phase k basically consists of a mixture of two operations: Op_{d-k} and Op_{d-k-1}. The functions α and τ represent respectively the expected net increase in Y_{k+1} in an Op_{d-k}, and the expected net decrease in Y_{k+1} in an Op_{d-k-1}. From these quantities, one may estimate the proportions of these operations being performed at any stage. The randomized algorithm referred to in the theorem uses roughly the same mixture of operations. This in turn lets us calculate the expected changes in the variables, and the result is (3.4), which leads to the differential equation (3.3) analogous to the equations derived in [4].

4. Algorithm analysis

Motivated by the algorithm for finding a small independent dominating set as described in section 2, we specify the tasks in Op_r, $1 \leqslant r \leqslant d$. Here Op_r must first select a random bucket, u, of degree $d-r$. The set of randomized tasks consists of choosing a bucket v, exposing pairs in v and all neighbouring buckets and adding v and any accidental isolates to I (which denotes the independent dominating set). An 'accidental isolate', when dealing with the pairing generation, refers to any bucket (other than v and its neighbours) that attains degree d. According to the rule given in the description of the algorithm, if u has any neighbours of strictly smaller degree, v is chosen randomly from the buckets neighbouring u that have the smallest degree amongst the neighbours of u, otherwise $v = u$ (recalling the relationship between bucket degree and vertex degree).

We may verify the hypotheses of Theorem 3.1. First we will show that (3.10) holds when $Y_d(t) > \epsilon n$ (for any $\epsilon > 0$). From here onwards in this description, *v-degree* refers to

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vertex degree, so v-degree i means bucket degree $d-i$. Let y_i denote Y_i/n and

$$\chi_{q,r} = \left(\mathrm{S}_1^q\right)^r - \left(\mathrm{S}_1^{q-1}\right)^r,$$

where

$$\mathrm{S}_a^b = \sum_{j=a}^{b} \mathrm{P}_j, \qquad \mathrm{P}_j = \frac{jy_j}{s} \quad \text{and} \quad s = \sum_{i=1}^{d} iy_i.$$

Then, when performing an instance of Op_r, the probability that the neighbours of u have maximum v-degree exactly q is $\chi_{q,r} + o(1)$ (see [4] for similar arguments). Similarly, for Op_r, the probability that u has b neighbours of v-degree q, given that the maximum v-degree amongst the neighbours of u is q, is $\beta_{b,q,r}/\chi_{q,r} + o(1)$ where

$$\beta_{b,q,r} = (\mathrm{P}_q)^b \binom{r}{b} (\mathrm{S}_1^{q-1})^{r-b}.$$

Also, the expected number of neighbours of u that have v-degree j, $1 \leqslant j \leqslant q-1$, given that u has b neighbours of v-degree q and q is the maximum v-degree of all neighbours of u, is $\gamma_{b,j,q,r}/\beta_{b,q,r} + o(1)$ where

$$\gamma_{b,j,q,r} = (\mathrm{P}_q)^b \binom{r}{b} (\mathrm{S}_1^{q-1})^{r-b-1} (r-b)\mathrm{P}_j.$$

Let q denote the maximum v-degree of the neighbours of u. If $q > r$, Op_r selects a bucket, v, u.a.r. from the neighbours of u of v-degree q to add to the dominating set. This is Case A. Otherwise (all neighbours of u have v-degree at most r), Op_r selects u to add to the dominating set. This is Case B. For both cases, once the dominating set bucket has been chosen, all pairs incident with the chosen bucket and its neighbours are exposed.

The pairs incident with u are always exposed and the effect, on the expected change in Y_i, of changing the v-degree of u to 0 is just

$$-\delta_{i=r}$$

where, here and in the following, $\delta_R = 1$ if the statement R is true, 0 otherwise.

For Case A, the effect, on the expected change in Y_i, of changing the v-degrees of v and its neighbours (other than u) by exposing all pairs incident with the neighbours of v, is just

$$-\delta_{i=q} + (q-1)\mu_i + o(1)$$

where

$$\mu_i = -\mathrm{P}_i + \rho_i \sum_{x=2}^{d} (x-1)\mathrm{P}_x \qquad \text{and}$$

$$\rho_i = -\mathrm{P}_i + \mathrm{P}_{i+1}\delta_{i+1\leqslant d}.$$

For each of the other neighbours of u, the effect, on the expected change in Y_i, of changing its v-degree from j to $j-1$ is just

