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ARTICLE

The Rank-Ramsey problem and the Log-Rank conjecture

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Abstract

A graph is called Rank-Ramsey if (i) Its clique number is small, and (ii) The adjacency matrix of its complement has small rank. We initiate a systematic study of such graphs. Our main motivation is that their constructions, as well as proofs of their non-existence, are intimately related to the famous log-rank conjecture from the field of communication complexity. These investigations also open interesting new avenues in Ramsey theory. We construct two families of Rank-Ramsey graphs exhibiting *polynomial separation* between order and complement rank. Graphs in the first family have bounded clique number (as low as 41). These are subgraphs of certain strong products, whose building blocks are derived from triangle-free strongly-regular graphs. Graphs in the second family are obtained by applying Boolean functions to Erdős-Rényi graphs. Their clique number is logarithmic, but their complement rank is far smaller than in the first family, about $\mathcal{O}(n^{2/3})$. A key component of this construction is our matrix-theoretic view of lifts. We also consider lower bounds on the Rank-Ramsey numbers, and determine them in the range where the complement rank is 5 or less. We consider connections between said numbers and other graph parameters, and find that the two best known explicit constructions of triangle-free Ramsey graphs turn out to be far from Rank-Ramsey.

Keywords: Ramsey; log rank; communication complexity

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1. Introduction

A graph G is called *Rank-Ramsey* if both its clique number and the rank of its complement are small.[1] Rank-Ramsey graphs are clearly Ramsey graphs, because $\alpha(G) \leqslant \operatorname{rank}(\overline{G})$ holds for every graph G. Indeed, an independent set in G corresponds to a clique in its complement, which has full rank. So, what changes when we replace the traditional *independence number* $\alpha(G)$, with the complement rank?

This new notion originates in our long-lasting failure to understand the structure of low-rank matrices. The most ambitious attempt at this mystery is the famous *log-rank* conjecture [46] from communication complexity, which attempts to characterise low-rank *binary matrices*. This conjecture (some would call it a *problem*) posits, in a form due to Nisan and Wigderson [50], that any low-rank binary matrix must contain a large monochromatic minor. An equivalent graph-theoretic formulation of this problem, due to Lovász and Saks (also in [50]), asks whether there exists an absolute constant c such that:

$$\log \chi(G) \leqslant \mathcal{O}\left(\log^c \operatorname{rank}(G)\right),$$

for every graph G. Here $\chi(G)$ is the chromatic number of G.

[1]As usual, the rank of the graph G is the real rank of its adjacency matrix A_G.

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Constructions of Rank-Ramsey graphs as well as impossibility results are deeply connected to the log-rank conjecture, as first suggested in [47]. As we show here, any construction of Rank-Ramsey graphs yields a separation in the log-rank conjecture, and conversely, under certain conditions, proving the impossibility of such graphs may validate the conjecture. We elaborate on this connection in Section 3.

The difficulty of characterising low-rank matrices is profound. Even understanding *typical* low-rank matrices is a mystery. Indeed, culminating 60 years of excellent research, Tikhomirov [56] showed that an $n \times n$ matrix with random ± 1 entries is singular with probability $(\frac{1}{2} + o_n(1))^n$, which is clearly tight up to the little-oh term. There is a rich literature of theorems of the same vein, showing that full-rank matrices are the rule rather than the exception.

Random constructions are key to the study of Ramsey numbers. Already over 75 years ago, Erdős [21] discovered that asymptotically almost all graphs are Ramsey. In contrast, random graphs typically have *logarithmic* independence number, but *full* complement rank. This poses new non-trivial challenges in the construction of Rank-Ramsey graphs and in understanding the range of possible values. We also consider two of the most interesting and deep constructions of Ramsey Graphs [3, 15], and both appear of little value in the study of Rank-Ramsey graphs.

As mentioned, our basic definition hinges on the observation that the submatrix that corresponds to an independent set in G has full complement rank. Such an interplay between cardinality (here, of independent sets in G) and rank (of the corresponding submatrix in the adjacency matrix of $\overline{G}$) is an old and fundamental theme in combinatorics. Babai and Frankl exhibit in their book [6] numerous fascinating examples of proofs that compare between cardinality and rank. Typically, one proves a desired lower bound on the cardinality of a set by showing that a certain associated matrix has large rank. The historically oldest example known to us of this method is Hanani's proof [32] of the de Bruijn-Erdős Theorem.[2] It is interesting to note that in contrast with such classical proofs, in the study of the Rank-Ramsey problem, rank bounds cardinality *from above*.

Our results. We first present some constructions of Rank-Ramsey graphs. As mentioned, using the probabilistic method to this end is not an option, since we lack a natural distribution over low-rank matrices. In this view, we opt for *explicit* constructions of Rank-Ramsey graphs, yet relying at times on some probabilistic machinery. We begin with two constructions of Rank-Ramsey graphs exhibiting a *polynomial separation* between complement rank and order.

We need to introduce some notation first. We consider the least complement rank of an n-vertex graph whose clique number is at most d.[3]

Definition. *For every positive d, let $\nu_d : \mathbb{N} \to \mathbb{N}$ be the function*

$$\nu_d(n) := \min_G \operatorname{rank}(A_G + I)$$

minimising over all n-vertex graphs G with clique number $\leqslant d$.

As a brief illustration, note that $\nu_d(n) \leqslant \lfloor n/d \rfloor$: Take the disjoint union of $\lfloor n/d \rfloor$ cliques K_d (and a clique on the remaining $n \pmod d$ vertices). Notwithstanding the simplicity of this inequality, it is not easy to beat. In particular, no graph with fewer than 10 vertices can accomplish this.

With this notation, here is our first result:

[2] While Hanani's paper dates to 1951, there are records that it was first written (without publishing) in the late 1930's.

[3] Throughout the paper we measure $\operatorname{rank}(A_G + I)$ only, rather than $\operatorname{rank}(\overline{G})$. Note that $A_G + I = J - A_{\overline{G}}$, where J is the all-ones matrix, therefore the two quantities differ by at most 1. Nevertheless, we stress that even when we use (for convenience) the term complement rank', we are always referring only to $\operatorname{rank}(A_G + I)$.

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Theorem 1.1. *The following bounds hold:*

1. $\nu_{41}(n) = \mathcal{O}(n^{1-\frac{1}{10000}})$.
2. *For any sufficiently large d, there holds* $\nu_d(n) = \mathcal{O}\left(n^{\log_{296}(232)}\right)$, *where* $\log_{296}(232) \approx 0.957$.

The proof of Theorem 1.1 starts with a base-graph, that beats the above *trivial bound*. We then take repeated strong products of this graph, to amplify its already low complement rank, and show that one can find a *large subgraph* of the product, with no cliques above a certain size. The search for a base-graph turns out to be challenging. To this end, we turn to strongly-regular graphs (SRGs): Highly-structured regular graphs with only three distinct eigenvalues, one of which, the Perron eigenvalue, is simple.

A particularly good base-graph would have both low complement rank *and* small clique number, say, be triangle-free. However, we encounter two immediate barriers. Only seven triangle-free strongly regular graphs are known: C_5, Petersen, Clebsch, Hoffman-Singleton, Gewirtz, Mesner, and the Higman-Sims graph. Whether this list is exhaustive or not is a famous open question (Cf. [8, 7]). Moreover, another problem arises. We show that triangle-free SRGs must have full complement rank.

To handle these difficulties, we use a 'clique tensoring trick', allowing us to produce new graphs with low complement rank from graphs whose spectrum has certain properties, while retaining the clique number. Among the aforementioned list of triangle-free SRGs, a unique good candidate emerges: the Clebsch graph (and for K_4-free, the complement of the Schläfli graph). To our knowledge this is at least the second time the Clebsch graph had appeared in the context of Ramsey theory, the first being the proof of the multicolour Ramsey number, $R(3, 3, 3) = 17$, due to Greenwood and Gleason [25].

Our second result is another construction of a Rank-Ramsey family.

Theorem 1.2. *For any two constants* $c, \varepsilon > 0$ *with* $c > 2\left(\frac{2}{3\varepsilon} + 1\right)^2$, *there holds* $\nu_{c\log n}(n) \leqslant \widetilde{\mathcal{O}}(n^{\frac{2}{3}+\varepsilon})$.

The graphs in this second family have complement rank near $\mathcal{O}(n^{2/3})$, and logarithmic clique number. One can take, e.g., $\varepsilon = 10^{-3}$ and $c = 10^6$.

A key component in the proof of Theorem 1.2 is a matrix-theoretic view of lifts that we adopt. As we elaborate below (Subsection 2.1) *lifting* is a well-established proof technique in the field of communication complexity. However, we apply it a little differently than usual, and consider the composition of a Boolean function $f : \{0, 1\}^n \to \{0, 1\}$, and a *collection* of binary matrices, not necessarily all identical. We show a fundamental connection between the expansion of f as a real multilinear polynomial, and the resulting lifted matrix. This implies a lifting theorem of-sorts: The rank of the lifted matrix is determined by the expansion of f, and by the ranks of its constituent matrices.

The Boolean function used in Theorem 1.2 is the well-known NAE : $\{0, 1\}^3 \to \{0, 1\}$. This function appeared first in the context of the log-rank conjecture in Nisan and Wigderson's seminal paper [50]. The vertices of the lifted graph can be naturally identified with the points of the cube, $[k]^3$. We show that within this graph, cliques emerge either (i) From correlations between the lifted matrices, or (ii) From certain degenerate subsets of the cube. The former case is handled by lifting three i.i.d. Erdős-Rényi graphs. The latter issue is dealt with by means of a probabilistic argument. We apply a random construction to find a large subset S of the cube, such that *every* subset $T \subset S$ of a certain size, is far from 'cube-like'. That is, T has *some* axis-projection of size *linear* in its cardinality. Our lifts are unorthodox in two ways. Firstly, all *gadgets* (i.e., matrices) are distinct, and in fact, uncorrelated. Secondly, the traditional roles are reversed; the function has *constant size*, while the gadgets *grow asymptotically*.

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To present our third result, we require yet another bit of notation. As usual, the Ramsey number $R(s,t)$ is the least number n such that every n-vertex graph has either an s-clique or a t-anticlique. Extending the analogy, we define the 'KRamsey' numbers.[4]

Definition. *$R^k(s,t)$ is the least number n such that every order-n graph G has either an s-clique, or*

$$\operatorname{rank}(A_G + I) \geqslant t.$$

We characterise the KRamsey numbers, when the rank is small.

Theorem 1.3. *For $2 \leqslant t \leqslant 5$ and every $s > 1$, there holds $R^k(s,t) = (s-1)(t-1)+1$.*

Unlike the classical Ramsey numbers, KRamsey numbers are not symmetric in their parameters: it is much harder to keep the rank low than it is to avoid large cliques. Indeed, the triangle-free base-graphs used in Theorem 1.1, together with the characterisation of Theorem 1.3, imply that $R^k(3,n) > R^k(n,3)$ for every sufficiently large n. Also, it is not hard to see that $(s-1)(t-1) < R^k(s,t) \leqslant R(s,t)$ for every s and t. Another stark contrast is, therefore, that while Ramsey numbers always strictly exceed this trivial bound (whenever $s,t > 2$), for KRamsey numbers this is evidently untrue.

The proof of Theorem 1.3 relies on *blowup* of graphs, where vertices are replaced with anticliques and edges by complete bipartite graphs. We remark that for every rank r, there exists a finite list of graphs $\mathcal{G}_r$ such that every connected graph of rank r is a blowup of a graph in $\mathcal{G}_r$. These lists appear in our proof, in particular those of ranks $r=4$ [12] and $r=5$ [13], found by Chang, Huang and Ye.

Next we consider triangle-free Rank-Ramsey graphs. Following a long line of research [34], the Ramsey number $R(3,t)$ is now known up to a small multiplicative factor [23]. In this view, in order to improve the bounds on $R^k(3,t)$, we must resort to graph parameters other than independence number. Recall that an *orthonormal representation* of a graph G is an assignment of unit vectors to its vertices such that vectors of non-adjacent vertices are orthogonal. Let M be the Gram matrix of these vectors. The least dimension of such a representation of G is denoted $\operatorname{msr}(G)$. The matrices M and $A_G + I$ agree both on the main diagonal, and on the entries of non-edges of G. The Lovász number $\vartheta(G)$ [44] is another well-known graph parameter, related to orthonormal representations, whose definition we omit here for brevity. An important relation between these two quantities and with Ramsey graphs is the well-known 'sandwich theorem' (see [39]), which states that for every graph G there holds

$$\alpha(G) \leqslant \vartheta(G) \leqslant \operatorname{msr}(G) \leqslant \chi(\overline{G}).$$

What is the relation between these different graph parameters, that bound the independence number? We show the following separations for triangle-free graphs.

Theorem 1.4. *For infinitely many $n > 1$, there exist n-vertex triangle-free graphs G_1 and G_2, with*

1. *$\vartheta(G_1) = \Theta(n^{2/3})$ and $\operatorname{rank}(A_{G_1} + I) = n$.*
2. *$\operatorname{msr}(G_2) \geqslant n/2$ and $\operatorname{rank}(A_{G_2} + I) = (3/8 + o(1))n$.*

The first result comes from an explicit construction of triangle-free Ramsey graphs, due to Alon [3]. These n-vertex graphs have $\vartheta(G) = \Theta(n^{2/3})$, so by the Sandwich Theorem, these are Ramsey graphs. The bound on $\vartheta(G)$ is best possible, so they are in fact optimal ϑ-Ramsey graphs.[5] However, as we show, these graphs are as bad as possible for the Rank-Ramsey property in that they satisfy $\operatorname{rank}(A_G + I) = n$. These are Cayley graphs of the Abelian group $\mathbb{Z}_N$, and so our proof

[4] We leave it to the reader's taste and judgement how to pronounce this new name.

