# 10

# Interlacing and Spectral Classes

*We derive various interlacing results under particular manipulations of graphs and simplicial complexes, such as contraction, duplication, or graph doubling. We also define asymptotic spectral classes using the Wasserstein distance.*

## 10.1 Interlacing

Interlacing theorems provide inequalities between the eigenvalues of graphs that are related by some geometric operations, such as adding or deleting an edge, or more generally, some subgraph.

Interlacing theorems originally arose in an algebraic, rather than a geometric context, concerned with the eigenvalues of two linear operators $A, B$ (which are assumed symmetric to make their eigenvalues real) that are related by some operation. An interlacing theorem then states that for some integers $n_1, n_2$, the eigenvalues $\alpha_k$ of $A$ and the eigenvalues $\beta_k$ of $B$, both arranged in increasing order, satisfy an inequality of the form

$$\alpha_{k-n_1} \leq \beta_k \leq \alpha_{k+n_2} \quad \text{for all } k. \tag{10.1.1}$$

The original example, the *Cauchy interlacing theorem*, controls the eigenvalues of a symmetric matrix $B$ obtained by deleting some rows and columns of a symmetric $A$ in terms of those of $A$. Let us recall the result and its proof, as they will also guide our subsequent constructions.

**Theorem 10.1.1.** *Let $A$ be a symmetric $(N \times N)$-matrix with eigenvalues $\alpha_1 \leq \ldots \leq \alpha_N$, and let $P\colon \mathbb{R}^N \to \mathbb{R}^M$ be an orthogonal projection. $B := PAP^\star$ then is a symmetric $(M \times M)$-matrix, and its eigenvalues $\beta_1 \leq \ldots \leq \beta_M$ satisfy*

$$\alpha_k \leq \beta_k \leq \alpha_{k+N-M}. \tag{10.1.2}$$



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*Proof* We shall apply Theorem 2.6.1. Thus, take $H = \mathbb{R}^N$ with the Euclidean scalar product, let $b_i$ be an eigenvector of $B$ for the eigenvalue $\beta_i$, and let $S_k$ be spanned by $b_1, \ldots, b_k$. Then, by (2.6.2), assuming $g \neq 0$ throughout,

$$\begin{aligned}\beta_k &= \max_{g \in S_k} \frac{(Bg,g)}{(g,g)} \\ &= \max_{g \in S_k} \frac{(PAP^\star g,g)}{(g,g)} \\ &\geq \min_{H_k \in \mathcal{H}_k} \max_{g \in H_k} \frac{(AP^\star g,P^\star g)}{(g,g)} \\ &= \alpha_k . \end{aligned} \tag{10.1.3}$$

For the upper bound, let $V_{M-k+1}$ be spanned by $b_k, \ldots, b_M$. By (2.6.2) again,

$$\begin{aligned}\beta_k &= \min_{g \in V_{M-k+1}} \frac{(Bg,g)}{(g,g)} \\ &= \min_{g \in V_{M-k+1}} \frac{(PAP^\star g,g)}{(g,g)} \\ &= \min_{g \in V_{M-k+1}} \frac{(AP^\star g,P^\star g)}{(g,g)} \\ &\leq \max_{H_{M-k+1} \in \mathcal{H}_{M-k+1}} \min_{g \in H_{M-k+1}} \frac{(AP^\star g,P^\star g)}{(g,g)} \\ &= \max_{H_{M-k+1} \in \mathcal{H}_{M-k+1}} \min_{g \in H_{M-k+1}} \frac{(AP^\star g,P^\star g)}{(P^\star g,P^\star g)} \frac{(P^\star g,P^\star g)}{(g,g)} \\ &\leq \alpha_{k+N-M}, \end{aligned} \tag{10.1.4}$$

since

$$\frac{(P^\star g,P^\star g)}{(g,g)} \leq 1. \tag{10.1.5}$$

□

In fact, one even has equality in the last line in (10.1.4), but the interpolation of $\frac{(P^\star g,P^\star g)}{(P^\star g,P^\star g)}$ and the use of (10.1.5) will be important for our subsequent reasoning.

We may, of course, directly apply Theorem 10.1.1 to the adjacency matrix of a graph and derive interpolation theorems for algebraic Laplacians. But, we want to develop a general scheme for interlacing inequalities for Laplacians of graphs and simplicial complexes. For that purpose, we shall systematically expand the scheme of the previous proof, following [137].

We can then systematically study the effects of various operations or modifications of simplicial complexes on Laplacian spectra, like the deletion, collapsing, and contracting of subcomplexes, simplicial maps, and coverings.

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In the remainder of this chapter, we shall mainly derive interlacing equalities for up-Laplacians $L_{up}$ of simplicial complexes. The same techniques can of course also yield such results for down-Laplacians. But we know from (4.8.6) that the nonzero spectrum of $L^{q+1}_{down}$ equals that of $L^q_{up}$, and therefore, it suffices to treat the latter. Anyway, in the most important case, that of graphs, the vertex Laplacian is an up-Laplacian.

### 10.1.1 Deletion of Subcomplexes

The main result of this section is

**Theorem 10.1.2.** *Let $(\Sigma, w_\Sigma)$ and $(\Sigma', w_{\Sigma'})$ be $(q+1)$-dimensional, weighted simplicial complexes for which their proper difference (Definition 2.5.11) $(\Theta, w_\Theta)$ exists. Let $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_N$ and $\theta_1 \le \theta_2 \le \ldots \le \theta_N$ be the eigenvalues of the Laplace operators $L^q_{up}(\Sigma)$ and $L^q_{up}(\Theta)$, respectively. Then, we have*

$$\theta_k \le \lambda_{k+D_{\Sigma'}} \tag{10.1.6}$$

*for $k = 1,2,\ldots,N - D_{\Sigma'}$, and for $k = 1,2,\ldots,N$*

$$\lambda_{k-D_W} \le \theta_k, \tag{10.1.7}$$

*where $D_W = \dim C^{q+1}(\Sigma') - \dim H^{q+1}(\Sigma')$, $D_{\Sigma'} = \dim C^q(\Sigma')$, and where we put $\lambda_j = 0$ whenever $j \le 0$.*

In fact, we also get an upper bound (10.1.6) for all $k$, if we define $\lambda_j$ for $j > N$ appropriately as the general upper bound for the eigenvalues. For the normalized Laplacian, we have

$$\lambda_j = N + 2 \quad \text{for } j > N, \tag{10.1.8}$$

while for the combinatorial Laplacian,

$$\lambda_j = |V| \text{ (the cardinality of the vertex set)} \quad \text{for } j > N. \tag{10.1.9}$$

According to our general convention, we shall denote the set of $q$-simplices of a simplicial complex $\Lambda$ by $\Lambda_q$.

Before entering the proof, we need some technical constructions, using the setting of Section 2.5.2.

For a weighted simplicial complex $(\Sigma, w_\Sigma)$, a subcomplex $(\Sigma', w_{\Sigma'})$, and their proper difference $(\Theta, w_\Theta)$, we let $i_q : C^q(\Theta) \to C^q(\Sigma)$ be the inclusion map (i.e., $(i_q g)[F] = g([F])$, for every $F \in S_q(\Sigma)$, $g \in C^q(\Theta)$) and let $\pi_{q+1} : C^{q+1}(\Sigma) \to C^{q+1}(\Theta)$ be the projection map (i.e., $(\pi_{q+1}\bar{f})[\bar{G}] = \bar{f}([\bar{G}])$,

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for all $\bar{G} \in S_{q+1}(\Theta)$ and $\bar{f} \in C^{q+1}(\Sigma)$). Their (formal, because the scalar product may be degenerate) adjoints $i_q^* \colon C^q(\Sigma) \to C^q(\Theta)$ and $\pi_{q+1}^* \colon C^{q+1}(\Theta) \to C^{q+1}(\Sigma)$ of the inclusion and projection maps satisfy

$$i_q^* f([G]) = \begin{cases} \frac{w_\Sigma(G)}{w_\Theta(G)} f(i_q[G]) = \frac{w_\Sigma(G)}{w_\Theta(G)} f(i_q[G]) & \text{if } w_\Theta(G) > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{10.1.10}$$

and

$$\pi_{q+1}^* \bar{g}([\bar{F}]) = \frac{w_\Theta(\bar{F})}{w_\Sigma(\bar{F})} \bar{g}(\pi_{q+1}[\bar{F}]) = \frac{w_\Theta(\bar{F})}{w_\Sigma(\bar{F})} \bar{g}([\bar{F}]). \tag{10.1.11}$$