$$\delta_{i=j-1} - \delta_{i=j}.$$

For Case B, the expected number of neighbours of u of degree $j \leqslant r$ is

$$r(\mathrm{S}_1^r)^{r-1}\mathrm{P}_j + o(1)$$

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and the effect, on the expected change in Y_i, of the changing v-degree of one such neighbour w, and all its neighbours (other than u), by exposing all pairs incident with w, is just

$$-\delta_{i=j} + (j-1)\rho_i + o(1).$$

So we have that (3.10) holds with

$$\begin{aligned} f_{i,r} = &-\delta_{i=r} + \sum_{q=r+1}^{d} \chi_{q,r}(-\delta_{i=q} + (q-1)\mu_i) \\ &+ \sum_{q=r+1}^{d} \sum_{b=1}^{r} \left[\beta_{b,q,r}(b-1)(\delta_{i=q-1} - \delta_{i=q}) + \sum_{j=1}^{q-1} \gamma_{b,j,q,r}(\delta_{i=j-1} - \delta_{i=j}) \right] \\ &+ \sum_{j=1}^{r} r(\mathrm{S}_1^r)^{r-1} \mathrm{P}_j((j-1)\rho_i - \delta_{i=j}). \end{aligned} \tag{4.1}$$

It also follows that (3.10) also holds for $i = d+1$ with $f_{d+1,r}$ defined as $1 + f_{0,r}$, since in each Op_r, an extra vertex is added to the independent dominating set $\mathscr{I}$ and the expected number of accidental isolates is $f_{0,r}$ as defined in (4.1).

Hypothesis (i) of Theorem 3.1 is immediate since in any operation only the pairs involving points in one bucket and its neighbours are exposed, and a bounded number of vertices are added to $\mathscr{I}$ (as there are certainly fewer than d^2 accidental isolates). The functions $f_{i,r}$ satisfy (ii) because from (4.1) their (possible) singularities satisfy $s = 0$, which lies outside $\mathscr{D}_\epsilon$ since in $\mathscr{D}_\epsilon$, $s \geqslant y_d \geqslant \epsilon$. Hypothesis (iii) follows from (4.1) again using $s \geqslant y_d \geqslant \epsilon$ and the boundedness of the functions y_i (which follows from the boundedness of $\mathscr{D}_\epsilon$). Thus, defining $\tilde{\mathbf{y}}$ as in (3.6) with $t_0 = 0$, $Y_d(0) = n$ and $Y_i(0) = 0$ for $i \neq d$, we may solve (3.3) numerically to find m, verifying (3.8) and (3.9) at the appropriate points of the computation.

It turns out that these hold for each d in Table 1, and that in each case $m = d - 1$, for sufficiently small $\epsilon > 0$. For such ϵ, the value of $\tilde{y}_{d+1}(x_m)$ may be computed numerically (the result is shown as the constant $\mathscr{C}_u(d)$ in Table 1), and then by Theorem 3.1, this is the asymptotic value of the size of the independent dominating set $\mathscr{I}$ at the end of some randomized algorithm. So the conclusion is that a random d-regular graph a.a.s. has an independent dominating set of size at most $n\tilde{y}_{d+1}(x_m) + o(n)$. Note also that (by the theorem) $\tilde{y}_i(x) \equiv 0$ in phase k for $1 \leqslant i \leqslant d-k-1$, and by the nature of the differential equation, $\tilde{y}_i(x)$ will be strictly positive for $i > d-k$. So by (3.6) and (3.7), the end of the process (for ϵ arbitrarily small) occurs in phase $d-1$ when $\tilde{y}_1$ becomes 0.

5. Lower bounds

Zito [16] considered the expected number of independent dominating sets of size λn for a random d-regular graph on n vertices, $d \geqslant 3$. He showed that this expectation is at most

$$n^{O(1)} \left\{ \frac{(1-2\lambda)^{\frac{d(1-2\lambda)}{2}} \, (d\lambda)^{d\lambda} \, ((1+x)^d - 1)^{1-\lambda}}{\lambda^\lambda \, (1-\lambda)^{1-\lambda} \, d^{d\lambda} \, x^{d\lambda}} \right\}^n$$

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where x takes the value that minimizes

$$\frac{((1+x)^d - 1)^{1-\lambda}}{x^{d\lambda}}.$$

Finding the value of λ below which this expression tends to zero gives a lower bound on the expected size of an independent dominating set of a random d-regular graph on n vertices, thus giving the constants $\mathscr{C}_\ell(d)$ reported in Table 1.

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