[5] By ϑ-Ramsey we mean graphs with small clique number and low *Lovász number*, rather than *independence number*.

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is Fourier-analytic. As for the latter result, concerning msr(G), the lower bound follows from a result of Deaett [17], and our converse follows from the Clebsch-derived family of graphs.

The relations between minimum semi-definite rank, Lovász number, and complement rank are still not sufficiently well-understood. We observe several similarities between the Lovász number and the rank of $A_G + I$: they are derived from closely related matrices, both are multiplicative in the strong graph product, and both bound the Shannon capacity from above. The possibility that $\mathrm{rank}(A_G + I) \geqslant \Omega(\vartheta(G))$ is especially intriguing, and we give some supporting evidence in this direction. For example, in every graph with fewer than 10 vertices the Lovász number is at least as big as the complement rank. We know of no counterexample. We remark that, if this holds true, then Alon and Kahale's [1] extension of Kashin and Konyagin's bound [35] on the Lovász number of graphs with bounded independence number implies a polynomial improvement in our lower-bounds on $\nu_d(n)$, for every positive constant d. In Appendix B we draw connections between this problem and other well-known questions in algebraic graph theory, related to the Hoffman and Cvetković bounds on the independence number.

More key results in Ramsey theory, and regarding the log-rank conjecture, can be observed through the lens of the Rank-Ramsey problem. For instance, there are additional interesting *explicit* constructions of triangle-free Ramsey graphs, such as Frankl and Wilson's set intersections [24], or Chung's fibrations [9]. They too can be shown not to be Rank-Ramsey. In Appendix A we sketch a shortened version of a noteworthy elementary construction, due to Codenotti, Pudlák and Resta [15], yielding triangle-free Ramsey graphs with $\mathrm{rank}(A_G - 2I) = \mathcal{O}(n^{3/4})$. These graphs appear close in spirit to our Rank-Ramsey problem,[6] with one key difference, which we believe to be central: $A_G - 2I$ is not binary. This is crucially important with regards to the log-rank conjecture, where it is well known how to construct non-binary matrices with *constant rank* and *full partition number*.

We also give a brief analysis of the Nisan-Wigderson construction [50] of matrices exhibiting a gap in the log-rank problem. We show that these matrices have large monochromatic principal minors, and therefore yield poor Rank-Ramsey graphs as-is. We stress that this is no contradiction: exhibiting a log-rank separation does not preclude a matrix from *having* a large monochromatic rectangle.[7]

1.1 Paper organisation

Log-Rank. To start we introduce two central topics that permeate this paper: Matrix-theoretic lifts (Subsection 2.1) and Rank-Ramsey graphs (Subsection 2.2). In Subsection 2.2 we also introduce the Rank-Ramsey numbers, and compute them precisely for graphs of low complement rank. In Section 3, we draw connections between the Rank-Ramsey problem and the log-rank conjecture in communication complexity.

Constructions. Sections 4 and 5 are dedicated to constructions of Rank-Ramsey families. The first utilises minors of Kronecker powers of families derived from SRGs with particular properties, and yields a *polynomial separation* between complement rank and order for graphs with *constant* clique number. The latter relies on matrix-theoretic lifts of Erdős-Rényi graphs with the well-known Boolean function NAE, and produces graphs whose clique number is *logarithmic*.

Analysis and bounds. Section 6 revolves around lower bounds on the complement rank of triangle-free Rank-Ramsey graphs. There, we draw connections to other known graph parameters, and also analyse the two best known explicit constructions of triangle-free Ramsey graphs, due to Alon [3], and Codenotti, Pudlák and Resta [15], showing they are far from Rank-Ramsey.

[6] Of course, $\mathrm{rank}(A - 2I) = o(n)$ implies $\mathrm{rank}(A_G + I) = (1 - o(1))n$, i.e., these too are far from Rank-Ramsey.

[7] However, due to the relation between log-rank and Rank-Ramsey, a graph family *related* to said matrices must contain large Rank-Ramsey *subgraphs*.

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In Section 7 we consider the Nisan-Wigderson construction [50], which first exhibited a polynomial separation for the log-rank conjecture, from a Rank-Ramsey perspective. We find large monochromatic principal minors in said matrices, implying that they induce poor Rank-Ramsey graphs.

2. Preliminaries

Graphs. By default, all graphs in this paper are undirected and simple. We occasionally also consider directed graphs, as well as graphs with self-loops. The *order* of a graph, $v(G)$, is the number of vertices, and its *size*, $e(G)$, is the number of edges. If two vertices $u, v \in V(G)$ are adjacent, we denote this by $u \sim_G v$ and resp. $u \to_G v$ (in the undirected, resp. directed case). The subscript G is omitted when the graph is clear from the context.

Associated with every order-n graph $G = ([n], E)$ is its *adjacency matrix*:

$$A_G \in M_n(\mathbb{R}), \text{ where } \forall i,j \in [n] : (A_G)_{i,j} = \mathbb{1}\left\{i \sim j\right\} \quad \text{(respectively, } i \to j \text{, if } G \text{ is directed)}$$

The *spectrum* of a graph, denoted $\mathrm{spec}(G)$, is the multiset of eigenvalues corresponding to its adjacency matrix. If G is undirected, its spectrum is real (since A_G is symmetric). The multiplicity of an eigenvalue λ in a (diagonalisable) matrix A is denoted by $\mu_A(\lambda)$. The *rank* of a graph is the real rank of its adjacency.

The subgraph induced by a set of vertices $S \subseteq V(G)$ is denoted $G[S]$. The *clique number*, the *independence number*, and *chromatic number* of a simple graph G are denoted $\omega(G)$, $\alpha(G)$ and $\chi(G)$, respectively. We denote the complement of a graph G, by $\overline{G}$. The adjacency matrix of $\overline{G}$ is $J - (A_G + I)$. Observe that

$$\left|\mathrm{rank}(\overline{G}) - \mathrm{rank}(A_G + I)\right| \leqslant 1,$$

therefore, where an additive error of ± 1 is insignificant, we sometimes refer to $\mathrm{rank}(A_G + I)$ as the 'complement rank' of G.

A *blowup* of a graph G is a graph attained by replacing each vertex by a nonempty anticlique and each edge by a complete bipartite graph. If $w(i)$ is the size of the anticlique that replaces vertex i, we denote the resulting graph by G^w. For matrices, the blowup A^w of a matrix A replaces each entry $A_{i,j}$ with a $(w(i) \times w(j))$-block of entries $A_{i,j}$ (so $A_{G^w} = A^w_G$). Importantly, $\mathrm{rank}(A^w) = \mathrm{rank}(A)$ for any A and w.

Two vertices are called *twins* if they are non-adjacent and have the same set of neighbours. Discarding twins from a graph is called a *reduction*, and is the inverse operation to blowup. Both blowup and reduction affect neither the chromatic number, nor the rank of a graph.

Boolean functions. It is well known that any Boolean function $f : \{0,1\}^n \to \{0,1\}$ is uniquely representable as a multilinear polynomial over the Reals, viz.,

$$f(x) = \sum_{S \subseteq [n]} a_S \prod_{i \in S} x_i \in \mathbb{R}[x_1, \ldots, x_n]$$

As usual, the *support* of f is the index set of the non-zero coefficients in this representation, denoted $\mathrm{mon}(f) := \{S : \emptyset \neq S \subseteq [n], a_S \neq 0\}$. We denote the size of f's support by $\mathrm{spar}(f) := |\mathrm{mon}(f)|$, and the *degree* of f is $\deg(f) := \max\{|S| : S \in \mathrm{mon}(p)\}$. By convention, the constant functions $f \equiv 0$ and $f \equiv 1$ are defined to have degree 0.

Matrices. As usual, the identity matrix is denoted I, and the all-ones matrix J. The trace of a square matrix is the sum of entries on its main diagonal. Aside from the regular matrix product, we use the Kronecker and Hadamard products which we now recall. Let $A, C \in M_{m \times n}(\mathbb{R})$, and $B \in M_{k \times l}(\mathbb{R})$ be three matrices. The Kronecker product $A \otimes B \in M_{mk \times nl}(\mathbb{R})$ and Hadamard

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(element-wise) product $A \odot C \in M_{m\times n}(\mathbb{R})$ are defined by:

$$(A \otimes B)_{i,j,x,y} := A_{i,j}B_{x,y}, \text{ and } (A \odot C)_{i,j} := A_{i,j}C_{i,j}$$

It is well known that rank is multiplicative under the Kronecker product, and sub-multiplicative under the Hadamard product. We also require the standard *Kronecker* and the *Strong* graph products.

Definition 2.1. *The Kronecker Product, $G \otimes H$, and the Strong Product, $G \boxtimes H$, of two simple graphs $G = ([n], E_G)$ and $H = ([k], E_H)$, are graphs with vertex set $[n] \times [k]$, where*

$$(i,a) \underset{G\otimes H}{\sim} (j,b) \iff (i \underset{G}{\sim} j) \wedge (a \underset{H}{\sim} b)$$

and

$$(i,a) \underset{G\boxtimes H}{\sim} (j,b) \iff (i = j \wedge a \underset{H}{\sim} b) \vee (a = b \wedge i \underset{G}{\sim} j) \vee \left((i \underset{G}{\sim} j) \wedge (a \underset{H}{\sim} b)\right)$$

In other words, their adjacency matrices are $A_G \otimes A_H$ and $A_G \otimes A_H + A_G \otimes I_k + I_n \otimes A_H$, respectively.

Unlike the usual notion of matrix minors, *Generalised Minors* allow repeated indices.

Definition 2.2. *Let $A \in M_k(\mathbb{R})$ be a matrix. The Generalised Minor of A corresponding to the indices $(i_1, \ldots, i_d) \in [k]^d$ is the matrix $A[i_1, \ldots, i_d] \in M_d(\mathbb{R})$ defined by*

$$(A[i_1, \ldots, i_d])_{s,t} := A_{i_s,i_t},$$

for every $s, t \in [d]$.

Note that the *rank* of a generalised minor cannot exceed the rank of its 'parent' matrix.

Proposition 2.3. *If $A \in M_k(\mathbb{R})$ be a matrix and $(i_1, \ldots, i_d) \in [k]^d$ are indices, then*

$$\text{rank}(A[i_1, \ldots, i_d]) \leqslant \text{rank}(A)$$

Proof. Let P be a $d \times k$ binary matrix with a 1 in position (x, y) iff $i_x = y$. The claim follows from the observation that $A[i_1, \ldots, i_d]$ coincides with PAP^T, and the fact that $\text{rank}(PAP^T) \leqslant \text{rank}(A)$. □

Communication complexity. We follow the standard notation and terminology in communication complexity, as introduced in the excellent book [38]. Let $A \in M_{m\times n}(\mathbb{R})$ be a binary matrix. A *combinatorial rectangle* of A is a minor with row set $R \subseteq [m]$ and column set $C \subseteq [n]$. For a bit $b \in \{0, 1\}$, a *b-rectangle cover* of A is a set of b-monochromatic combinatorial rectangles whose union comprises all b-entries in A. The b-cover number of A, and its b-nondeterministic communication, are defined as follows:[8]

$$\chi^b(A) := \min_{\mathcal{R}} |\mathcal{R}|, \text{ and } \mathrm{N}^b(A) := \lceil \log \chi^b(A) \rceil$$

where the minimum is taken over all b-rectangle covers of A.

As usual, the *deterministic* communication complexity of $\mathrm{D}(A)$, is the least cost of a deterministic communication protocol computing A (where the cost of a protocol is the maximum, over all inputs, of the number of bits transmitted).

2.1 Matrix representation of lifts

Lifting is a powerful technique in the study of communication complexity, first introduced by Raz and McKenzie [51]. An application of this method starts with two Boolean functions, $f : \{0, 1\}^n \to \{0, 1\}$ and $g : \{0, 1\}^b \times \{0, 1\}^b \to \{0, 1\}$. Typically b is much smaller than n, and g is commonly

[8] All logarithms in this paper are *binary* (base 2), unless stated otherwise.

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called a *gadget*. The corresponding lift is the composition $f \circ g^n : (\{0,1\}^b)^n \times (\{0,1\}^b)^n \to \{0,1\}$:

$$(f \circ g^n)\big((x_1, y_1), \ldots, (x_n, y_n)\big) := f\big(g(x_1, y_1), \ldots, g(x_n, y_n)\big),$$

where $x_i, y_j \in \{0,1\}^b$.

A lifting theorem establishes a relation between the *communication complexity* of $f \circ g^n$, the *query complexity* of f and some property of g. This is done for a particular choice of query and communication models. The logic behind such theorems is this: Generally speaking it is hard to prove lower bounds for a communication model. Lifting theorems accomplish this task by relying on a lower bound in a query model, which is easier to come by. There are numerous examples of such theorems, c.f. [14, 26, 28, 29, 51].

We take a matrix-theoretic view of lifts. Rather than compose *two* Boolean functions, we compose a Boolean function with a *collection* of binary matrices,

Definition 2.4. *Let $A_1, \ldots, A_n \in M_{m\times k}(\mathbb{R})$ be binary matrices and let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function. The f-lift of $A_1, \ldots, A_n$, denoted $f\big(A_1, \ldots, A_n\big)$, is the $m^n \times k^n$ Boolean matrix,*

$$f\big(A_1, \ldots, A_n\big)_{i,j} := f\big((A_1)_{i_1,j_1}, \ldots, (A_n)_{i_n,j_n}\big),$$

for $i = (i_1, \ldots, i_n) \in [m]^n$ and $j = (j_1, \ldots, j_n) \in [k]^n$.