**Lemma 10.1.1.** *For* $0 \le q < \dim \Sigma$,

$$\delta_\Theta^q = \pi_{q+1} \delta_\Sigma^q i_q \tag{10.1.12}$$

$$(\delta_\Theta^q)^\star = i_q^\star (\delta_\Sigma^q)^\star \pi_{q+1}^\star. \tag{10.1.13}$$

*Moreover,*

$$i_q^* i_q g([G]) = \begin{cases} \frac{w_\Sigma(G)}{w_\Theta(G)} g([G]) & \textit{for } w_\Theta(G) > 0, \\ 0 & \textit{otherwise} \end{cases} \tag{10.1.14}$$

*and*

$$\pi_{q+1}^* \pi_{q+1} \bar{f}([\bar{F}]) = \frac{w_\Theta(\bar{F})}{w_\Sigma(\bar{F})} \bar{f}([\bar{F}]). \tag{10.1.15}$$

(In fact, $i_q^*$ is not quite the adjoint of $i_q$, but what we need is (10.1.14) whose factor interpolates between the scalar products on $C^q(\Sigma)$ and $C^q(\Theta)$ on all $q$-simplices that have a positive weight in $\Theta$.)

*Proof* (10.1.12) and (10.1.13) follow from the commutativity of the diagrams

$$\begin{array}{ccc} C^{q+1}(\Sigma) & \xleftarrow{\delta_\Sigma} & C^q(\Sigma) \\ \downarrow{\scriptstyle \pi_{q+1}} & & \uparrow{\scriptstyle i_q} \\ C^{q+1}(\Theta) & \xleftarrow{\delta_\Theta} & C^q(\Theta) \end{array} \tag{10.1.16}$$

$$\begin{array}{ccc} C^{q+1}(\Sigma) & \xrightarrow{\delta_\Sigma^*} & C^q(\Sigma) \\ \uparrow{\scriptstyle \pi_{q+1}^*} & & \downarrow{\scriptstyle i_q^*} \\ C^{q+1}(\Theta) & \xrightarrow{\delta_\Theta^*} & C^q(\Theta) \end{array} . \tag{10.1.17}$$

Of course, the second is a consequence of the first, since $\pi_{q+1}^*$ is the adjoint of $\pi_{q+1}$, and $i_q^*$ satisfies this role for $i_q$ on the image of $\delta_\Sigma^*$.

(10.1.15) and (10.1.14) follow from (10.1.11) and (10.1.10). □

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We shall now use the diagrams (10.1.16) and (10.1.17) to factorize the Rayleigh quotient of $L^q_{up}(\Theta)$. For simplicity of notation, we omit the indices $q$ of $i$ and $L$ and $q+1$ of $\pi$. We have, recalling (2.6.6) and the convention that $R(g) = 0$ if $g = 0$,

$$\begin{aligned} R_{L_{up}(\Theta)}(g) = \frac{(\delta^*_\Theta \delta_\Theta g, g)}{(g,g)} &= \frac{(i^*\delta^*_\Sigma \pi^* \pi \delta_\Sigma i g, g)}{(g,g)} \\ &= \frac{(\pi^*\pi\delta_\Sigma i g, \delta_\Sigma i g)}{(\delta_\Sigma i g, \delta_\Sigma i g)} \frac{(\delta_\Sigma i g, \delta_\Sigma i g)}{(ig, ig)} \frac{(ig, ig)}{(g,g)} \\ &= R_{\pi^*\pi}(\delta_\Sigma i g) R_{L_{up}(\Sigma)}(ig) R_{i^*i}(g). \end{aligned} \tag{10.1.18}$$

We now look at the three Rayleigh quotients in (10.1.18) in turn.

We start with the operator $\pi^*\pi$.

**Lemma 10.1.2.** *The eigenvalues of the operator* $\pi^*\pi \colon C^{q+1}(\Sigma) \to C^{q+1}(\Sigma)$ *are*

$$\left\{ \frac{w_\Theta(\bar F)}{w_\Sigma(\bar F)} \mid \bar F \in \Sigma_{q+1} \right\},$$

*hence* $R_{\pi^*\pi}(f) \le 1$. *Moreover,*

$$R_{\pi^*\pi}(\delta_\Sigma i g) = 1 \text{ if and only if } g_{|C^q(\Sigma')} \in \ker \delta_{\Sigma'}. \tag{10.1.19}$$

*These are the functions* $g \in C^n(\Theta)$ *that are orthogonal to* $W := \operatorname{coker} \delta_{\Sigma'}$, *and*

$$D_W := \dim W = \dim C^{q+1}(\Sigma') - \dim H^{q+1}(\Sigma'). \tag{10.1.20}$$

*Proof* The first part follows from (10.1.18).

By (10.1.15) $\bar f := \delta_\Sigma i g$ satisfies

$$R_{\pi^*\pi}(\bar f) = \frac{\sum_{\bar F \in \Theta_{q+1}} w_\Theta(\bar F) \bar f([\bar F])^2}{\sum_{\bar F \in \Sigma_{q+1}} w_\Sigma(\bar F) \bar f([\bar F])^2}. \tag{10.1.21}$$

This is equal to 1 precisely if $w_{\Sigma'}(\bar F)\bar f([\bar F])^2 = (w_\Sigma(\bar F) - w_\Theta(\bar F))\bar f([\bar F])^2 = 0$ for every $\bar F \in \Sigma_{q+1}$, that is, if $\bar f$ is identically equal to zero on $\Sigma'$. This means that the restriction of $g \in C^q(\Theta)$ on $C^q(\Sigma')$ is in the kernel of $\delta_{\Sigma'}$. Therefore, the function $g$ is orthogonal to $W := \operatorname{coker} \delta_{\Sigma'}$. Now

$$\dim \operatorname{coker} \delta^q = \dim \operatorname{im} \delta^q = \dim \ker \delta^{q+1} - \dim H^{q+1}, \tag{10.1.22}$$

and since $\Sigma'$ and $K$ are $(q+1)$-dimensional, $\ker \delta^{q+1} = C^{q+1}(\Sigma')$ and

$$\dim \operatorname{im} \delta_{\Sigma'} = \dim C^{n+1}(\Sigma') - \dim H^{n+1}(\Sigma') \tag{10.1.23}$$

and we get (10.1.20). □

We now turn to the eigenvalues of $i^*i$.

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**Lemma 10.1.3.** *Let $g \in C^q(\Theta)$. Then,*

*(i)* $(g,g) > 0 \Rightarrow R_{i^*i}(g) \geq 1$.
*(ii)* $R_{i^*i}(g) = 1$ *if and only if* $g \perp C^q(\Sigma')$.

*Proof* From (10.1.14),

$$\frac{\sum_{G\in\Theta_q} w_K(G) g([G])^2}{\sum_{G\in\Theta_q} w_\Theta(G) g([G])^2}. \tag{10.1.24}$$

Since $w_K(G) \geq w_\Theta(G)$, $(i)$ holds. By (10.1.24), $R_{i^*i}(g) = 1$ if and only if $g([G]) = 0$ for every $G \in \Sigma'_q$, yielding $(ii)$. □

*Proof of Theorem 10.1.2* We shall systematically use the fact that maximization (minimization) over a smaller set increases the maximum (decreases the minimum).

Let $\mathcal{H}_k$ be the set of $k$-dimensional subspaces of $C^q(\Sigma)$ and with $N = \dim C^q(\Sigma)$, let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ and $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_N$ be the eigenvalues of the operators $L^q_{up}(\Sigma)$ and $L^q_{up}(\Theta)$, respectively. Then, recalling $W := \operatorname{coker} \delta_{\Sigma'}$ with $D_W = \dim W = \dim C^{q+1}(\Sigma') - \dim H^{q+1}(\Sigma')$ from (10.1.20),

$$\begin{aligned}
\theta_k &= \min_{H_k\in\mathcal{H}_k} \max_{g\in H_k} R_{L^q_{up}(\Theta)}(g) \\
&= \min_{H_k\in\mathcal{H}_k} \max_{g\in H_k} R_{\pi^*\pi}(\delta_\Sigma ig) R_{L^q_{up}(\Sigma)}(ig) R_{i^*i}(g) \quad \text{by (10.1.18)} \\
&\geq \min_{H_k\in\mathcal{H}_k} \max_{g\in H_k} R_{\pi^*\pi}(\delta_\Sigma ig) R_{L^{up}_q(\Sigma)}(ig) \quad \text{by Lemma 10.1.3} \\
&\geq \min_{\mathcal{H}_k} \max_{g\in H_k, g\perp W} R_{L^q_{up}(\Sigma)}(ig) \quad \text{by Lemma 10.1.2} \\
&\geq \min_{\mathcal{H}_k} \max_{g\in H_{k-D_W}} R_{L^q_{up}(\Sigma)}(ig) \\
&= \min_{\mathcal{H}_{k-D_W}} \max_{g\in H_{k-D_W}} R_{L^q_{up}(\Sigma)}(ig) \\
&= \lambda_{k-D_W}.
\end{aligned}$$

This proves (10.1.7).
For (10.1.6), we use (10.1.18) and Theorem 2.6.1 again, getting

$$\begin{aligned}
\theta_k &= \max_{\mathcal{H}_{N-k+1}} \min_{f\in H_{N-k+1}} R_{L^q_{up}(\Theta)}(f) \\
&= \max_{\mathcal{H}_{N-k+1}} \min_{f\in H_{N-k+1}} R_{\pi^*\pi}(\delta_\Sigma if) R_{L^q_{up}(\Sigma)}(if) R_{i^*i}(f) \\
&\leq \max_{\mathcal{H}_{N-k+1}} \min_{g\in H_{N-k+1}} R_{\Delta_\Sigma}(ig) R_{i^*i}(g) \quad \text{by Lemma 10.1.2} \\
&\leq \max_{\mathcal{H}_{N-k+1}} \min_{g\in H_{N-k+1}, g\perp C^q(\Sigma')} R_{\Delta_\Sigma}(ig) \quad \text{by Lemma 10.1.3}
\end{aligned}$$

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$$\leq \max_{\mathcal{H}_{N-k+1}} \min_{g \in H_{N-k+1-D_{\Sigma'}}} R_{\Delta_\Sigma}(ig) \tag{10.1.25}$$

$$= \max_{\mathcal{H}_{N-k+1-D_{\Sigma'}}} \min_{g \in H_{N-k+1-D_{\Sigma'}}} R_{\Delta_\Sigma}(ig)$$

$$= \lambda_{k+D_{\Sigma'}}. \tag{10.1.26}$$

□

Since we are particularly interested in the normalized graph Laplacian, we formulate the version of Theorem 10.1.2 for graphs explicitly.

**Corollary 10.1.1.** *Let $\Gamma'$ be a weighted subgraph of the weighted graph $\Gamma$ with $N$ vertices and eigenvalues $\lambda_1 \leq \ldots \leq \lambda_N$ of the normalized Laplacian, and let $\Theta$ be their proper difference, with eigenvalues $\theta_1 \leq \ldots \leq \theta_N$ of its normalized Laplacian. Assume that the positive weight part of $\Theta$ is a (possibly disconnected) graph. Suppose that $\Gamma'$ has $N'$ vertices and $M'$ connected components. Then,*

$$\lambda_{k-N'+M'} \leq \theta_k \leq \lambda_{k+N'} \quad \text{for } 1 \leq k \leq N, \tag{10.1.27}$$

*if we put $\lambda_j = 0$ for $j \leq 0$ and $\lambda_j = 2$ for $j \geq N+1$.*

*Proof* Using the Euler formula $\sum_{i=0,1}(-1)^i \dim C^i(G) = \sum_i (-1)^i \dim H^i(G)$ for every graph, we get $D_W = \dim C^1(\Gamma') - \dim H^1(\Gamma') = \dim C^0(\Gamma') - \dim H^0(\Gamma') = N' - M'$, and we also have $D_{\Gamma'} = \dim C^0(\Gamma') = N'$. The result thus follows from Theorem 10.1.2, recalling (10.1.8). □

The result for the combinatorial Laplacian is almost the same, except that the largest eigenvalue is now bounded by $N$ instead of 2.

**Corollary 10.1.2.** *Let $\Gamma'$ be a weighted subgraph of the weighted graph $\Gamma$ with $N$ vertices and eigenvalues $\lambda_1 \leq \ldots \leq \lambda_N$ of the combinatorial Laplacian, and let $\Theta$ be their proper difference, with eigenvalues $\theta_1 \leq \ldots \leq \theta_N$ of its normalized Laplacian. Assume that the positive weight part of $\Theta$ is a (possibly disconnected) graph. Suppose that $\Gamma'$ has $N'$ vertices and $M'$ connected components. Then,*

$$\lambda_{k-N'+M'} \leq \theta_k \leq \lambda_{k+N'} \quad \text{for } 1 \leq k \leq N, \tag{10.1.28}$$

*if we put $\lambda_j = 0$ for $j \leq 0$ and $\lambda_j = N$ for $j \geq N+1$.* □

**Example:** Suppose that $\Gamma' = (v, v')$ is a single edge, where $v$ is a pending vertex of $\Gamma$, that is, its only neighbor is $v'$. Assume $w_{\Gamma'}(v, v') = w_\Gamma(v, v')$. Then, the proper difference $\Theta$, obtained by removing the edge $(v, v')$, has $v$ as an isolated vertex, and $\deg_\Theta v' = \deg_\Gamma v' - 1$. Then,

$$\theta_k \geq \lambda_{k-1} \quad \text{for all } k. \tag{10.1.29}$$

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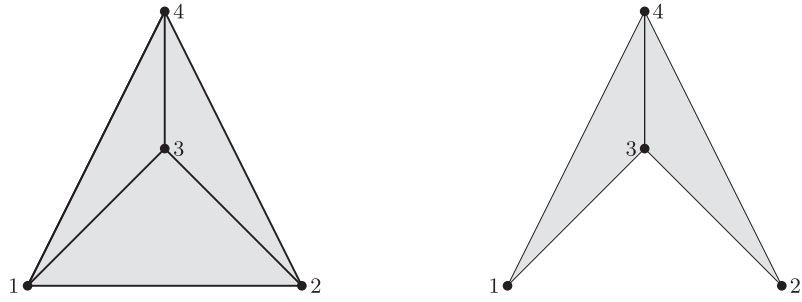



Figure 10.1 A collapse.

In particular, $\theta_2 = \theta_1 = 0$, but $\theta_3 \geq \lambda_2$.

We now present another operation, which, however, can also be interpreted as the deletion of a subcomplex.

**Definition 10.1.1.** Let $\Sigma$ be a simplicial complex and $(\bar{F}, F)$ a pair of simplices with $F \in \partial\bar{F}$. The face $\bar{F}$ is called a *free face*, if it is not a facet of any other simplex in $\Sigma$. A simplicial complex $\Sigma'$ obtained from $\Sigma$ by deleting $\bar{F}$ and $F$ and the simplicial map $\varphi\colon \Sigma \to \Sigma'$ that represents such an operation yields an *elementary collapse* of $\Sigma$. A sequence of elementary collapses is called a *collapse*.

In Figure 10.1, the edge $[1,2]$ and the simplex $[1,2,3]$ are collapsed.

An elementary collapse can also be seen as the deletion of a subcomplex, according to Definition 2.5.11, at least when the weights are properly chosen. Therefore, interlacing inequalities for elementary collapses follow from Theorem 10.1.2 when the Laplacians are either combinatorial (all weights $\equiv 1$) or normalized (the weight of a $q$-simplex is the sum of the weights of the $(q+1)$-simplices of which it is a facet).