This notion of lifting binary matrices with a Boolean function $f : \{0,1\}^n \to \{0,1\}$ is intimately related to the multilinear representation of f over the Reals, and to the Kronecker product of matrices.

Proposition 2.5. *Let $A_1, \ldots, A_n \in M_{m\times k}(\mathbb{R})$ be binary matrices and let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function whose multilinear expansion is $f(x) = \sum_{S\subseteq[n]} a_S \cdot \prod_{l\in S} x_l \in \mathbb{R}[x_1, \ldots, x_n]$. Then, the f-lift of $A_1, \ldots, A_n$ can be written as:*

$$f\big(A_1, \ldots, A_n\big) = \sum_{S\subseteq[n]} a_S \cdot \bigotimes_{l\in[n]} Z_l^S$$

where $J_{m\times k}$ is the $(m \times k)$ all-ones matrix and $Z_l^S = \begin{cases} A_l & l \in S \\ J_{m\times k} & l \notin S \end{cases}$

Proof. By the definition of the Kronecker product, $\forall (i,j) \in [m]^n \times [k]^n$ we have:

$$\begin{aligned}\left(\sum_{S\subseteq[n]} a_S \cdot \bigotimes_{l\in[n]} \begin{cases} A_l & l \in S \\ J_{m\times k} & l \notin S \end{cases}\right)_{i,j} &= \sum_{S\subseteq[n]} a_S \cdot \prod_{l\in[n]} \left(\begin{cases} A_l & l \in S \\ J & l \notin S \end{cases}\right)_{i_l,j_l} \\ &= \sum_{S\subseteq[n]} a_S \cdot \prod_{l\in S} (A_l)_{i_l,j_l} \\ &= f\big((A_1)_{i_1,j_1}, \ldots, (A_n)_{i_n,j_n}\big) = f\big(A_1, \ldots, A_n\big)_{i,j}\end{aligned}$$

□

This relationship implies the following powerful bound on the rank of a lifted matrix.

Lemma 2.6. *Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function and let $A_1, \ldots, A_n \in M_{m\times k}(\mathbb{R})$ be binary matrices. Then,*

$$rank\left(f(A_1, \ldots, A_n)\right) \leqslant \sum_{S\in mon(f)} \prod_{i\in S} rank(A_i)$$

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Proof. Let $\sum_{S\in\mathrm{mon}(f)} a_S \prod_{i\in S} x_i$ be the multilinear representation of f over the Reals. By Definition 2.4,

$$f(A_1,\ldots,A_n) = \sum_{S\in\mathrm{mon}(f)} a_S \bigotimes_{i\in[n]} \begin{cases} A_i & i\in S \\ J_{m\times k} & i\notin S \end{cases}$$

The proof now follows by recalling that $\mathrm{rank}(J)=1$, and that rank is subadditive, and multiplicative under the Kronecker product. □

2.2 Rank-Ramsey graphs

The Ramsey number $R(s,t)$ is the smallest n such that every graph G of order n contains either an s-clique or an t-anticlique. In other words, either $\omega(G)\geqslant s$ or $\alpha(G)\geqslant t$. Observe that $\mathrm{rank}(A_G+I)\geqslant\alpha(G)$, since the minor corresponding to an anticlique in G is all-zeros in A_G, and an identity submatrix in A_G+I. This suggests a search for graphs where both the clique number $\omega(G)$ and $\mathrm{rank}(A_G+I)$ are small. We call graphs with such properties *Rank-Ramsey*. To proceed, we introduce some notation.

Definition 2.7. *For every $d\geqslant 1$, let $\nu_d:\mathbb{N}\to\mathbb{N}$ be the function*

$$\nu_d(n) := \min_G \mathrm{rank}(A_G+I)$$

minimising over all order-n graphs G with $\omega(G)\leqslant d$.

A similar notion for directed graphs is of interest as well,

Definition 2.8. *For every $d\geqslant 1$, let $\eta_d:\mathbb{N}\to\mathbb{N}$ be the function*

$$\eta_d(n) := \min_{A\in M_n(\mathbb{R})} \mathrm{rank}(A+I)$$

minimising over all $n\times n$ binary matrices A with zeros on the main diagonal, and such that $(A+I)$ has no J_{d+1} principal minor.

There is a simple relation between Ramsey numbers and the function ν_d:

Proposition 2.9. *For any two positive integers d and n, there holds $n<R(d+1,\nu_d(n)+1)$.*

Proof. If G is a graph attaining $\nu_d(n)$, then by definition $\omega(G)<d+1$. Also, as mentioned, $\alpha(G)<\mathrm{rank}(A_G+I)+1=\nu_d(n)+1$. Therefore, the order of G is smaller than $R(d+1,\nu_d(n)+1)$, as claimed. □

Corollary 2.10 [47]. *For any two positive integers $n>d$, there holds*

$$\nu_d(n)=\Omega\left(n^{1/d}\cdot\left(\frac{\log n}{d}\right)^{\frac{d-1}{d}}\right)$$

Proof. Apply the classical bounds on Ramsey numbers, due to Ajtai, Komlós and Szemerédi [2]. □

The following lemma aggregates several useful properties of our two quantities.

Lemma 2.11. *For every natural number d, it holds that:*

1. *Both quantities $\nu_d(n)$ and $\eta_d(n)$ are non-increasing in d.*
2. *$\eta_d(n)\leqslant\nu_d(n)\leqslant\eta_d(n)^2$.*
3. *Both ν_d and η_d are submultiplicative: $\nu_d(kn)\leqslant k\cdot\nu_d(n)$, and $\eta_d(kn)\leqslant k\cdot\eta_d(n)$.*

and moreover, for every d_1,d_2 and n_1,n_2, we have:

4. *$\nu_{d_1d_2}(n_1n_2)\leqslant\nu_{d_1}(n_1)\nu_{d_2}(n_2)$, and similarly $\eta_{d_1d_2}(n_1n_2)\leqslant\eta_{d_1}(n_1)\eta_{d_2}(n_2)$.*

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Proof. We prove each property in turn,

1. Obvious: If $b > a$, then a graph with no a-clique has no b-clique either.
2. The lower bound is obvious. For the upper bound, let A be the matrix attaining $\eta_d(n)$ and consider the Hadamard product $(A+I) \odot (A+I)^T$.
3. Let G be a graph attaining $\nu_d(n)$, and let H be the disjoint union of k copies of G. Clearly $\omega(H) = \omega(G)$ and $\text{rank}(A_H + I) = k \cdot \text{rank}(A_G + I)$. The proof for η_d is identical.
4. Let G_1 and G_2 be graphs attaining the minimum for $\nu_{d_1}(n_1)$ and $\nu_{d_2}(n_2)$, respectively. Let Γ be the graph whose adjacency matrix is $A_\Gamma = (A_{G_1} + I) \otimes (A_{G_2} + I) - I$. By multiplicativity,

$$\text{rank}(A_\Gamma + I) = \text{rank}\left((A_{G_1} + I) \otimes (A_{G_2} + I)\right) = \text{rank}(A_{G_1} + I) \cdot \text{rank}(A_{G_2} + I)$$

   But for any clique S in Γ, we have $|S| \leqslant \omega(G_1) \cdot \omega(G_2)$, since its projections on either coordinate are cliques in G_1 and G_2, respectively. The same proof applies to η. □

2.2.1 Rank-Ramsey numbers

Maintaining the analogy with Ramsey numbers, we define the *Rank-Ramsey numbers*,

Definition 2.12. *$R^k(s,t)$ is the smallest integer N such that for every graph of order N,*

$$\omega(G) \geqslant s \text{ or } \text{rank}(A_G + I) \geqslant t$$

Clearly, $(s-1)(t-1) < R^k(s,t) \leqslant R(s,t)$. The lower bound follows by taking the disjoint union of $(t-1)$ copies of K_{s-1}. For small numbers s and t, Rank-Ramsey numbers are *strictly smaller* than Ramsey numbers. In fact they match the trivial bound $R^k(s,t) = (s-1)(t-1)+1$ for all $2 \leqslant t \leqslant 5$ and any $1 < s$.

Theorem 2.13. *For $2 \leqslant t \leqslant 5$ and every $s > 1$, there holds $R^k(s,t) = (s-1)(t-1)+1$.*

This equality fails for substantially larger values of t. For example, we show later on (Corollary 4.6) that $R^k(3, 6l+11) > 16l$ for every $l > 2$. For $l = 6$ this yields $R^k(3,47) > 96 > 93 = 2 \cdot 46 + 1$. Consequently, $R^k(3,n) > R^k(n,3)$, for every sufficiently large n, and in particular, unlike Ramsey numbers, Rank-Ramsey numbers are not symmetric in their parameters. To wit, it is much harder to keep the rank low than to avoid large cliques.

It is not hard to show that a simple connected graph G has rank 2 iff it is a complete bipartite graph, i.e., a *blowup* of an edge. Likewise, $\text{rank}(A_G) = 3$ if and only if G is a blowup of a triangle. Chang, Huang and Ye [12, 13], showed that every connected graph of rank 4 or 5, is the blowup of a graph from an explicit finite list of graphs, $\mathcal{G}_4$ and $\mathcal{G}_5$. This is, in fact, true in general. Indeed, if a graph has a pair of twin vertices and we remove one of the two, the rank remain unchanged (as does connectivity). Such reductions can be applied repeatedly. Also, note that such a reduction is the *inverse of a blowup*.

Proposition 2.14. *For every positive integer r, there is a finite list of graphs $\mathcal{G}_r$ such that every connected graph of rank r is a blowup of a member of $\mathcal{G}_r$.*

Proof. As shown by Kotlov and Lovász [36], a twin-free graph of rank r has at most $\mathcal{O}(2^{r/2})$ vertices. The claim now follows with

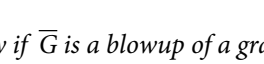
$$\mathcal{G}_r = \{\, \Gamma : \text{rank}(A_\Gamma) = r \text{ and } \Gamma \text{ is twin-free} \,\}.$$

□

An adaptation of the results on $\mathcal{G}_4$ and $\mathcal{G}_5$ leads us to the following lemma.

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Lemma 2.15. *Let G be a simple graph whose complement is connected. Then,*

1. *$\mathrm{rank}(A_G + I) = 4$ if and only if $\overline{G}$ is a blowup of a graph in* .
2. *$\mathrm{rank}(A_G + I) = r$ for $r \in \{2, 3\}$ if and only if $\overline{G}$ is a blowup of an r-clique.*

Proof. Let G be an n-vertex graph with $\mathrm{rank}(A_G + I) = r$ where $r \in \{2, 3, 4\}$, and $\overline{G}$ connected. Then,

$$l := \mathrm{rank}(A_{\overline{G}}) = \mathrm{rank}(J - (A_G + I)) \in \{r - 1, r, r + 1\}$$

So, by [12, 13], $\overline{G} = H^w$ is a blowup of a graph $H \in \mathcal{G}_l$ with some weights w. But then,

$$\begin{aligned} r = \mathrm{rank}(A_G + I_n) &= \mathrm{rank}(J_n - J_n + A_G + I_n) \\ &= \mathrm{rank}(J_n - A_{H^w}) \\ &= \mathrm{rank}(J_n - A_H^w) \\ &= \mathrm{rank}((J_{v(H)} - A_H)^w) = \mathrm{rank}(J_{v(H)} - A_H) \end{aligned}$$

therefore $H \in \mathcal{G}_{r-1} \sqcup \mathcal{G}_r \sqcup \mathcal{G}_{r+1}$ and is such that $\mathrm{rank}(J - A_H) = r$. This yields the above lists. □

We also require the following lemma.

Lemma 2.16. *Let G be an n-vertex graph whose complement is a disjoint union of blowups of cliques and isolated vertices. Then, $\mathrm{rank}(A_G + I) \cdot \omega(G) \geqslant n$.*

Proof. Let k be the number of connected components in $\overline{G}$, say

$$\overline{G} = \overline{G}_1 \sqcup \ldots \sqcup \overline{G}_k, \text{ where } \overline{G}_i = K_{l_i}^{w_i} \text{ for every } 1 \leqslant i \leqslant k$$

(where here, with slight abuse of notation, isolated vertices are denoted by K_1). The clique number is determined by the blowup weights,

$$\omega(G) = \alpha(\overline{G}) = \sum_{i=1}^{k} \alpha(\overline{G}_i) = \sum_{i=1}^{k} \|w_i\|_\infty ,$$

and conversely, denoting $l^\star := \max\left\{l_i : 1 \leqslant i \leqslant k\right\}$ and $L := l_1 + \ldots + l_k$, and letting w be the concatenation of the weights for each of the components, we may bound the rank as follows

$$\begin{aligned} \mathrm{rank}(A_G + I_n) = \mathrm{rank}(J_n - A_{\overline{G}}) &= \mathrm{rank}(J_L^w - A_{K_{l_1}^{w_1} \sqcup \ldots \sqcup K_{l_k}^{w_k}}) \\ &= \mathrm{rank}(J_L - A_{K_{l_1} \sqcup \ldots \sqcup K_{l_k}}) \\ &\geqslant \mathrm{rank}(J_{l^\star} - A_{K_{l^\star}}) = \mathrm{rank}(I_{l^\star}) = l^\star \end{aligned}$$

where the last inequality follows by taking a principal minor. The claim now follows, as

$$n = \sum_{i=1}^{k} \langle w_i, \mathbb{1} \rangle \leqslant \sum_{i=1}^{k} l_i \|w_i\|_\infty \leqslant l^\star \sum_{i=1}^{k} \|w_i\|_\infty \leqslant \mathrm{rank}(A_G + I) \cdot \omega(G)$$

□

The proof of Theorem 2.13 now follows.

Proof. Let G be an n-vertex graph with $n > 1$ and $\mathrm{rank}(A_G + I) = t - 1$, and write

$$\overline{G} = \overline{G}_1 \sqcup \ldots \sqcup \overline{G}_k, \text{ and } r_i := \mathrm{rank}(A_{\overline{G}_i}) \text{ for every } 1 \leqslant i \leqslant k$$

where $k \geqslant 1$ is the number of components in $\overline{G}$. Clearly, $r_1 + \ldots + r_k = \mathrm{rank}(A_{\overline{G}}) = \mathrm{rank}(J - (A_G + I)) \leqslant t$. The cases $t = \{2, 3, 4\}$ are thus handled, since if $k = 1$ then $\overline{G}$ is a blowup of a clique

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(recall Lemma 2.15), and if $k > 1$ each *component* is an isolated vertex or blowup of a clique, which is covered by Lemma 2.16.

It remains to handle $t = 5$, in the cases where not all the non-trivial components of $\overline{G}$ are blowups of cliques. Let $\mathcal{H}$ be the list of graphs in Lemma 2.15, excluding K_4. Only one possibility still remains to be dealt with: $\overline{G}$ is a blowup H^w of some graph $H \in \mathcal{H}$, together with (possibly) some isolated vertices. We can rule out the existence of isolated vertices, since then:

$$\operatorname{rank}(A_G + I) = \operatorname{rank}(J - A_{\overline{G}}) = \operatorname{rank}\left(\begin{array}{c|c} J - A_{H^w} & J \\ \hline J & J \end{array}\right) = \operatorname{rank}\left(\begin{array}{c|c} J - A_H & \mathbb{1}^T \\ \hline \mathbb{1} & 1 \end{array}\right)$$

and for all $H \in \mathcal{H}$, the latter rank is $5 > \operatorname{rank}(A_G + I) = t - 1 = 4$, a contradiction.

Finally we deal with the case $\overline{G} = H^w$. Maximal independent sets in H^w correspond to blowups of anticliques in H. Therefore, the following LP bounds the independence number of blowups of H:

$$\begin{aligned} &\text{minimize } a \\ &\text{subject to } x \geqslant 0 \\ &\qquad \langle \mathbb{1}, x \rangle = 1 \\ &\qquad \langle \mathbb{1}_S, x \rangle \leqslant a \quad \forall S \in I(H) \end{aligned}$$

where $I(H)$ is the set of all independent sets of H. Let $\alpha_H^\star$ be the optimum of the above programme. The least optima of this LP over the base-graphs in $\mathcal{H}$ is exactly $1/3$, by direct computation. Therefore,

$$\frac{\omega(G)}{n} = \frac{\alpha(H^w)}{\langle \mathbb{1}, w \rangle} \geqslant \alpha_H^\star \geqslant \frac{1}{3} = \frac{1}{\operatorname{rank}(A_G + I) - 1}$$

and the proof follows by re-arranging. □

2.2.2 Warmup: Simple constructions of Rank-Ramsey graphs

As Proposition 2.9 shows, *any* explicit (constructive) sublinear bound on $\nu_d(n)$ yields an explicit Ramsey graph. Moreover, the bound on the independence number is replaced by the possibly stricter requirement concerning the rank. How difficult is it to accomplish this? The directed quantity η_1 is easy to bound. Shigeta and Amano [52] obtained an asymptotically tight estimate, $\eta_1(n) = \Theta(n^{1/2+o_n(1)})$. As a brief warmup, we give the following weaker claim.

Claim 2.17. *For every positive integer n, there holds $n^{1/2} \leqslant \eta_1(n) \leqslant \mathcal{O}\left(n^{\log_4 3}\right)$.*

Proof. For the lower bound, if A is the adjacency matrix of a simple directed graph with no 2-cycle, then $(A + I) \odot (A + I)^T = I$, and by sub-multiplicativity indeed $\operatorname{rank}(A + I) \geqslant n^{1/2}$. For the upper bound, note the following properties of A_{D_4} the adjacency matrix of the directed 4-cycle,

- $\operatorname{rank}(A_{D_4} + I) = 3$.
- $A_{D_4} + I$ has no J_2 principal minor.

Consequently $\eta_1(4) \leqslant 3$, and the proof follows by repeatedly applying property 4 of Lemma 2.11. □

The upper bound can be likewise improved, e.g., to $\eta_1(n) \leqslant \Theta(n^{\log_6 4})$, as an exhaustive search through digraphs of order 6 yields $\eta_1(6) = 4$. Turning our attention to the *undirected* quantity $\nu_d(n)$, we note that $\nu_1(n) = n$ (the graph must have no edges). Instead, we provide, as part of the warmup, a trivial construction for $\nu_d(n)$. Better estimates provided in the sequel require new ideas.

Proposition 2.18. *For every two positive integers d and n, we have $\nu_d(n) \leqslant \lceil \frac{n}{d} \rceil$.*

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Proof. Let G be the disjoint union of $\lfloor \frac{n}{d} \rfloor$ copies of K_d and a single copy of $K_{n \bmod d}$. By construction, $\omega(G) = d$. Furthermore, $(A_G + I)$ is a block matrix of $\lceil \frac{n}{d} \rceil$ matrices J, thus $\operatorname{rank}(A_G + I) = \lceil \frac{n}{d} \rceil$. □

3. Rank-Ramsey graphs and the Log-Rank conjecture

In this section we briefly overview the connections between the log-rank problem in communication complexity, and Rank-Ramsey graphs. This discussion is mostly expository and not very technical.

On the one hand, we observe that any Rank-Ramsey construction *witnesses* a gap in the graph-theoretic formulation of the log-rank conjecture. Conversely we show that, *under certain conditions*, exhibiting a gap in the log-rank problem implies the construction of Rank-Ramsey graphs.

3.1 The log-rank conjecture

What is the relation between the *rank* of a binary matrix and its *communication complexity*? As observed by Mehlhorn and Schmidt [49], every deterministic communication protocol induces a partition of the communication matrix into monochromatic rectangles. This clearly implies that $\mathrm{D}(A) \geqslant \log \operatorname{rank}(A)$ for every binary matrix A, where $\mathrm{D}(A)$ is the deterministic communication complexity of A. Does a converse hold? This is one of the most notorious open questions in computational complexity. The *log-rank conjecture* states that

$$\mathrm{D}(A) \leqslant \mathcal{O}\left(\log^c \operatorname{rank}(A)\right) \tag{1}$$

for *every* binary matrix A, where $c > 1$ is some absolute constant.

Our current best lower bound on c is $c \geqslant 2$, due to Göös, Pitassi and Watson [28]. The upper bounds that we have remain exponentially far. The current record, due to Sudakov and Tomon [54] (see also [45]), states that $\mathrm{D}(A) = \mathcal{O}(\operatorname{rank}^{1/2}(A))$.

3.2 The graph-theoretic perspective

There are several known equivalent formulations of the log-rank conjecture, eschewing the use of terminology from communication complexity. For example, Nisan and Wigderson [50] proved that the conjecture is equivalent to the following statement: Define $\operatorname{mono}(A) := \max |B|/|A|$, over all monochromatic minors B of A.[9] The conjecture is that for every binary matrix A there holds

$$-\log(\operatorname{mono}(A)) \leqslant \mathcal{O}(\log^c \operatorname{rank}(A)),$$

where $c > 1$ is a universal constant.

Here, we consider the *graph-theoretic formulation*, due to Lovász and Saks (see [46, 50]), which states that there exists an absolute constant $c > 0$ such that for *every* graph G,[10]

$$\log \chi(G) \leqslant \mathcal{O}\left(\log^c \operatorname{rank}(A_G)\right)$$

To motivate the next definition, let us think of the clique number $\omega(G)$ as the largest cardinality of a set of vertices with independence number 1. We consider, likewise, the largest order of an induced subgraph of G, of independence number at most d:

[9] Here we denote by $|Z|$ the 'volume' of the matrix Z, i.e., the product of Z's dimensions.

[10] The constants c appearing in these three formulations may differ. The constant of the graph-theoretic formulation is known to be equivalent to that in a formulation comparing rank and χ^1 (i.e., the 1-*cover number*), which, by known results [4], is at most a factor of 2 away from the constant for deterministic communication complexity. We refer the reader to [47] for details of the graph-theoretic reduction.

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Definition 3.1. *Let G be a graph, and d be a positive integer. Then,*

$$\psi_d(G) := \max_{\substack{U \subseteq V(G) \\ \alpha(G[U]) \leqslant d}} |U|$$

An interesting feature of $\psi_d(G)$ is that it captures the chromatic number, up to log factors. This goes back to the classical results on integrality gaps for covering problems [43]. A similar argument appears in the analysis [19], of *extended fooling sets*.[11]

Claim 3.2. *For every graph G of order n, it holds that:*

$$\max_{1 \leqslant d \leqslant \alpha(G)} \frac{\psi_d(G)}{d} \leqslant \chi(G) \leqslant \ln n \cdot \max_{1 \leqslant d \leqslant \alpha(G)} \frac{\psi_d(G)}{d} + 1$$

Proof. Let H be an induced subgraph of G attaining $\psi_d(G)$ for some $1 \leqslant d \leqslant \alpha(G)$. Then,

$$\chi(G) \geqslant \chi(H) \geqslant \frac{\psi_d(G)}{\alpha(H)} \geqslant \frac{\psi_d(G)}{d}.$$

The middle inequality follows from the fact that $\alpha(F) \cdot \chi(F) \geqslant v(F)$ for every graph F. The following colouring algorithm yields the upper bound.

- Let $U_1 := V(G)$, and repeat until $U_{i+1} = \emptyset$:
- Let $U_{i+1} := U_i \setminus S_i$, where S_i is a largest independent set in the induced subgraph $G[U_i]$.
- Colour the vertices of S_i with colour i.

If this algorithm terminates after t iterations, then it yields a t-colouring of G, whence $t \geqslant \chi(G)$. Note also that $|U_i| \leqslant \psi_{|S_i|}(G)$. Denote $B := \max_{1 \leqslant i \leqslant t} |U_i|/|S_i|$, i.e., $|S_i| \geqslant \frac{1}{B}|U_i|$, for $i = 1, \ldots, t$. Thus, for all i, we have

$$|U_{i+1}| = |U_i| - |S_i| \leqslant |U_i| \cdot \left(1 - \frac{1}{B}\right).$$

Iterating, and using this for $i = t - 1$, we get

$$1 \leqslant |U_t| \leqslant n \cdot \left(1 - \frac{1}{B}\right)^{t-1} < n \cdot e^{-\frac{t-1}{B}}$$

Taking logs, we have,

$$\chi(G) \leqslant t < B \cdot \ln n + 1 \leqslant \max_{1 \leqslant i \leqslant t} \left(\frac{\psi_{|S_i|}(G)}{|S_i|}\right) \cdot \ln n + 1$$

□

The following two corollaries connect between the growth rates of the functions $v_d(n)$, and the log-rank conjecture. The first corollary shows that Rank-Ramsey graphs witness a gap in the graph-theoretic formulation of the log-rank conjecture. Conversely, the second corollary shows how under certain circumstances, log-rank separations can imply constructions of Rank-Ramsey graphs (see also Subsection 5.3 for an example of the interplay between the log-rank conjecture and Rank-Ramsey graphs).

Corollary 3.3. *For any two positive integers d and n, there exists a graph G such that:*

$$\chi(G) \geqslant \frac{n}{d}, \text{ and } \operatorname{rank}(A_G) \leqslant v_d(n) + 1$$

[11] It clearly suffices to consider the log-rank conjecture for twin-free graphs, as *reductions* affect neither chromatic number, nor rank. The main result of [36] then implies one can replace the $(\ln n)$-factor in Claim 3.2, with $(\ln \sqrt{2} + o(1))\operatorname{rank}(A_G)$.

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Proof. Let $\overline{G}$ be an order-n graph attaining $\nu_d(n)$. Then, $\chi(G)\alpha(G) \geqslant n$ and $\alpha(G) = \omega(\overline{G}) \leqslant d$. And,

$$\operatorname{rank}(A_G) = \operatorname{rank}(J - A_{\overline{G}} - I) \leqslant \operatorname{rank}(A_{\overline{G}} + I) + 1 = \nu_d(n) + 1$$

□

Corollary 3.4. *For every graph G, there exists some $1 \leqslant d \leqslant \alpha(G)$ such that,*

$$\chi(G) \leqslant \ln n \cdot \frac{\psi_d(G)}{d} + 1, \text{ and } \operatorname{rank}(A_G) \geqslant \nu_d(\psi_d(G)) - 1$$

Proof. Let $1 \leqslant d \leqslant \alpha(G)$ be a number for which the upper bound in Claim 3.2 holds. This gives us the left inequality. For the right inequality, let H be an induced subgraph of G attaining $\psi_d(G)$. Then,

$$\operatorname{rank}(A_G) \geqslant \operatorname{rank}(A_H) = \operatorname{rank}(J - A_{\overline{H}} - I) \geqslant \operatorname{rank}(A_{\overline{H}} + I) - 1 \geqslant \nu_d(\psi_d(G)) - 1$$

□

4. Rank-Ramsey graphs from minors of Kronecker powers

In this section we obtain a polynomial separation between $\nu_d(n)$ and n, for all $d \geqslant 41$.

Theorem 4.1. *The following bounds hold:*

1. $\nu_{41}(n) = \mathcal{O}(n^{1-\frac{1}{10000}})$.
2. *For any sufficiently large d, $\nu_d(n) = \mathcal{O}(n^{\log_{296}(232)})$, where $\log_{296}(232) \approx 0.957$.*

As we know, such gaps are implied by the existence of low-rank Boolean matrices with the desired properties. However, whereas most (non-explicit) constructions of Ramsey graphs found in the literature use the probabilistic method, this avenue is closed for us. This is due to the fact that we have no natural distribution over low-rank Boolean matrices. Consequently, we opt for *explicit* constructions.