**Theorem 10.1.3.** *Let $\varphi\colon \Sigma \to \Sigma'$ be an elementary collapse, and let $\lambda_1 \leq \ldots \leq \lambda_{N_\Sigma}$ and $\theta_1 \leq \ldots \theta_{N_{\Sigma'}}$ be the eigenvalues of the combinatorial or the normalized Laplacians on $\Sigma$ and $\Sigma'$. Then,*

$$\lambda_k \leq \theta_k \leq \lambda_{k+q+3}, \tag{10.1.30}$$

*where $N = \lambda_{N_\Sigma+1} = \ldots = \lambda_{N_\Sigma+q+3}$, and $N$ is the number of vertices of $\Sigma$.*

□

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### 10.1.2 Simplicial Maps

**Definition 10.1.2.** A *simplicial map* $\varphi\colon \Sigma \to \Sigma'$ is a map from the vertices of the simplicial complex $\Sigma$ to those of $\Sigma'$ with the property that $\varphi(v_1),\dots,\varphi(v_k)$ span a simplex whenever $v_1,\dots,v_k$ do.

We consider such a simplicial map $\varphi\colon \Sigma \to \Sigma'$ and assume that we have a weight function $w_\Sigma$ on $\Sigma$, and we then want to use $\varphi$ to construct a weight function on $\Sigma'$ so that the Laplacians on $\Sigma$ and $\Sigma'$ are naturally related, and we can prove interlacing inequalities for their eigenvalues. In fact, there are two natural possibilities to define $w_{\Sigma'}$. The first option is

$$w_{\Sigma'}(G) = \sum_{\substack{F\in\Sigma_q:\\ \varphi(F)=G}} w_\Sigma(F), \text{ for every } G \in \Sigma'_q. \tag{10.1.31}$$

The second option is

$$w_{\Sigma'}(G) = \sum_{\substack{F\in\Sigma_q:\\ \varphi(F)=G}} w_\Sigma(F) - \sum_{\bar F\in\Sigma_{q+1}:\, \varphi(\bar F)=G} w_\Sigma(\bar F), \text{ for } G \in \Sigma'_q. \tag{10.1.32}$$

In fact, it suffices for our purposes to consider $q$- and $(q+1)$-simplices in $\Sigma$ that are mapped to a $q$-simplex in $\Sigma'$ because this suffices for determining the eigenvalues of $L^q_{up}$, which is the operator we want to study here.

The weight function $w_{\Sigma'}$ in (10.1.32) is then motivated by the following consideration. Let $(\Sigma, w_\Sigma)$ be a complex with normalized weights, that is, $w_\Sigma(F) = \sum_{\bar F:\, F\in\bar F} w_\Sigma(\bar F)$, and let $\varphi\colon \Sigma \to \Sigma'$ be a simplicial map. We consider the situation where $\bar F_1,\dots,\bar F_k$ are $(q+1)$-faces of $\Sigma$, but their $\varphi$-images are only $q$-dimensional, that is, $\varphi(\bar F_i) = G_i \in \Sigma'_q$. $\Sigma'$ then is a complex in which $\varphi(\bar F_i)$ are loops; thus,

$$\deg G = \sum_{\substack{\bar G\in\Sigma'_{q+1}:\\ G\in\partial\bar G}} w_{\Sigma'}(\bar G) + \sum_{\substack{F_i\in\bar\Sigma_{q+1}:\\ G\in\partial\varphi(\bar F_i)}} w_\Sigma(\bar F_i).$$

If now (10.1.31) holds, then $\deg G = \sum_{\substack{\bar G\in\Sigma'_{q+1}:\\ G\in\partial\bar G}} w_{\Sigma'}(\bar G) + \sum_{\substack{F_i\in\bar\Sigma_{q+1}:\\ G\in\partial\varphi(\bar F_i)}} w_\Sigma(\bar F_i)$, and

$$\begin{aligned} w_{\Sigma'}(G_i) &= \sum_{\substack{F\in\Sigma_q:\\ \varphi(F)=G}} w_\Sigma(F) \\ &= \sum_{\substack{\bar F\in\Sigma_{q+1},\,\bar F\neq\bar F_i:\\ F\in\partial\bar F}} w_\Sigma(\bar F) + 2w_\Sigma(\bar F_i) \end{aligned}$$

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$$= \sum_{\substack{\bar{F}\in\Sigma_{q+1}:\\ F\in\partial\bar{F}}} w_\Sigma(\bar{F}) + w_\Sigma(\bar{F}_i)$$
$$= \deg G_i + w_\Sigma(\bar{F}_i).$$

Therefore the weight function $w_{\Sigma'}$ is not normalized. With definition (10.1.32) for $w_{\Sigma'}$, however, we get $w_{\Sigma'}(G) = \deg G$, and $w_{\Sigma'}$ is normalized. Consequently, with this choice of weights, if $L^q_{up}(\Sigma, w_\Sigma)$ is the normalized Laplacian of $\Sigma$, then $L^q_{up}(\Sigma', w_{\Sigma'})$ is the normalized Laplacian of $\Sigma'$.

Having defined the weights, we obtain the adjoints $(\varphi^*)^q$ of the cochain maps $\varphi^q$ defined in Section 2.5.1.

**Lemma 10.1.4.** *When $\Sigma'$ is equipped with the weight function (10.1.31) or (10.1.32), the diagram*

$$\begin{array}{ccc} C^q(\Sigma) & \xleftarrow{L_{up}(\Sigma)} & C^q(\Sigma) \\ \downarrow{\scriptstyle \varphi^*} & & \uparrow{\scriptstyle \varphi} \\ C^q(\Sigma') & \xleftarrow{L_{up}(\Sigma')} & C^q(\Sigma') \end{array} \tag{10.1.33}$$

*commutes. In particular, for $G \in \Sigma'_q$, we have the relation*

$$L^q_{up}(\Sigma')e_{[G]} = \varphi^* L^q_{up}(\Sigma)\varphi e_{[G]} = \sum_{\substack{F\in\Sigma_q:\\ \varphi(F)=G}} \sum_{\substack{F'\in\Sigma_q:\\ F,\Phi\in\partial\bar{F}}} \operatorname{sgn}([F],[G]) \operatorname{sgn}([F],$$
$$\partial[\bar{F}]) \operatorname{sgn}([\Phi],[\varphi(\Phi)]) \operatorname{sgn}([\Phi],\partial[\bar{F}]) \frac{w_\Sigma(\bar{F})}{w_{\Sigma'}(\varphi(\Phi))} e_{[\varphi(\Phi)]}. \tag{10.1.34}$$

*Proof* We first assume that $\varphi$ preserves dimensions of simplices, that is, all $q$-simplices are mapped to $q$-simplices. In other words, $\varphi$ is injective on the vertices $v_0, \dots, v_q$ of every $q$-simplex $F$. In that case, (10.1.31) and (10.1.32) agree.

Let $[\bar{F}] = [v_0, \dots, v_{q+1}]$ be a positively oriented $(q+1)$-simplex in $\Sigma$, and

$$[F] = [v_0, \dots, \hat{v_i}, \dots v_{q+1}],\ [\Phi] = [v_0, \dots, \hat{v_j}, \dots v_{q+1}]$$

its oriented facets. Let $\bar{G} = \{\varphi(v_0), \dots, \varphi(v_{q+1})\}$, $G = \{\varphi(v_0), \dots, \varphi(\hat{v_i}), \dots, \varphi(v_{q+1})\}$, and $\varphi(\Phi) = \{\varphi(v_0), \dots, \varphi(\hat{v_j}), \dots, \varphi(v_{q+1})\}$ be the images of $\bar{F}, F$, and $\Phi$, respectively. Assume $[\bar{G}]$, $[G]$, and $[\varphi(\Phi)]$ are all positively oriented. Then,

$$[\bar{G}] = [\varphi(v_0), \dots, \varphi(v_{q+1})] \operatorname{sgn}([\bar{F}],[\bar{G}]),$$

$$[G] = [\varphi(v_0), \dots, \varphi(\hat{v_i}), \dots, \varphi(v_{q+1})] \operatorname{sgn}([F],[G])$$

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and

$$[\varphi(\Phi)] = [\varphi(v_0),\dots,\varphi(\hat{v}_j),\dots,\varphi(v_{q+1})]\,\mathrm{sgn}([F],[G]).$$

Therefore,

$$\begin{aligned}\mathrm{sgn}([G],\partial[\bar{G}]) &= \mathrm{sgn}\Big([\varphi(v_0),\dots,\varphi(\hat{v}_i),\dots,\varphi(v_{q+1})]\,\mathrm{sgn}([F],[G]),\\ &\quad [\varphi(v_0),\dots,\varphi(v_{q+1})]\,\mathrm{sgn}([\bar{F}],[\bar{G}])\Big)\\ &= (-1)^i\,\mathrm{sgn}([F],[G])\,\mathrm{sgn}([\bar{F}],[\bar{G}])\\ &= \mathrm{sgn}([F],\partial[\bar{F}])\,\mathrm{sgn}([F],[G])\,\mathrm{sgn}([\bar{F}],[\bar{G}]). \qquad (10.1.35)\end{aligned}$$

Therefore, (10.1.34) is

$$\varphi^* L^q_{up}(\Sigma)\varphi e_{[G]} =$$

$$= \sum_{\substack{F\in\Sigma_q:\\ \varphi(F)=G}} \sum_{\substack{\Phi\in\Sigma_q:\\ F,\Phi\in\partial\bar{F}}} \mathrm{sgn}([G],\partial[\bar{G}])\,\mathrm{sgn}([\varphi(\Phi)],\partial[\bar{G}])\frac{w_\Sigma(\bar{F})}{w_{\Sigma'}(\varphi(\Phi))}e_{[\varphi(\Phi)]} \qquad (10.1.36)$$

$$= \sum_{\substack{F\in\Sigma_q:\\ \varphi(F)=G}} \sum_{\substack{\bar{F}\in\Sigma_{q+1}:\\ F\in\partial\bar{F}}} w_\Sigma(\bar{F}) \sum_{\Phi:\,\Phi\in\partial\bar{F}} \mathrm{sgn}([G],\partial[\bar{G}])\,\mathrm{sgn}([\varphi(\Phi)],\partial[\bar{G}])\frac{e_{[\varphi(\Phi)]}}{w_{\Sigma'}(\varphi(\Phi))}. \qquad (10.1.37)$$

Since by (10.1.31), $w_{\Sigma'}(G) = \sum_{\substack{F\in\Sigma_q:\\ \varphi(F)=G}} w_\Sigma(F)$, (10.1.36) yields

$$\begin{aligned}\varphi^* L^q_{up}(\Sigma)\varphi e_{[G]} &= \sum_{\substack{\varphi(\Phi)\in\Sigma'_q:\\ G,\varphi(\Phi)\in\partial\bar{G}}} \mathrm{sgn}([G],\partial[\bar{G}])\,\mathrm{sgn}([G]',\partial[\bar{G}])\frac{w_{\Sigma'}(\bar{G})}{w_{\Sigma'}(\varphi(\Phi))}e_{[\varphi(\Phi)]}\\ &= L^q_{up}e_{[G]}. \qquad (10.1.38)\end{aligned}$$

Thus, diagram (10.1.33) commutes.

We now consider the situation where there exists some $q$-simplex in $\Sigma$, whose image under map $\varphi$ is $m$-dimensional, $m < q$. Then, we find $[\bar{F}] = [v_0,\dots,v_{q+1}]$ with $\varphi(v_i) = \varphi(v_j)$ for some $i \neq j$, and so, $[F_i] = [v_0,\dots,\hat{v_i},\dots v_{q+1}]$, $[F_j] = [v_0,\dots,\hat{v_j},\dots v_{q+1}]$, and $\varphi(\bar{F}) = \varphi(F_i) = \varphi(F_j) = G$. Then,

$$\sum_{F\in\{F_i,F_j\}} \sum_{\substack{\bar{F}\in\Sigma_{q+1}:\\ F,\Phi\in\partial\bar{F}}} \mathrm{sgn}([F],[G])\,\mathrm{sgn}([F],\partial[\bar{F}])[\Phi,\varphi(\Phi)]\,\mathrm{sgn}([F]',$$

$$\partial[\bar{F}])\frac{w_\Sigma(\bar{F})}{w_{\Sigma'}(\varphi(\Phi))}e_{[\varphi(\Phi)]} \qquad (10.1.39)$$

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$$
\begin{aligned}
&= \frac{w_\Sigma(\bar{F})}{w_{\Sigma'}(G)} \operatorname{sgn}([F_i],[G])(-1)^i \operatorname{sgn}([F_j],[G])(-1)^j e_{[G]} \\
&+ \frac{w_\Sigma(\bar{F})}{w_{\Sigma'}(G)} \operatorname{sgn}([F_i],[G])(-1)^i \operatorname{sgn}([F_i],[G])(-1)^i e_{[G]} \\
&+ \frac{w_\Sigma(\bar{F})}{w_{\Sigma'}(G)} \operatorname{sgn}([F_i],[G])(-1)^i \operatorname{sgn}([F_j],[G])(-1)^j e_{[G]} \\
&+ \frac{w_\Sigma(\bar{F})}{w_{\Sigma'}(G)} \operatorname{sgn}([F_j],[G])(-1)^j \operatorname{sgn}([F_j],[G])(-1)^j e_{[G]} \\
&= \frac{w_\Sigma(\bar{F})}{w_{\Sigma'}(G)} (\operatorname{sgn}([F_i],[G])(-1)^i (-1)^j \operatorname{sgn}([F_i],[G])(-1)^{j-i+1} e_{[G]} + e_{[G]}) \\
&+ \frac{w_\Sigma(\bar{F})}{w_{\Sigma'}(G)} (\operatorname{sgn}([F_j],[G])(-1)^j (-1)^i \operatorname{sgn}([F_j],[G])(-1)^{j-i+1} e_{[G]} + e_{[G]}) \\
&= 0. \qquad (10.1.40)
\end{aligned}
$$

Therefore, $L^q_{up}(\Sigma')e_{[G]} = \varphi^* L^q_{up}(\Sigma)\varphi e_{[G]}$, and the diagram (10.1.33) commutes for both choices (10.1.32) and (10.1.31) for the weight function $w_{\Sigma'}$. □

We can now derive interlacing results for simplicial maps. We first state

**Theorem 10.1.4.** *Let $\Sigma, \Sigma'$ be simplicial complexes with $N_\Sigma$ and $N_{\Sigma'}$ $q$-simplices, and let $\Sigma$ have $N$ vertices. Let $\varphi\colon \Sigma \to \Sigma'$ be a simplicial map, and let the weight functions $w_\Sigma$ and $w_{\Sigma'}$ be related by (10.1.31). Let $\lambda_1 \le \ldots \le \lambda_{N_\Sigma}$ and $\theta_1 \le \ldots \le \theta_{N_{\Sigma'}}$ be the eigenvalues of $L^q_{up}(\Sigma, w_\Sigma)$ and $L^q_{up}(\Sigma', w_{\Sigma'})$. Then,*

$$\lambda_k \le \theta_k \le \lambda_{k+N_\Sigma - N_{\Sigma'}}, \qquad (10.1.41)$$

*with $\lambda_{N_\Sigma+1} = \ldots = \lambda_{2N_\Sigma - N_{\Sigma'}} = N$.*

*Proof* For $g \in C^q(\Sigma)$, we have

$$
\begin{aligned}
(\varphi g, \varphi g) &= \sum_{G \in \Sigma'_{q+1}} \sum_{\substack{F \in \Sigma_q : \\ \varphi(F) = G}} (\operatorname{sgn}([F],[G]) g(\varphi F))^2 w_\Sigma(F) \qquad (10.1.42) \\
&= \sum_{G \in \Sigma'_q} g(G)^2 w_{\Sigma'}(G) = (g,g). \qquad (10.1.43)
\end{aligned}
$$

Here and in the sequel, $\mathcal{H}_\ell$ will be the collection of all $\ell$-dimensional subspaces of $C^q(\Sigma')(\cong \mathbb{R}^{N_{\Sigma'}})$ and we shall take the maximum (or minimum) over all vector spaces $H_\ell \in \mathcal{H}_\ell$, as prescribed by Theorem 2.6.1.