The proof of Theorem 4.1 is divided into two parts: first, we find a *constant* size graph G, beating the trivial bound from Proposition 2.18 for some small d. Then, we show how to locate a good principal minor within large Kronecker powers of $(A_G + I)$, such that the clique number remains low.

4.1 Part 1: Finding a base graph

Our first goal is to find *some* graph G, such that $\operatorname{rank}(A_G + I)$ is strictly smaller than $\frac{\nu(G)}{\omega(G)}$. It is even better if G has a *small* clique number (say, $\omega(G) = 2$), as this improves the bounds in the blowup procedure (of part 2). This is equivalent to seeking a graph where the eigenvalue -1 has a large multiplicity, since $\operatorname{rank}(A_G + I) = \nu(G) - \mu_{A_G}(-1)$.

An exhaustive search reveals that there is no such graph with fewer than 10 vertices. Therefore, we seek larger, highly structured graphs. *Strongly-regular* graphs are natural candidates for such a search. These are regular, highly symmetric graphs with only *three* distinct eigenvalues (the Perron eigenvalue, equal to the degree, and two other distinct eigenvalues):

Definition 4.2. *Let $v > k > \max\{\lambda, \mu\} \geqslant 0$ be integers. We say that G is an* $\operatorname{srg}(v, k, \lambda, \mu)$ *(i.e., strongly regular graph with parameters v, k, λ and μ) if*

1. *G has order v.*
2. *It is k-regular.*
3. *Every two adjacent vertices in G have λ common neighbours.*
4. *Every two non-adjacent vertices in G have μ common neighbours.*[12]

[12] Here we use the standard notation μ for the number of common neighbours of non-adjacent neighbours in an SRG. This is not to be confused with $\mu_A(\lambda)$, the notation used for the multiplicity of λ in A.

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We cannot use strongly regular graphs as base graphs for $d = 2$, because apart from K_2 and the complete bipartite graphs $K_{n,n}$ (whose spectrum is *symmetric*, and thus cannot beat the trivial bound of Proposition 2.18), only seven triangle-free strongly regular graphs are known: C_5, Petersen, Clebsch, Hoffman-Singleton, Gewirtz, Mesner, and the Higman-Sims graph. Whether this list is exhaustive or not is a famous open question (Cf. [8, 7]). But even if such a graph exists, another problem arises.

Proposition 4.3. *$G = K_2$ is the only triangle-free strongly-regular graph for which $-1 \in \mathrm{spec}(G)$.*

Proof. Say G is in $\mathrm{srg}(v, k, \lambda, \mu)$, let A be its adjacency matrix, and let $r > s$ be its two distinct non-Perron eigenvalues. Observe that $\lambda = 0$ since G is triangle-free. The following relation among the parameters of strongly-regular graph are well known (e.g., [7]):

$$(1)\ k - \mu = -rs,\ \ (2)\ \lambda = \mu + r + s,\ \text{and}\ \ (3)\ (v - k - 1)\mu = k(k - \lambda - 1)$$

If $r = -1$ then $(1) \implies k - \mu = s < r < 0$, which implies that $k < \mu$, a contradiction. Alternatively, if $s = -1$, then $(2) \implies \mu = 1 - r < 2$ and either $\mu = 0$ (and $(3) \implies k = 1$, therefore $G = K_2$), or $\mu = 1$ (and $r = 0$). In the latter case, it follows (see [18]) that the graph must be a Moore graph, since substituting back into (3) yields:

$$v = k^2 + 1 = 1 + k \sum_{i=0}^{\mathrm{diam}(G)-1} (k-1)^i,$$

thus G is a *Moore graph* (which is clearly connected of diameter two, as it is strongly-regular). It remains to rule out the Moore graphs. Firstly, G can be neither the complete graph, nor an odd cycle over > 5 vertices, as they are not strongly-regular. Moreover, G cannot be C_5, the Petersen graph, the Hoffman-Singleton graph, or the famed (hypothesised) 57-regular Moore graph, as their spectra are all fully determined (c.f. [8, 30]), and do not match $r = 0,\ s = -1$. □

Proposition 4.3 *appears* to rule out the use of strongly regular graphs as base graphs, if we insist on $d = 2$. However, we have a way out: A triangle-free strongly regular graph *can* have the eigenvalue 1. This suggests that we find a way to negate the spectrum of such graphs, while retaining triangle-freeness. As the following two simple lemmas show, this is indeed possible.

Lemma 4.4. *For any two graphs, G and H, $\omega(G \otimes H) \leqslant \min\{\omega(G), \omega(H)\}$.*

Proof. Let $S \subseteq V(G) \times V(H)$ be a clique in $G \otimes H$. By definition of the Kronecker product, the projections of S onto G is a clique, and likewise for H. As the graphs are simple, every vertex appears in these projections exactly once. □

Lemma 4.5. *Let A be the adjacency matrix of a graph G and let $l > 2$ be an integer. Then:*

$$\mu_{A \otimes K_l}(-1) = (l-1) \cdot \mu_A(1)$$

Proof. Recall that the spectrum of the Kronecker Product of two matrices is the pairwise product of their respective eigenvalues. Since $\mathrm{spec}(A_{K_l}) = \{l^{(1)}, -1^{(l-1)}\}$, it follows that

$$\mu_{A \otimes K_l}(-1) = (l-1) \cdot \mu_A(1) + \mu_A\left(\frac{1}{l-1}\right)$$

but $\mu_A(\frac{1}{l-1}) = 0$, since A is symmetric and binary, and thus all its eigenvalues are algebraic integers. □

In combination, Lemmas 4.4 and 4.5 yield some base-graphs and upper bounds on ν_2, and ν_3.

Corollary 4.6. *For any $l > 2$, we have $\nu_2(16l) \leqslant 6l + 10$, and $\nu_3(27l) \leqslant 7l + 20$.*

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Proof. Let $\mathcal{C}$ be the Clebsch graph, and let $\mathcal{S}$ be the Schläfli graph. It is known that:

$$\omega(\mathcal{C}) = 2,\ \ \mu_{A_{\mathcal{C}}}(1) = 10,\ \ \nu(\mathcal{C}) = 16 \quad \text{and} \quad \omega(\overline{\mathcal{S}}) = 3,\ \ \mu_{A_{\overline{\mathcal{S}}}}(1) = 20,\ \ \nu(\overline{\mathcal{S}}) = 27$$

Consider the graph $G = \mathcal{C} \otimes K_l$. By construction, $\nu(G) = \nu(\mathcal{C}) \cdot \nu(K_l) = 16l$, and by Lemma 4.5, it follows that $\mu_{A_G}(-1) = \mu_{A_{\mathcal{C}}}(1) \cdot (l-1) = 10(l-1)$, and therefore $\operatorname{rank}(A_G + I) = \nu(G) - \mu_{A_G}(-1) = 6l + 10$. Moreover, $\mathcal{C}$ is triangle-free, and thus by Lemma 4.4, so is G. Similarly $\overline{\mathcal{S}}$ yields the claim about ν_3. □

Remark 4.7. *In a very recent paper, Bamberg, Bishnoi, Ihringer and Ravi [5] constructed a family of Cayley graphs of $\mathbb{F}_{2^n}^2$ that improves upon the constant of our triangle-free construction, lowering the bound from $\nu_2(n) \leqslant (3/8 + o(1))n$ (see Corollary 4.6), to $\nu_2(n) \leqslant (1/4 + o(1))n$.*

4.2 Part 2: Principal minors of Kronecker powers

Next we use the base-graphs from Part 1 to construct infinite families of graphs with bounded clique numbers, which exhibit a polynomial separation between the order and rank of the complement. Let G be a base-graph, and let H be the graph with adjacency matrix $(A_G + I)^{\otimes n} - I$. Clearly $\omega(H) = \omega(G)^n$ (take the direct product of n copies of a largest clique of G, plus self loops). However, as shown below, if $\omega(G)$ is sufficiently *small*, then there is, nevertheless, a *large principal minor* of A_H with no *large* cliques.

Theorem 4.8. *Let $G = ([k], E)$ be a graph, $d > 1$ a positive integer, and $r := \operatorname{rank}(A_G + I)$. Then, for every large enough n there exists a graph H such that:*

1. $\omega(H) \leqslant d$.
2. $\operatorname{rank}(A_H + I) \leqslant r^n$
3. $\nu(H) = \frac{k^n}{d}\left(\sum_{t=1}^{\omega(G)} \#K_t(G) \cdot t^{d+1}\right)^{-\frac{n}{d+1}} - 1$, *where $\#K_t(G)$ is the number of copies of K_t in G.*

Proof. Let $M = (A_G + I)^{\otimes n}$, and let $X_1, \ldots, X_N \overset{\text{iid}}{\sim} [k]^n$ be indices in M, sampled independently and uniformly at random (the value of N will be fixed later). We find an upper bound on the probability that the principal (generalised) minor $W = M[X_1, \ldots, X_N]$ of M, contains a principal $(d+1) \times (d+1)$ all-ones minor. By union bound:

$$\mathbb{P}\left[\exists S \in \binom{[N]}{d+1} : W[\{X_i\}_{i \in S}] = J_{d+1}\right] \leqslant N^{d+1} \cdot \mathbb{P}_{X_1,\ldots,X_{d+1} \sim [k]^n}\left[W[X_1, \ldots, X_{d+1}] = J_{d+1}\right].$$

The choice of $X_1, \ldots, X_{d+1}$ determines a $n \times (d+1)$ matrix with entries $x_{i,j} \in [k]$.

By definition of the Kronecker product the above event occurs exactly when

$$\forall i, j \in [d+1] : \forall l \in [n] : (A_G + I)_{x_{l,i}, x_{l,j}} = 1$$

and by independence, the probability of this event equals

$$\mathbb{P}_{z_1,\ldots,z_{d+1} \sim [k]}\left[\forall i, j \in [d+1] : (A_G + I)_{z_i, z_j} = 1\right]^n.$$

The latter expression counts the ways to place integer weights on G's vertices (with self-loops) such that all vertices in the support (i.e., with non-zero weights) form a clique. This is easily bounded by:[13]

$$\mathbb{P}_{z_1,\ldots,z_{d+1} \sim [k]}\left[\forall i, j \in [d+1] : (A_G + I)_{z_i, z_j} = 1\right] \leqslant \frac{1}{k^{d+1}} \cdot \sum_{t=1}^{\omega(G)} \#K_t(G) \cdot t^{d+1}.$$

[13] An inclusion-exclusion refinement of the following argument yields only a negligible improvement in the results.

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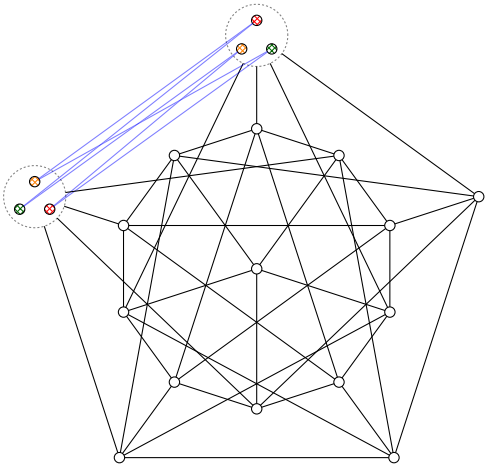


Figure 1. Illustration of the Kronecker product $\mathcal{C} \otimes K_3$, where $\mathcal{C}$ is the Clebsch graph. Similarly to *blowup*, vertices are replaced by anticliques, and edges by bipartite graphs. The key difference is that here edges are replaced by complete bipartite graphs *minus* the identity matching (so the graph is twin-free).

If this expression is bounded away from 1, then a good minor W exists. When this is so, a bound on N follows:

$$N^{d+1} \cdot \left(\frac{1}{k^{d+1}} \cdot \sum_{t=1}^{\omega(G)} \#K_t(G) \cdot t^{d+1} \right)^n < 1 \implies N \leqslant \left(\frac{k^{d+1}}{\sum_{t=1}^{\omega(G)} \#K_t(G) \cdot t^{d+1}} \right)^{\frac{n}{d+1}}$$

We can, therefore, *fix* some values for $X_1, \ldots, X_N$, such that $W = M[X_1, \ldots, X_N]$ contains no all-ones principal minor of size $d+1$. By Proposition 2.3, $\operatorname{rank}(W) \leqslant \operatorname{rank}(M) = r^n$, since W is a generalised minor of M. The proof is concluded by letting $A_H = W - I$. □

We are now ready to prove Theorem 4.1.

Proof. Fix two constants $d > 1$ and $l > 2$. As in the proof of Corollary 4.6, let $G = \mathcal{C} \otimes K_l$, where $\mathcal{C}$ is the Clebsch graph (see Figure 1). As shown there,

$$v(G) = v(\mathcal{C}) \cdot v(K_l) = 16l, \; e(G) = e(\mathcal{C}) \cdot e(K_l) = 40\binom{l}{2}, \text{ and } \operatorname{rank}(A_G + I) = 6l + 10$$

By applying Theorem 4.8 to G, we obtain, for every large N a K_{d+1}-free graph H with:

$$\operatorname{rank}(A_H + I) \leqslant (6l+10)^N, \text{ and } v(H) = \Theta\left(\frac{(16l)^{d+1}}{16l + 40\binom{l}{2}\left(2^{d+1} - 2\right)} \right)^{\frac{N}{d+1}}$$

Denoting $n = v(H)$ and re-arranging, we may write:

$$\operatorname{rank}(A_H + I) \leqslant \mathcal{O}\left(n^{\delta}\right), \text{ where } \delta = \frac{(d+1)\ln(6l+10)}{(d+1)\ln(16l) - \ln\left(16l + 40\binom{l}{2}(2^{d+1} - 2)\right)}$$

For the first bound, take $d = 41$ and $l = 26$.

For the second bound, note that for any fixed l, the expression for δ decreases with d.

$$\lim_{d\to\infty} \delta = \lim_{d\to\infty} \frac{(d+1)\ln(6l+10)}{(d+1)\ln(16l) - \ln\left(16l + 40\binom{l}{2}(2^{d+1} - 2)\right)} = \frac{\ln(6l+10)}{\ln(16l) - \ln 2}.$$

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The minimum of this expression over the integers occurs at $l = 37$. This yields,

$$\delta = \frac{\ln(232)}{\ln(296)} + o_d(1)$$

as claimed. □

5. Rank-Ramsey graphs from lifts of Erdős-Rényi graphs

So far, we have constructed Rank-Ramsey graphs with constant d. In this section we proceed to the range $d = \Theta(\log n)$, and prove the following theorem:

Theorem 5.1. *For any two constants $c, \varepsilon > 0$ with $c > 2\left(\frac{2}{3\varepsilon} + 1\right)^2$, there holds $\nu_{c \log n}(n) \leqslant \widetilde{\mathcal{O}}(n^{\frac{2}{3}+\varepsilon})$.*

These are graphs with a *logarithmic* clique number, and approaching $\mathrm{rank}(A_G + I) = \mathcal{O}\left(n^{2/3}\right)$. For example, one can instantiate Theorem 5.1 to obtain:

$$\nu_{10^6 \log n}(n) \leqslant \mathcal{O}\left(n^{\frac{2}{3}+\frac{1}{1000}}\right)$$

The key ingredient in our proof of Theorem 5.1 is our matrix-theoretic view of *lifting* (Subsection 2.1).

5.1 The functions NAE and AE

The graphs constructed in Theorem 5.1 are the lifts of matrices with the following Boolean functions.

Definition 5.2. *The Boolean functions* $\mathrm{NAE} : \{0,1\}^3 \to \{0,1\}$ *and* $\mathrm{AE} : \{0,1\}^3 \to \{0,1\}$ *are defined by:*

$$\mathrm{NAE}(x_1, x_2, x_3) := \begin{cases} 0 & x_1 = x_2 = x_3 \\ 1 & \textit{otherwise} \end{cases}, \textit{ and } \mathrm{AE} := 1 - \mathrm{NAE}$$

The seminal paper of Nisan and Wigderson [50] exhibited a gap for the log-rank conjecture using recursive compositions of NAE and lifts thereof (with the AND-gadget). For more on this, see Section 7.

Since the NAE function depends only on the Hamming weight of the input, its unique expression as a multilinear polynomial is a linear combination of elementary symmetric functions,

$$\mathrm{NAE}(x_1, x_2, x_3) = e_2(x_1, x_2, x_3) - e_1(x_1, x_2, x_3) + 1 = x_1x_2 + x_2x_3 + x_1x_3 - x_1 - x_2 - x_3 + 1.$$

5.2 Our construction

It is more convenient to formulate the proof in terms of AE, rather than NAE. Our aim is to bound from above two quantities of the graphs that we construct: (i) Rank, and (ii) Maximum Clique Size.

The rank. The polynomial representation of AE (in particular, its low degree), in conjunction with Lemma 2.6, imply that the lift of *any* three matrices $A_1, A_2, A_3 \in M_k$,

$$M := \mathrm{AE}(A_1, A_2, A_3) \in M_{k^3}(\mathbb{R})$$

has *low rank*, namely $\mathrm{rank}(M) \leqslant 3k^2 + 3k = \mathcal{O}(k^2)$.

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The clique number. In our construction A_1, A_2 and A_3 are adjacency matrices of simple graphs, G_1, G_2 and G_3 respectively. This guarantees that M is a symmetric binary matrix with ones on the diagonal. As an illustration (and an aside), this class of graphs includes the *trivial construction* of Proposition 2.18: if $G_1 = K_k$ and $G_2 = G_3 = \overline{K}_k$, the resulting lifted graph is a disjoint union of cliques.[14]

Consider a set of vertices $S = \{(x_1, y_1, z_1), \ldots, (x_s, y_s, z_s)\}$ in $V(G_1) \times V(G_2) \times V(G_3)$. By definition of AE, this set forms a clique in the lifted graph if and only if the graphs (with multiplicities, i.e., generalised minors) induced over the constituent graphs are *identical*. Therefore, informally, cliques of the lifted graph emerge from *correlations* between G_1, G_2 and G_3. To avoid correlations, we will sample our base graphs i.i.d. from $G(k, 1/2)$, as in the classical lower bound on diagonal Ramsey numbers.

Large projections. We view the vertices of the lifted graph as lattice points in $[k] \times [k] \times [k]$. To control the clique size of the lifted graph, we seek to avoid subsets $S \subseteq [k] \times [k] \times [k]$ with small axis projections $|\pi_x(S)|$, $|\pi_y(S)|$ and $|\pi_z(S)|$. Since our constituent graphs are random, sets with large projections will guarantee 'sufficient randomness', and will therefore be less likely to induce cliques in the lifted graph. This intuition is made concrete in the proof of Theorem 5.1. Beforehand, we use a probabilistic argument to show that there exist *large* subsets of the cube, in which *every* subset above a certain cardinality has *some* large axis projection.[15]

Lemma 5.3. *Let $a > 0$ and $1 > \eta > 0$ be constants. For any sufficiently large k, there exists a subset $T \subset [k]^3$, of cardinality $|T| = \widetilde{\Theta}\left(k^{3(1-\eta)}\right)$, such that for every subset $S \subset T$ of cardinality $|S| = a \log k$,*

$$\max\left\{|\pi_x(S)|, |\pi_y(S)|, |\pi_z(S)|\right\} \geqslant \eta|S| = \eta a \log k$$

Proof. Let T be a sample of N points, chosen i.i.d and uniformly from $[k]^3$, with N to be determined later. What is the probability that T fails to satisfy the statement in the lemma? Say that a subset $S \subset T$ of cardinality $|S| = a \log k$ is *x-bad* if $|\pi_x(S)| \leqslant \eta a \log k$. Similarly define *$y$* and *$z$*-bad subsets. Call S *bad* if it is x, y and z-bad. A sample T with no bad subsets is called *appropriate*. If the expected number of bad subsets $S \subset T$ is negligible, then by Markov's inequality, whp T is appropriate.

The event that S is bad at a certain axis (e.g., x-bad) is equivalently formulated as follows: We throw $a \log k$ balls independently into k bins, and all our balls fall into at most $\eta a \log k$ bins. The probability of this bad event is

$$\binom{k}{\eta a \log k}\left(\frac{\eta a \log k}{k}\right)^{a \log k} \leqslant k^{(\eta-1)a \log k + \mathcal{O}(\log\log k)}.$$

So, the probability that S is bad is at most

$$k^{3(\eta-1)a \log k + \mathcal{O}(\log\log k)}.$$

Consequently, the expected number of bad subsets that are contained in our random set T is at most

$$\binom{N}{a \log k} k^{3(\eta-1)a \log k + \mathcal{O}(\log\log k)}.$$

[14] Given graphs G_1, G_2, G_3 of *small order*, lifts $\mathrm{AE}(G_1, G_2, G_3)$ can be used as base-graphs for Theorem 4.8. For instance, $M = \mathrm{AE}(K_4, K_4, M_2) \in M_{64}(\mathbb{R})$, where M_2 is a perfect matching, has $\omega(G) = 2$ and $\mathrm{rank}(A_G) = 31 < \frac{64}{\omega(G)} = 32$. However, this does not beat the base-graph of Subsection 4.1.

[15] The Loomis-Whitney inequality provides some lower bounds on projection sizes. Namely, for every $S \subset [k] \times [k] \times [k]$, we have $|\pi_{xy}(S)||\pi_{yz}(S)||\pi_{xz}(S)| \geqslant |S|^2$ and $|\pi_x(S)||\pi_y(S)||\pi_z(S)| \geqslant |S|$.

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This expression is $o(1)$ provided that

$$N = k^{3(1-\eta)-o(1)}$$

in which case we can conclude that there *exists* an appropriate sample T of size N.

Since T is chosen *with repetitions*, we have to account for the possibility that the same point is selected more than once. But no point can be chosen more than $a(1-\eta)\log k$ times. For otherwise any set of $a\log k$ triples containing these duplicates would be bad and render T inappropriate. It follows that there exists an appropriate set $T' \subset [k]^3$ of cardinality at least $\frac{N}{a(1-\eta)\log k}$. □

Appropriate sets allow us to identify a large principal minor of the lifted graph, in which every set of cardinality $a\log k$ is 'sufficiently random'. With it, we are ready to present our construction.

Proof of Theorem 5.1. As mentioned, our construction proceeds by sampling three graphs G_1, G_2, and G_3, independently and uniformly at random from $G\left(k, 1/2\right)$. Let A_1, A_2 and A_3 be their respective adjacency matrices, and let $M := \mathrm{AE}(A_1, A_2, A_3)$ be their lift.

Pick $a > 0$ and $1 > \eta > 0$, construct a subset $T \subset [k]^3$ as in Lemma 5.3, and consider $W := M[T]$. Since G_1, G_2 and G_3 have no self-loops, the main diagonal of M (and hence W as well) is all-ones. Also,

$$\mathrm{rank}(W) \leqslant \mathrm{rank}(M) \leqslant 3k^2 + 3k$$

in view of Lemma 2.6 and the fact that AE has total degree 2.

So, we now have the *rank* under control and move to deal with the *clique number*. It remains to show that, with positive probability (over the choice of G_1, G_2 and G_3), the matrix W has no all-ones principal minors of order $a\log k$. Let X be the random variable that counts the number of such principal minors. If its expectation is $\mathbb{E}(X) = o(1)$, then whp the resulting graph has no $(a\log k)$-clique.

Consider a subset $S \subset T$ of cardinality $|S| = a\log k$. What is the probability that $W[S]$ is an all-ones matrix? According to Lemma 5.3, S has a large projection on some axis, say $|\pi_x(S)| \geqslant \eta a\log k$ and for convenience we assume that the first $\eta a\log k$ x's are all distinct. By definition of the AE function, the adjacency relations among the $\eta a\log k$ *distinct* vertices of $\pi_x(S)$ and the corresponding vertices of $\pi_y(S), \pi_z(S)$, must coincide with each other. That is,

$$A_1\left[x_1, \ldots, x_{\eta a\log k}\right] = A_2\left[y_1, \ldots, y_{\eta a\log k}\right] = A_3\left[z_1, \ldots, z_{\eta a\log k}\right].$$

Since $A_1[x_1, \ldots, x_{\eta a\log k}]$ is a *proper* minor of A_1, with no repeated indices, it is the adjacency matrix of a $G(\eta a\log k, 1/2)$ graph. Therefore, for any fixed choice of $A_2\left[y_1, \ldots, y_{\eta a\log k}\right]$, it holds that the event $A_1\left[x_1, \ldots, x_{\eta a\log k}\right] = A_2\left[y_1, \ldots, y_{\eta a\log k}\right]$ occurs with probability exactly

$$2^{-\binom{\eta a\log k}{2}} = k^{-\left(\frac{\eta^2 a^2}{2} - o(1)\right)\log k}$$

Consequently

$$\mathbb{E}(X) \leqslant \binom{|T|}{a\log k} k^{-\left(\frac{\eta^2 a^2}{2} - o(1)\right)\log k} = k^{\left(3a(1-\eta) - \frac{\eta^2 a^2}{2} + o(1)\right)\log k}.$$

It follows that $\mathbb{E}(X) = o(1)$ when $a > \frac{6(1-\eta)}{\eta^2}$. The proof now follows by fixing

$$n := |T| = \widetilde{\Theta}(k^{3(1-\eta)}), \quad c := \frac{a}{3(1-\eta)}, \quad \varepsilon := \frac{2\eta}{3(1-\eta)}$$

□

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5.3 Rank-Ramsey graphs with logarithmic clique number

In Subsection 5.2, we construct Rank-Ramsey graphs with *logarithmic* clique number and *polynomial* rank, approaching $n^{2/3}$. Conversely, in the study of 'classical' Ramsey graphs, Erdős [21] showed that asymptotically almost all n-vertex graphs have logarithmic clique *and* independence numbers, and this is indeed best-possible up to constants (c.f. [11, 22, 55]). This contrast between Ramsey and Rank-Ramsey graphs raises the question: what is the growth rate of $\nu_{a\log n}(n)$?

Our construction witnesses that for every constant $a > 0$, there holds $\nu_{a\log n}(n) = \widetilde{\mathcal{O}}(n^{2/3} + K(a))$, where $K(a)$ is a constant depending on a. Conversely, we remark that if the log-rank conjecture holds, then by Corollary 3.3, there exists a universal constant $c > 1$, such that for every constant $a > 0$,

$$\nu_{a\log n}(n) = \Omega\left(2^{\sqrt[c]{\log\left(\frac{n}{a\log n}\right)}}\right)$$

6. In search of better bounds on $\nu_2(n)$

The Ramsey numbers $R(3, n)$ have fascinated combinatorialists for decades. Culminating a long line of excellent research, Kim [34] proved that $R(3, n) = \Theta(n^2/\log n)$. Even the implicit constant is known, up to a factor of $(4 + o(1))$ [23]. What about $R^k(3, n)$, or equivalently, $\nu_2(n)$? Trivially,

$$\Omega\left(\sqrt{n\log n}\right) \leqslant \nu_2(n) \leqslant \left\lfloor \frac{n}{2} \right\rfloor$$

where the lower-bound follows from the Ramsey numbers $R(3, n)$.