By Theorem 2.6.1, (10.1.34), and (10.1.42), we then have

$$\theta_k = \max_{H_{k-1} \in \mathcal{H}_{k-1}} \min_{g \perp H_{k-1}} \frac{(L_{up}(\Sigma')g, g)}{(g,g)} \qquad (10.1.44)$$

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$$= \max_{H_{k-1}} \min_{g \perp H_{k-1}} \frac{(\varphi^* L_{up}(\Sigma)\varphi g, g)}{(g,g)} \tag{10.1.45}$$

$$= \max_{H_{k-1}} \min_{g \perp H_{k-1}} \frac{(L_{up}(\Sigma)\varphi g, \varphi g)}{(\varphi g, \varphi g)}. \tag{10.1.46}$$

Let $\bigcup_j F_{ji} = \varphi^{-1}(G_i)$ and $\bigcup_i G_i = \Sigma'_q$ and $W$ be the vector space generated by

$$\operatorname{sgn}([F_{ji}],[G_i])e_{[F_{ji}]} - \operatorname{sgn}([F_{(i+1)j}],[G_i])e_{[F_{(i+1)j}]}.$$

The dimension of $W$ is $N_\Sigma - N_{\Sigma'}$. We let $\mathcal{V}_\ell$ be the collection of all $\ell$-dimensional subspaces of $C^q(\Sigma)(\cong \mathbb{R}^{N_\Sigma})$, and we shall take the maximum (or minimum) over all vector spaces $V_\ell \in \mathcal{V}_\ell$, again as prescribed by Theorem 2.6.1.

Let $\varphi g = f$. From (10.1.46), we get

$$\theta_k = \max_{V_{k-1} \in \mathcal{V}_{k-1}} \min_{f \perp V_{k-1}, f \perp W} \frac{(L_{up}(\Sigma)f, f)}{(f,f)} \tag{10.1.47}$$

$$\geq \max_{V_{k-1}} \min_{f \perp V_{k-1}} \frac{(L_{up}(\Sigma)f, f)}{(f,f)} \tag{10.1.48}$$

$$\geq \lambda_k. \tag{10.1.49}$$

Inequality (10.1.48) holds since we are minimizing over a larger set. From (10.1.47), we also obtain the upper interlacing inequality

$$\theta_k \leq \max_{H_{k-1+N_\Sigma-N_{\Sigma'}}} \min_{f \perp H_{k-1+N_\Sigma-N_{\Sigma'}}} \frac{(L_{up}(\Sigma)f, f)}{(f,f)} \tag{10.1.50}$$

$$= \lambda_{k+N_\Sigma-N_{\Sigma'}}. \tag{10.1.51}$$

Recalling (10.1.40), we also obtain the result when the dimension of simplices is not necessarily preserved. □

For the second choice of the weight function of $\Sigma'$, we have

**Theorem 10.1.5.** *Let $(\Sigma, w_\Sigma)$ and $(\Sigma', w_{\Sigma'})$ be weighted simplicial complexes with their weights related by (10.1.32), and $\varphi\colon \Sigma \to \Sigma'$ a simplicial map. Let $Z$ be the vector space with basis $\{e_{[G]} \mid \exists \bar{F} \in \Sigma_{q+1}\colon \varphi(\bar{F}) = G\}$, let $z$ be its dimension, and let $W$ be the vector space spanned by*

$$\{\operatorname{sgn}([F_{ji}],[G_i])e_{[F_{ji}]} - \operatorname{sgn}([F_{(i+1)j}],[G_i])e_{[F_{(i+1)j}]} \mid \bigcup_j F_{ji} = \varphi^{-1}(G_i), \text{ and} \bigcup_i G_i = \Sigma'_q\}.$$

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*Then, the eigenvalues* $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{N_\Sigma}$ *of* $L^q_{up}(\Sigma)$ *and the eigenvalues* $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_{N_\Sigma}$ *of* $L^q_{up}(\Sigma')$ *satisfy*

$$\lambda_k \leq \theta_k \leq \lambda_{k+z+N_\Sigma - N_{\Sigma'}}. \tag{10.1.52}$$

*Proof* We shall state inequalities here that are analogous to those in Theorem 10.1.4, and therefore, we do not repeat all details.

Let $W$ be, as before, a vector space spanned by

$$\{\mathrm{sgn}([F_{ji}],[G_i])e_{[F_{ji}]} - \mathrm{sgn}([F_{(i+1)j}],[G_i])e_{[F_{(i+1)j}]} \mid \bigcup_j F_{ji} = \varphi^{-1}(G_i), \text{ and} \bigcup_i G_i = \Sigma'_q\}.$$

Let $\phi\colon C^q(\Sigma') \to C^q(\Sigma')$ be an operator with

$$\phi e_{[G]} = \sum_{\bar{F}\in\Sigma_{q+1}\colon \varphi(\bar{F})=G} \frac{w_\Sigma(\bar{F})}{w_{\Sigma'}(G)} e_{[G]},$$

and let $Z$ be a vector space of dimension $z$ spanned by vectors

$$\{e_{[G]} \mid G \in \Sigma'_q, \exists \bar{F} \in \Sigma_{q+1}\colon \varphi(\bar{F}) = G\}.$$

Assume that $\varphi\colon \Sigma \to \Sigma'$ is a simplicial map, and $w_\Sigma$ and $w_{\Sigma'}$ are weight functions satisfying 10.1.32.

As before, $\mathcal{H}_\ell$ will be the collection of all $\ell$-dimensional subspaces of $C^q(\Sigma')$, and we shall take the maximum (or minimum) over all vector spaces $H_\ell \in \mathcal{H}_\ell$, as prescribed by Theorem 2.6.1.

With these conventions, we have

$$\theta_k = \max_{H_{k-1}\in\mathcal{H}_{k-1}} \min_{g\perp H_{k-1}} \frac{(L_{up}(\Sigma')g,g)}{(g,g)} \tag{10.1.53}$$

$$= \max_{H_{k-1}} \min_{g\perp H_{k-1}} \frac{(\varphi^* L_{up}(\Sigma)\varphi g,g)}{(g,g)} \tag{10.1.54}$$

$$= \max_{H_{k-1}} \min_{g\perp H_{k-1}} \frac{(L_{up}(\Sigma)\varphi g,\varphi g)}{(\varphi g,\varphi g) - (\phi g,g)} \tag{10.1.55}$$

$$\geq \max_{H_{k-1}} \min_{g\perp H_{k-1}} \frac{(L_{up}(\Sigma)\varphi g,\varphi g)}{(\varphi g,\varphi g)} \tag{10.1.56}$$

$$= \max_{H_{k-1}} \min_{f\perp H_{k-1}, f\perp \mathcal{W}} \frac{(L_{up}(\Sigma)f,f)}{(f,f)} \tag{10.1.57}$$

$$\geq \max_{H_{k-1}} \min_{g\perp H_{k-1}} \frac{(L_{up}(\Sigma)f,f)}{(f,f)} \tag{10.1.58}$$

$$\geq \lambda_k. \tag{10.1.59}$$

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On the other hand, the upper interlacing inequality follows from (10.1.55). As before, $\mathcal{V}_\ell$ is the collection of all $\ell$-dimensional subspaces of $C^q(\Sigma)(\cong \mathbb{R}^{N_\Sigma})$.

$$\theta_k \le \max_{V_{k-1}\in\mathcal{V}_{k-1}} \min_{g\perp V_{k-1}} \frac{(L_{up}(\Sigma)\varphi g,\varphi g)}{(\varphi g,\varphi g)} \tag{10.1.60}$$

$$\le \max_{V_{k-1}} \min_{g\perp V_{k-1}, g\perp Z} \frac{(L_{up}(\Sigma)\varphi g,\varphi g)}{(\varphi g,\varphi g)} \tag{10.1.61}$$

$$\le \max_{V_{k+z-1}} \min_{g\perp V_{k+z-1},} \frac{(L_{up}(\Sigma)\varphi g,\varphi g)}{(\varphi g,\varphi g)} \tag{10.1.62}$$

$$= \max_{V_{k+z-1}} \min_{f\perp V_{k+z-1}, f\perp W} \frac{(L_{up}(\Sigma)f,f)}{(f,f)} \tag{10.1.63}$$

$$\le \max_{V_{k+z+N_\Sigma-N_{\Sigma'}-1}} \min_{f\perp V_{k+z+N_\Sigma-N_{\Sigma'}-1}} \frac{(L_{up}(\Sigma)f,f)}{(f,f)} \tag{10.1.64}$$

$$\le \lambda_{k+N_\Sigma-N_{\Sigma'}+z} \tag{10.1.65}$$

□

### 10.1.3 Coverings

In a topological setting, a continuous map $f\colon M \to N$ between two compact spaces is a *covering* if every point in the image has a neighborhood $V$ whose preimage $f^{-1}(V)$ consists of disjoint sets $U_1,\dots,U_m$ that are all homeomorphically mapped to $V$ by $f$. In particular, in the context of Riemannian manifolds, we require $f$ to be an isometry, and every Laplacian eigenfunction of $N$ can then be pulled back by $f$ to a Laplacian eigenfunction of $M$. Consequently, the spectrum of $N$ is contained in that of $M$. In order to get such a result in the simplicial context, we first need to clarify the notion of a covering between simplicial complexes. A naive version would simply require for a simplicial map $\varphi\colon \Sigma \to \Sigma'$ between simplicial complexes to be a covering such that the preimage of every simplex $F$ is a disjoint union of simplices all mapped bijectively to $F$. That, however, does not suffice to let the spectrum of $\Sigma'$ be contained in that of $\Sigma$. Simply take a path with one edge (i.e., a single edge with its two vertices) and a path with two edges and join them at a vertex to get a path with three edges.

The union of the two original paths then would be a covering in that sense of the path with three edges, but the latter has the eigenvalues $1/2$ and $3/2$, which do not belong to the spectra of the shorter paths. In fact, an even simpler

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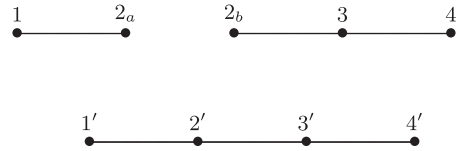



Figure 10.2 A covering, but not a strong covering.

example would join two paths of length 1 into a path of length 2. The latter has the eigenvalue 1, but the former does not.

The problem with that naive notion is that the boundaries of simplices may have different numbers of preimages than those simplices themselves. Since the Laplace operator involves boundary operations, of course, we can then not hope for strong results. Or putting it differently, the analogue of a neighborhood in a Riemannian manifold in the context of simplicial complexes should also involve the boundary facets of simplices. We thus formulate

**Definition 10.1.3.** A simplicial map $\varphi\colon \Sigma \to \Sigma'$ is a *strong covering* if for every $q$-simplex $G$ in $\Sigma'$ that is a facet of some $\bar{G}$ and every $F \in \varphi^{-1}(G)$, there exists $\bar{F} \in \Sigma_{q+1}$ such that $F$ is a facet of $\bar{F}$ and $\varphi(\bar{F}) = \bar{G}$.

This property is not satisfied in (Figure 10.2), because there, the vertex $2'$ in the image is a facet of the edge $[2',3']$, but its preimage $2_a$ is not a facet of an edge that is mapped to the edge $[2',3']$.

*Remark* 10.1.1. Thus, under a strong covering for any two $q$-simplices $G_1$ and $G_2$ in $\Sigma'$ that are $(q+1)$-up neighbors, and for every $F_1 \in \varphi^{-1}(G_1)$, there must exist $F_2 \in \Sigma_q$ such that $F_1$ and $F_2$ are $(q+1)$-up neighbors and $\varphi(F_2) = G_2$. Again, this is not satisfied in (Figure 10.2) because the vertices $2',3'$ are 1-up neighbors, but the preimage $2_a$ is not a 1-neighbor of any vertex that is mapped to $3'$.

**Lemma 10.1.5.** *Let $\varphi\colon \Sigma \to \Sigma'$ be a strong covering. Then, every $F \in \Sigma_q$ and every $G \in \Sigma'_q$ with $\varphi(F) = G$ are facets of the same number of $(q+1)$-simplices, that is,*

$$|\{\bar{F} \in \Sigma_{q+1} \mid F \in \partial\bar{F}\}| = |\{\bar{G} \in \Sigma'_{q+1} \mid G \in \partial\bar{G}\}|.$$

In particular, all preimages of a $q$-simplex in $\Sigma'$ are facets of the same number of $(q+1)$-simplices. Thus, the local structure is the same.

*Proof* Assume that $F \in \partial\bar{F}_i$ for $\bar{F}_i \in \Sigma_{q+1}$, for $i \in \{1,\dots,k\}$. Then, since $\varphi$ is a (strong) covering, $\varphi(\bar{F}_i) \in \Sigma'_{q+1}$ and $\varphi(F) \in \partial\varphi(\bar{F}_i)$, and $\varphi(\bar{F}_i) \neq \varphi(\bar{F}_j)$ for $i \neq j$. Consider $\bar{G} \in \Sigma'_{q+1}$ with $G \in \partial\bar{G}$. Since $\varphi$ is a strong covering, we

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can find $\bar{F} \in \Sigma_{q+1}$ with $F \in \partial\bar{F}$, such that $\varphi(\bar{F}) = \bar{G}$. Moreover, since $\varphi$ is a simplicial map, if $G \in \partial\bar{G}, \partial\bar{G}'$ and $\bar{G} \neq \bar{G}'$, then $\bar{F} \neq \bar{F}'$. □

**Lemma 10.1.6.** *Let $\varphi\colon \Sigma \to \Sigma'$ be a strong covering. Assume that $\Sigma'$ is $(q+1)$-up path connected for every $q$ (i.e., for any two $G, G' \in \Sigma'_q$, there is a sequence $G = G_1, \bar{G}_1, G_2, \bar{G}_2, \ldots, \bar{G}_m, G_{m+1} = G'$ such that $G_i$ and $G_{i+1}$ are facets of the $(q+1)$-simplex $\bar{G}_i$ for $i = 1, \ldots, m$). Then, all simplices in $\Sigma'$ have the same number of preimages, called the degree of the covering. That is*

$$|\{F \in \Sigma_q \mid F \in \varphi^{-1}(G), G \in \Sigma'_q\}| = c$$

*for some positive integer $c$, for every $F \in \Sigma$.*

*Proof* Let $G$ and $G'$ be $(q+1)$-up neighbors in $\Sigma'$. Then, there is some $\bar{G} \in \Sigma'_{q+1}$ with $G, G' \in \partial\bar{G}$. Since $\varphi$ is a strong covering, for every $F \in \varphi^{-1}(G)$, there exists $\bar{F}$ with $\varphi(\bar{F}) = \bar{G}$. Thus, there exists $F' \in \partial\bar{F}$ in the preimage of $G'$. Therefore, if $G$ and $G'$ are $(q+1)$-up neighbors, then $|\{F \in \Sigma_q \mid F \in \varphi^{-1}(G)\}| = |\{F \in \Sigma_q \mid F' \in \varphi^{-1}(G')\}|$. Since $\Sigma'$ is $(q+1)$-path connected, for $q$-faces $G$ and $G'$ of $\Sigma'$, the conclusion follows. □

**Theorem 10.1.6.** *Let $\varphi\colon \Sigma \to \Sigma'$ be a strong covering where $\Sigma'$ is $(q+1)$-up path connected, and let the weights in $\Sigma$ and $\Sigma'$ be related by $\frac{w_\Sigma(\bar{F})}{w_\Sigma(F)} = \frac{w_{\Sigma'}(\varphi(\bar{F}))}{w_{\Sigma'}(\varphi(F))}$, for every pair $\bar{F}, F$ with $F \in \partial\bar{F}$. Then, $\varphi L^q_{up}(\Sigma') = L^q_{up}(\Sigma)\varphi$, that is,*

$$\mathrm{Spec}(L^q_{up}(\Sigma')) \subset \mathrm{Spec}(L^q_{up}(\Sigma)\varphi). \tag{10.1.66}$$