The base-graphs used in Section 4, derived from the Clebsch graph, yields an improvement of the *upper bound* to $\nu_2(n) \leqslant (3/8 + o(1))n$. If we wish to improve the *lower bound* to $\nu_2(n) \geqslant \Omega(n^{1/2+\varepsilon})$ for some positive constant ε, then of course, bounds on the *independence number* will not do, as there *exist* triangle-free graphs with $\alpha(G) = \Omega(\sqrt{n\log n})$ [2]. Instead, we turn to consider other graph parameters. More concretely we investigate parameters that are related to *orthonormal representations* of graphs, on which a considerable body of research exists. This choice is motivated by the fact that the Gram matrix that corresponds to an orthonormal representation of G is reminiscent of $A_G + I$.

Definition 6.1. *An orthonormal representation* $\left\{w_v \in \mathbb{R}^N\right\}_{v\in V(G)}$ *of a graph* G *is an assignment of unit vectors to the vertices of* G*, such that*

$$\forall v, u \in V(G) : \langle w_v, w_u\rangle = \begin{cases} 1 & v = u \\ 0 & v \not\sim u \\ \star & \textit{otherwise} \end{cases}$$

The Gram matrix of the vectors w_v in any orthnormal representation of an n-vertex graph G, is an $(n \times n)$ positive semidefinite matrix W that *agrees* with $A_G + I$ on the main diagonal, and on G's non-edges. It is very suggestive that such matrices can tell us a lot about $A_G + I$. Two of the most studied measures related to M are the Lovász number $\vartheta(G)$, and the minimum semidefinite rank, msr(G), whose definitions we recall below. Like rank$(A_G + I)$, these two quantities are related to constructions of Ramsey graphs. By the well-known 'sandwich theorem' (see [39]), for every graph G there holds

$$\alpha(G) \leqslant \vartheta(G) \leqslant \mathrm{msr}(G) \leqslant \chi(\overline{G})$$

How does rank$(A_G + I)$ fit into this web of relations? We show the following.

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Theorem 6.2. *For infinitely many $n > 1$, there exist triangle-free graphs G_1 and G_2 of order n, with*

1. $\vartheta(G_1) = \Theta(n^{2/3})$ *and* $\operatorname{rank}(A_{G_1} + I) = n$.
2. $msr(G_2) \geqslant n/2$ *and* $\operatorname{rank}(A_{G_2} + I) = (3/8 + o(1))n$.

That is, we show *separations* between $\operatorname{rank}(A_G + I)$, and $\vartheta(G)$ and $\operatorname{msr}(G)$.[16] Understanding the relations between $\vartheta(G)$ and $\operatorname{rank}(A_G + I)$ is particularly interesting in view of striking similarities between the two parameters. See Subsection 6.1 for more.

It is conceivable to us that $\vartheta(G) = \mathcal{O}(\operatorname{rank}(A_G + I))$. For instance, $\vartheta(G) \leqslant \operatorname{rank}(A_G + I)$ for every graph of order less than 10, and we do not know any graph that fails this inequality. We remark that if this indeed holds true, then the results of [35, 1] regarding the Lovász number of graphs with bounded independence number, imply a *polynomial improvement* on the lower bounds on ν_d, namely we have $\nu_d(n) = \Omega(n^{2/(d+1)})$, versus the weaker bound $\Omega(n^{1/d})$ (see Corollary 2.10) that follow purely from Ramsey numbers. We refer the reader to Appendix B for a discussion on the connection between this open problem, and other well-known problems regarding the relationship between the well-known Hoffman's bound and Cvetković's bound.

6.1 Lovász number versus Rank

Let us recall the definition of the Lovász number $\vartheta(G)$ of a simple graph G:

Definition 6.3 [44]. *The Lovász number of a graph $G = (V, E)$ is*

$$\vartheta(G) := \min_{c,W} \max_{v \in V} \frac{1}{\langle c, w_v \rangle^2}$$

where $W : V \to \mathbb{R}^N$ is an orthonormal representation of G and $c \in \mathbb{R}^N$ is a unit vector.

The graph parameters $\vartheta(G)$ and $\operatorname{rank}(A_G + I)$ share many properties. E.g., both are multiplicative in the strong product (see Definition 2.1), and both are upper bounds on the Shannon capacity of a graph.

Proposition 6.4. *Let G and H be two graphs. Then,*

$$\vartheta(G \boxtimes H) = \vartheta(G)\vartheta(H), \text{ and } \operatorname{rank}(A_{G \boxtimes H} + I) = \operatorname{rank}(A_G + I) \operatorname{rank}(A_H + I)$$

Proof. Lovász proved ([44], Lemma 2 and Theorem 7) that ϑ is *multiplicative* in the strong product. As for the rank, by Definition 2.1 and the multiplicativity of rank under the Kronecker product,

$$\begin{aligned} \operatorname{rank}(A_{G \boxtimes H} + I) &= \operatorname{rank}(A_G \otimes A_H + A_G \otimes I + I \otimes A_H + I \otimes I) \\ &= \operatorname{rank}((A_G + I) \otimes (A_H + I)) = \operatorname{rank}(A_G + I) \operatorname{rank}(A_H + I) \end{aligned}$$

□

Corollary 6.5 (See also [44, 31]). *For every graph G it holds that*

$$\vartheta(G) \geqslant \Theta(G) \text{ and } \operatorname{rank}(A_G + I) \geqslant \Theta(G)$$

where $\Theta(G) := \sup_k \sqrt[k]{\alpha(G^{\boxtimes k})}$ is the Shannon capacity of G.

6.1.1 Best-possible separation of $\operatorname{rank}(A_G + I)$ from $\vartheta(G)$

The current best known *explicit construction* of triangle-free Ramsey graphs is due to Alon [3]. An order-n graph G in this family satisfies $\vartheta(G) = \Theta(n^{2/3})$. As we will presently show, there also holds that $\operatorname{rank}(A_G + I) = n$. In other words, these are *best-possible* ϑ-Ramsey graphs, yet they are *worst-possible* Rank-Ramsey graphs.

[16] Triangle-free graphs with $\operatorname{rank}(A_G + I)$ much larger than $\operatorname{msr}(G)$ are also not hard to come by. For example, the complete bipartite graph $G = K_{n,n}$ has $\operatorname{msr}(G) \leqslant \chi(\overline{G}) = \chi(K_n \sqcup K_n) = n$ and $\operatorname{rank}(A_G + I) = 2n$.

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Let us briefly recount Alon's construction: Let $k > 1$ be such that $3 \nmid k$ and let[17]

$$W_0 := \{x \in \mathrm{GF}(2^k) : \text{ leftmost bit of } x^7 \text{ is } 0\}, \quad W_1 := \{x \in \mathrm{GF}(2^k) : \text{ leftmost bit of } x^7 \text{ is } 1\}$$

and

$$U_0 := \{(w, w^3, w^5) : w \in W_0\} \subset \mathbb{Z}_2^{3k}, \quad U_1 := \{(w, w^3, w^5) : w \in W_1\} \subset \mathbb{Z}_2^{3k}$$

Then, $G_k := \mathrm{Cay}(\mathbb{Z}_2^{3k}, \{u_0 + u_1 : u_0 \in U_0, u_1 \in U_1\})$ is a graph on $n_k := |\mathbb{Z}_2^{3k}| = 2^{3k}$ vertices.

Theorem 6.6 [3]. *For every $k > 1$ with $3 \nmid k$, G_k is triangle-free, and $\vartheta(G_k) = \Theta(n_k^{2/3})$.*

This bound on ϑ is tight, as shown by Kashin and Konyagin.

Theorem 6.7 [35]. *For every n vertex triangle-free graph G, there holds $\vartheta(G) \geqslant (2n)^{2/3}$.*

Proof. Kashin and Konyagin [35] (see also [3]) proved $\vartheta(\overline{G}) \leqslant 2^{2/3} n^{1/3}$ for every triangle-free graph. The claim follows, since $\vartheta(G)\vartheta(\overline{G}) \geqslant n$ for every graph of order n (see [44], Corollary 2). □

In the following claim, we show that $\mathrm{rank}(A_{G_k} + I) = n_k$, for every admissible k.

Claim 6.8. *For every $k > 1$ with $3 \nmid k$, there holds* $\mathrm{rank}(A_{G_k} + I) = n_k$.

Proof. Since G_k is a Cayley graph of an Abelian group, $\mathbb{Z}_2^{3k}$, its eigenvalues correspond to sums of group characters, evaluated over the generating set. The characters of $\mathbb{Z}_2^t$ are the Fourier-Walsh functions, i.e., the *parity functions* $\chi_D : Z_2^t \to \{\pm 1\}$, for every set $D \subseteq [t]$. So, every $D \subseteq [n]$, yields an eigenvalue λ_D of A_{G_k}, where

$$\lambda_D = \sum_{\substack{u_0 \in U_0 \\ u_1 \in U_1}} \chi_D(u_0 + u_1) = \left(\sum_{u \in U_0} \chi_D(u) \right) \left(\sum_{u \in U_1} \chi_D(u) \right)$$

which is an even integer, because $|U_1| = |W_1| = 2^{k-1}$ (see [3]) are *even*. Consequently, all eigenvalues of A_{G_k} are even, and $\mathrm{rank}(A_{G_k} + I) = n_k - \mu_{A_{G_k}}(-1) = n_k$. □

6.2 Minimum semidefinite rank

Another obvious point of comparison is the *rank*. The minimum semidefinite rank, $\mathrm{msr}(G)$, is the smallest rank of an $n \times n$ orthonormal representation matrix of an order-n graph G. It is easy to see that triangle-free graphs have a large $\mathrm{msr}(G)$.[18] Deaett [17] showed that $\mathrm{msr}(G) \geqslant n/2$, for any connected triangle-free graph. In contrast, in Corollary 4.6 we prove that $\nu_2(16l) \leqslant 6l + 10$, which is attained by a family of *connected* graphs. Therefore,

Corollary 6.9. *For any sufficiently large n, there exists an order n triangle-free graph G, with*

$$msr(G) \geqslant n/2 \text{ and } \mathrm{rank}(A_G + I) = (3/8 + o(1))n$$

We do not know whether a converse of the form $\mathrm{rank}(A_G + I) = \Omega(\mathrm{msr}(G))$ holds.

[17] This refers to the binary representation of elements in $\mathrm{GF}(2^k)$, which are naturally identified with binary strings of length k, as usual.

[18] For instance, a trivial Cauchy-Schwarz argument implies that $\mathrm{msr}(G) \geqslant n/3$ for every triangle-free G.

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7. The Nisan-Wigderson construction

In a seminal paper, Nisan and Wigderson [50] constructed an infinite family of symmetric binary matrices A_k of dimension 2^{3^k}. These matrices exhibit a separation between log-rank and communication complexity, and show that $c \geqslant \log_2 3$ in the conjectured inequality (1).

As discussed in Section 3, such separations can be related to Rank-Ramsey graphs. Here, we analyse the Nisan-Wigderson matrices from this perspective, and show that these matrices A_k have *very large monochromatic principal minors* and yield, therefore, poor Rank-Ramsey graphs.

Claim 7.1. *For every $k \geqslant 1$ there exist subsets $S, T \subset [2^{3^k}]$ of cardinality $2^{3^k(1-o_k(1))}$ each, such that*

$$A_k[S] = \mathbf{0}, \text{ and } A_k[T] = J$$

We stress that this does *not* a contradict the discussion of Section 3: Exhibiting a log-rank separation *does not* preclude a matrix from *having* a large monochromatic rectangle.

7.1 The construction

The construction of [50] involves a *lift* using the AND gadget, with a recursive composition of the function NAE (as in Definition 5.2) with itself. To describe their result, we require some notation.

Definition 7.2. *For every $k > 1$, the Boolean function $\mathrm{NAE}^k : \{0,1\}^{3^k} \to \{0,1\}$ is defined:*

$$\mathrm{NAE}^k(x) := \mathrm{NAE}\big(\mathrm{NAE}^{k-1}(x_1, \ldots, x_{3^{k-1}}), \mathrm{NAE}^{k-1}(x_{3^{k-1}+1}, \ldots, x_{2\cdot 3^{k-1}}), \\ \mathrm{NAE}^{k-1}(x_{2\cdot 3^{k-1}+1}, \ldots, x_{3^k})\big)$$

and $\mathrm{NAE}^1 := \mathrm{NAE}$.

We denote by A_k the symmetric binary matrix $\mathrm{NAE}^k \circ \mathrm{AND}^{3^k}$. Moreover:

$$\forall b \in \{0,1\} : G_k^{(b)} := G\left(\{x \in \{0,1\}^{3^k} : \mathrm{NAE}^k(x) = b\}, \{\{x,y\} : \mathrm{NAE}^k(x \wedge y) = 1\}\right)$$

in other words, $G_k^{(1)}$ (resp. $G_k^{(0)}$) is the graph whose adjacency matrix is the principal minors of A_k, induced by those indices for which the main diagonal is 1 (resp. 0). With this notation, the result of Nisan and Wigderson says:[19]

Theorem 7.3 [50]. *Let $A_k := NAE^k \circ AND^{3^k} \in M_{3^k}(\mathbb{R})$. Then,*

$$\log \mathrm{rank}(A_k) \leqslant \mathcal{O}(2^k), \text{ and } D(A_k) = 3^k(1 - o_k(1))$$

It easy to bound $\mathrm{rank}(A_k)$: It is well known (e.g., [37]) that $\mathrm{rank}(f \circ \mathrm{AND}^n) = \mathrm{spar}(f)$ for any function $f : \{0,1\}^n \to \{0,1\}$. So here the rank is the number of monomials appearing in the expansion of NAE^k, which can be bounded by a simple inductive argument.

7.2 Finding large monochromatic principal minors

In search of large monochromatic principal minors of A_k, let us first estimate the orders of the two subgraphs, $G_k^{(0)}$ and $G_k^{(1)}$.

Proposition 7.4. *For every $k \geqslant 1$, we have that:*

$$v\left(G_k^{(0)}\right) = 2^{3^k} \cdot \left(\frac{1}{3} \pm o_k(1)\right), \text{ and } v\left(G_k^{(1)}\right) = 2^{3^k} \cdot \left(\frac{2}{3} \pm o_k(1)\right)$$

[19] One can even replace $\mathrm{D}(A_k)$ by $\mathrm{N}^0(A_k)$ (the 0-nondeterministic communication complexity of A_k) in the statement of the theorem, as the bound on $\mathrm{D}(A_k)$ follows from a reduction to the promise communication problem 'unique disjointness', for which Kaibel and Weltge [40] recently proved the bound $\chi^0(\mathrm{UDISJ}_n) \geqslant (3/2)^n$, where χ^0 is the 0-cover number.

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Proof. Let p_k be the probability that a uniformly random $x \sim \{0,1\}^{3^k}$ is in $V(G_k^{(1)})$. Then,

$$\begin{aligned} p_k &= \mathbb{P}_{x\sim\{0,1\}^{3^k}}\left[\left(\mathrm{NAE}^k \circ \mathrm{AND}^{3^k}\right)(x) = 1\right] \\ &= \mathbb{P}_{x_1,x_2,x_3\sim\{0,1\}^{3^{k-1}}}\left[\mathrm{NAE}\left(\mathrm{NAE}^{k-1}(x_1), \mathrm{NAE}^{k-1}(x_2), \mathrm{NAE}^{k-1}(x_3)\right) = 1\right] \\ &= 3p_{k-1}^2(1-p_{k-1}) + 3p_{k-1}(1-p_{k-1})^2 = 3p_{k-1}(1-p_{k-1}). \end{aligned}$$

By direct observation we have $p_1 = 3/4$. Furthermore, $x > 3x(1-x) > \frac{2}{3}$ for $\frac{3}{4} > x > \frac{2}{3}$. Thus, the sequence p_k is decreasing to its limit, the unique positive root of $x = 3x(1-x)$, namely $x = 2/3$. □

We now construct large monochromatic principal minors in $G_k^{(0)}$ and $G_k^{(1)}$.

Proposition 7.5. *The matrices A_k have large monochromatic principal minors. That is,*

$$\min\left\{\omega\left(G_k^{(1)}\right), \alpha\left(G_k^{(0)}\right)\right\} \geqslant 2^{3^k(1-o_k(1))}$$

Proof. For every positive k, let us denote

$$\alpha_k := \alpha\left(G_k^{(0)}\right), \text{ and } \omega_k := \omega\left(G_k^{(1)}\right)$$

We claim that $\alpha_k \geqslant \omega_{k-1}^3$ and that $\omega_k \geqslant 2^{3^{k-1}}\alpha_{k-1}\omega_{k-1}$, for any $k \geqslant 2$. Let $A, B \subset \{0,1\}^{3^{k-1}}$ be a largest clique of $G_{k-1}^{(1)}$ and largest anticlique of $G_{k-1}^{(0)}$, respectively. Observe that for every $x_1, x_2 \in A$, it holds that $\mathrm{NAE}^{k-1}(x_1 \wedge x_2) = 1$ (whether or not $x_1 = x_2$). Likewise it holds that $\mathrm{NAE}^{k-1}(y_1 \wedge y_2) = 0$ for every $y_1, y_2 \in B$. Therefore, the sets

$$B' := A \times A \times A \subset \{0,1\}^{3^k}, \text{ and } A' := \{0,1\}^{3^{k-1}} \times A \times B \subset \{0,1\}^{3^k}$$

are an anticlique of $G_k^{(0)}$, and a clique of $G_k^{(1)}$, respectively.

Combining the two bounds, it holds that $\omega_k \geqslant 2^{3^{k-1}}\alpha_{k-1}\omega_{k-1} \geqslant 2^{3^{k-1}}\omega_{k-1}\omega_{k-2}^3$. Taking logs and denoting $a_k := \log(\omega_k)$, we thus arrive at the linear recurrence $a_k = 3^{k-1} + a_{k-1} + 3a_{k-2}$. Denoting $a_k = 3^k(1-\varepsilon_k)$, the above translates into

$$3^k(1-\varepsilon_k) = 3^{k-1} + 3^{k-1}(1-\varepsilon_{k-1}) + 3^{k-1}(1-\varepsilon_{k-2})$$

i.e.,

$$\varepsilon_k = \frac{\varepsilon_{k-1} + \varepsilon_{k-2}}{3} \tag{2}$$

and we conclude that $a_k = 3^k(1-o_k(1))$, where the little-oh term is exponentially small in k. To see this consider $\lambda^2 - \frac{\lambda+1}{3}$, the characteristic polynomial of Equation 2, the roots of which are $\frac{1\pm\sqrt{13}}{6}$. □

8. Open problems

Many intriguing questions regarding the Rank-Ramsey problem remain unanswered. For starters, the growth rate of the function $\nu_d(n)$ is mostly unknown. For bounded d, we have shown (see Section 4) a polynomial separation between $\nu_d(n)$ and n, starting at $d = 41$. Is this true of all d? Do there exist triangle-free Rank-Ramsey graphs? Concretely,

Open Problem 8.1. *Is there a constant $c > 0$ such that $\nu_2(n) = \mathcal{O}(n^{1-c})$?*

The paper [42] advocates the perspective that rank is a complexity measure of sign matrices and mentions some additional measures of similar nature like γ_2, margin complexity and more.

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Rather than demand that the complement rank be small, one can similarly investigate graphs with low clique number for which $\gamma_2(\overline{G})$ is also small etc.[20]

We saw several connections between Rank-Ramsey numbers and various graph parameters. Our list is far from exhausting all possible interesting ties. Could there be any relation with the Colin de Verdière parameter [20])? With other orthonormal representations of graphs (e.g., see [48])?

In Section 6, we briefly consider minimum semidefinite rank, and show n-vertex triangle-free graphs G and H, with both $(\mathrm{msr}(G) - \mathrm{rank}(A_G + I)) = \Omega(n)$ and with $(\mathrm{rank}(A_H + I) - \mathrm{msr}(H)) = \Omega(n)$. The relation between the two is therefore likely nuanced. We find it interesting to understand what is the least distance, under the rank metric, between $A_G + I$ and a PSD representation matrix M for G. Another quantity which we consider in Section 6, originating in orthonormal representations, is the Lovász number. We give some evidence that the Lovász number bounds the complement rank from below, perhaps up to a multiplicative constant. Thus, we ask:

Open Problem 8.2. *What is the relation between* $\vartheta(G)$ *and* $\mathrm{rank}(A_G + I)$ *for a simple graph* G?

The regime of unbounded d, which we explore in Section 5 is also of great interest. Recall that we construct a graph of logarithmic clique number, and polynomial rank. This is in stark contrast to the classical Ramsey problem, where in almost all graphs both the clique number *and* independence number are logarithmic. Naturally, we think that maintaining low-rank should be much harder than controlling the independence number. It is therefore quite natural to ask,

Open Problem 8.3. *What is the growth rate of* $v_d(n)$, *as a function of* d *and* n (as $d = d(n) \to \infty$)?

In Subsection 2.2 we begin an exploration of the Rank-Ramsey numbers. Unlike the usual Ramsey numbers, which are symmetric (i.e., $R(s,t) = R(t,s)$ for every s and t), this does not hold for our numbers. In fact, we determine the numbers $R^k(s,t)$ for every $2 \leqslant t \leqslant 5$, and prove that $R^k(3,n) > R^k(n,3)$ for sufficiently large n. It would be interesting to better understand the interplay between the clique number and complement rank.

Another perspective of Rank-Ramsey numbers, stems from *twin-free graphs*. Recall that two vertices in a graph are called *twins* if they are non-adjacent and have the same set of neighbours. Pruning twins from a graph clearly affects neither the rank and nor chromatic number. Therefore, the sets $\mathcal{G}_r$ of all twin-free connected graphs of rank r, play a crucial role in understanding the log-rank conjecture. Indeed, the log-rank problem (in particular, its graph-theoretic formulation, see Section 3) can be *equivalently* re-formulated as follows:

Equivalent formulation of Log-Rank problem
Is there a constant $c > 0$ such that $\log \chi(G) \leqslant \mathcal{O}\left(\log^c r\right)$ for every $G \in \mathcal{G}_r$ and every $r > 1$?
It would therefore be interesting to investigate the sets $\mathcal{G}_r$, for a larger range of values r.

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A. The construction of Codenotti, Pudlák and Resta

Codenotti, Pudlák and Resta constructed [15] an interesting family of explicit triangle-free Ramsey graphs. The construction is elementary.

Theorem A.1 [15]. *For every sufficiently large n, there exists an n-vertex triangle-free graph* H_n *such that* $rank(A_{H_n} - 2I) = \mathcal{O}\left(n^{3/4}\right)$.

The construction is as follows: Let $G = (L \sqcup R, E)$ be a k-vertex bipartite graph with girth at least 8. Let $D = (E, \hat{E})$ be the directed graph whose vertices are the (undirected) edges of G, and

$$\forall (l, r), (a, b) \in E : (l, r) \to_D (a, b) \iff (l \sim_G b) \wedge (a \neq l) \wedge (b \neq r)$$

It is not hard to see that:

1. $\mathrm{rank}(A_D - I) \leqslant k$.
2. D contains no transitive triangle.
3. There are no back-and-forth edges in D.

The undirected graph H whose adjacency matrix is $A_H = A_D + A_D^T$ (i.e., omitting orientations) is therefore triangle-free, and has $\mathrm{rank}(A_H - 2I) \leqslant 2\mathrm{rank}(A_D - I) \leqslant 2k$. It remains to fix the base graph G to be bipartite, with large size relative to order, and of girth at least 8. For this, the graphs H_3 of Wenger [57] suffice: they have order k and size $n = \mathcal{O}\left(k^{4/3}\right)$.

As usual, $\alpha(H) \leqslant \mathrm{rank}(A_H + cI)$ for any $c \neq 0$, so this Theorem yields Ramsey graphs. While these graphs have *very high* multiplicity of the eigenvalue 2 (or -2, by the clique-tensoring trick of Lemma 4.4), they are not Rank-Ramsey, as $\mathrm{rank}(A + I) \geqslant n - \mathrm{rank}(A - 2I) = (1 - o(1))n$. Crucially, we remark that the matrix $A_H - 2I$ is not binary.

B. Hoffman, Cvetković, Lovász and (Rank-)Ramsey

Any Rank-Ramsey graph is also a Ramsey graph, since $\mathrm{rank}(A_G + I) \geqslant \alpha(G)$ holds for every graph G. How does this bound compare to other known upper bounds on the independence number of a graph? Two of the best-known bounds are Hoffman's ratio bound, and Cvetković's inertia bound, both of which relate to the *spectra* of *symmetric weighted adjacency matrices* of a graph G.

Definition B.1. *Let* G *be a graph and let* $w{:}E(G) \to \mathbb{R}$ *be a function. The weighted adjacency of* G *is the matrix:*

$$A_{G,w} := \left(w(\{x, y\}) \cdot \mathbb{1}\{x \sim y\}\right)_{x,y \in V(G)}$$

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Theorem B.2. *(Hoffman Bound) Let G be an n-vertex graph and let $w : E(G) \to \mathbb{R}$ be a weight function, such that $A_{G,w}$ has constant row sums. Then,*

$$\alpha(G) \leqslant h(A_{G,w}) := \left| \frac{\lambda_{min}(A_{G,w})}{\lambda_{max}(A_{G,w}) - \lambda_{min}(A_{G,w})} \right| \cdot n$$

Theorem B.3. *(Cvetković Bound [10, 16]) Let G be a graph and let $w : E(G) \to \mathbb{R}$ be a weight function. Then,*

$$\alpha(G) \leqslant c(A_{G,w}) := \left| \left\{ \lambda \in \mathrm{spec}(A_{G,w}) : \lambda \geqslant 0 \right\} \right|$$

For a graph G, let $h(G) := \inf_w h(A_{G,w})$ and $c(G) := \inf_w c(A_{G,w})$ be the best-possible bounds attained by Theorems B.2 and B.3, respectively. The Hoffman and Cvetković bounds had been at the centre of several longstanding problems in algebraic graph theory. Sinkovic [53] constructed the first example of a graph for which $c(G)$ is not tight, resolving decade-long an open question of Godsil [27]. The comparability of Hoffman and Cvetković had similarly been undetermined until very recently: Kwan and Wigderson [41] constructed a family of graphs for which $h(G) = o(c(G))$, and a week later, Ihringer [33] gave an example of a family for which $c(G) = o(h(G))$. Both of these separations are *polynomial*.

In this paper we consider two other bounds on the independence number $\alpha(G)$: $\mathrm{rank}(A_G + I)$, by means of Rank-Ramsey graphs, and the Lovász theta $\vartheta(G)$, as discussed in Section 6. These two bounds are intimately related to Hoffman and Cvetković. Clearly $\mathrm{rank}(A_G + I) = n - \mu_{A_G}(-1) \geqslant c(G)$. Moreover, it is well-known that $\vartheta(G) \geqslant h(G)$, for every graph G. Therefore, we may think of $\mathrm{rank}(A_G + I)$ as a less-tight analogue of Cvetković's bound, and of $\vartheta(G)$ is an analogue of Hoffman's bound, and as such the task of comparing $\mathrm{rank}(A_G + I)$ and $\vartheta(G)$ related to the well-known problem of comparing the ratio bound versus the inertia bound.

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