*Proof* We have

$$\varphi L_{up}(\Sigma')e_{[G]} = \tag{10.1.67}$$

$$= \sum_{\substack{\bar{G}\in\Sigma'_{q+1}:\\ G',G\in\partial\bar{G}}} \sum_{\substack{F'\in\Sigma_q:\\ \varphi(F')=G'}} \mathrm{sgn}([F'],[G'])\,\mathrm{sgn}([G],\partial[\bar{G}])\,\mathrm{sgn}([G'],\partial[\bar{G}])\,\frac{w_{\Sigma'}(\bar{G})}{w_{\Sigma'}(G')}e_{[F']}.$$

$$L_{up}(\Sigma)\varphi e_{[G]} = \tag{10.1.68}$$

$$= \sum_{\substack{F\in\Sigma_q:\\ \varphi(F)=G}} \sum_{\substack{\bar{F}\in\Sigma_{q+1}:\\ F,F'\in\partial\bar{F}}} \mathrm{sgn}([F],[G])\,\mathrm{sgn}([F],\partial[\bar{F}])\,\mathrm{sgn}([F'],\partial[\bar{F}])\,\frac{w_\Sigma(\bar{F})}{w_\Sigma(F')}e_{[F']}.$$

We now claim

$$\{F' \in \Sigma_q \mid \exists \bar{G} \in \Sigma'_{q+1}, : G, \varphi(F') \in \partial\bar{G}\} =$$
$$\{F' \in \Sigma_q \mid \exists \bar{F} \in \Sigma_{q+1}, : F, F' \in \partial\bar{F}, \text{ where } \varphi(F) = G\}. \tag{10.1.69}$$

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If a multiple of $e_{[F']}$ occurs in the sum in (10.1.68), then there exists $F \in \Sigma_q$ with $\varphi(F) = G$, and $F$ and $F'$ are $(q+1)$-up neighbors. Therefore, $\varphi(F')$ and $G$ are $(q+1)$-up neighbors as well, and $e_{[F']}$ is a summand in (10.1.67).

On the other hand, if a multiple of $e_{[F']}$ appears in (10.1.67), then $F'$ is a $q$-face of the simplicial complex $\Sigma$, such that $\varphi(F')$ and $G$ are $(q+1)$-up neighbors. Since $\varphi$ is a strong covering, there must exist $F \in \Sigma_q$ with $\varphi(F) = G$, and $F, F'$ are $(q+1)$-up neighbors; hence, $e_{[F']}$ is a summand in (10.1.68). This proves (10.1.69).

By (10.1.35), we have

$$\begin{aligned}
&\operatorname{sgn}([F],[G])\operatorname{sgn}([F],\partial[\bar{F}])\operatorname{sgn}([F'],\partial[\bar{F}]) \\
&\quad = \operatorname{sgn}([G],\partial[\bar{G}])\operatorname{sgn}([\bar{F}],[\bar{G}])\operatorname{sgn}([F'],\partial[\bar{F}]) \\
&\quad = \operatorname{sgn}([G],\partial[\bar{G}])\operatorname{sgn}([G'],\partial[\bar{G}])\operatorname{sgn}([F'],[G']),
\end{aligned}$$

which makes (10.1.67) and (10.1.68) equal. □

In particular, for the normalized Laplace operators, we obtain an inclusion of spectra.

**Corollary 10.1.3.** *If $\varphi\colon \Sigma \to \Sigma'$ is a strong covering, and $\Sigma'$ is $(q+1)$-up path connected, then for the normalized Laplace operators $\Delta^q_{up}$, we have*

$$\operatorname{Spec}(\Delta^q_{up}(\Sigma')) \subset \operatorname{Spec}(\Delta^q_{up}(\Sigma)). \tag{10.1.70}$$

□

### 10.1.4 Contraction

In this section, we analyze the effect of another operation, namely contraction, on the eigenvalues of the Laplace operator.

**Definition 10.1.4.** Let $\Sigma$ be a simplicial complex of dimension $q+1$, and let $\bar{F}_1$ be a $(q+1)$-simplex, for which precisely two of its facets $F, F'$, are incident to other $(q+1)$-simplices. An *elementary contraction* then is obtained by the construction of a simplicial complex $\Sigma'$ and a simplicial map $\varphi\colon \Sigma \to \Sigma'$ in the following manner. We first delete the simplex $\bar{F}_1$ and identify $F$ and $F'$. Thus, there exist vertices $v \in F$ and $v' \in F'$ with $\varphi(v) = \varphi(v')$, and $\varphi$ is injective on all other vertices of $\Sigma$. When there exist $2m$ $(q+1)$-simplices $\bar{F}_2, \bar{F}'_2, \ldots, \bar{F}_{m+1}, \bar{F}'_{m+1}$ of $\Sigma$ such that $F$ is a facet of the $\bar{F}_i$, and $F'$ is a facet of the $\bar{F}'_i$ and $\bar{F}_i$ and $\bar{F}'_i$ share another facet, we let $\varphi$ also identify $\bar{F}_i$ and $\bar{F}'_i$, for $i = 2, \ldots, m$. When $m > 0$, we say that the resulting elementary contraction is of type $(i)$. When there are no such pairs, we can let $\varphi$ be injective on $\Sigma_{q+1}$ and say that the resulting elementary contraction is of type $(ii)$.

A sequence of elementary contractions is called a *contraction*.

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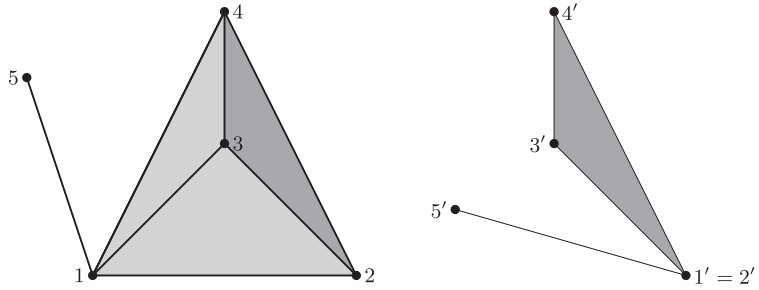



Figure 10.3 An elementary contraction of type (i).

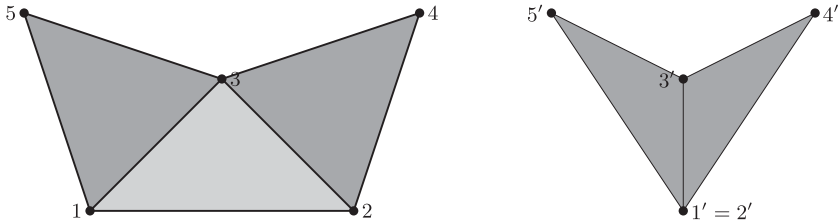



Figure 10.4 An elementary contraction of type (ii).

*Remark* 10.1.2. Assume that the elementary contraction $\varphi\colon \Sigma \to \Sigma'$ identifies the $q$-faces $F$ and $F'$ of the $(q+1)$-simplex $\bar{F}$. If $F'$ is not a facet of any other simplex, then this operation is a collapse, as in Definition 10.1.1, and therefore can be treated as such. Thus, to avoid redundancy, in the definition of an elementary contraction, we had required that both $F$ and $F'$ (but no further facets of $\bar{F}$) are incident to other $(q+1)$-simplices.

Figures 10.3 and 10.4 represent two main types of elementary contractions, type $(i)$ and type $(ii)$, respectively. In Figure 10.3, the simplex with vertices 1,2,3 is contracted, and then the simplex 1,3,4 has to be mapped onto the simplex $1' = 2', 3', 4'$. In contrast, in Figure 10.4, the simplex 1,2,3 is contracted without further ado by simply identifying the vertices 1 and 2.

Although collapses and contractions are simplicial maps, we have to recall that the interlacing theorems of Section 10.1.2 require specific choices of weights on $\Sigma'$, as specified in (10.1.32) and (10.1.31).

**Theorem 10.1.7.** *Let $L^q_{up}(\Sigma)$ be either the combinatorial Laplacians $L^q_{up}(\Sigma, w_\Sigma)$ with $w_\Sigma \equiv 1$, or the normalized Laplacian $\Delta^q_{up}(\Sigma)$, and analogously for the Laplacian on $\Sigma'$. Let $\varphi\colon \Sigma \to \Sigma'$ be an elementary contraction, and let $\lambda_1 \leq \ldots \leq \lambda_{N_\Sigma}$, $\theta_1 \leq \ldots \leq \theta_{N_{\Sigma'}}$ be the eigenvalues of $L^q_{up}(\Sigma)$ and $L^q_{up}(\Sigma')$, then*

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*(i)*

$$\lambda_{k-m(q+2)} \le \theta_k \le \lambda_{k+N_\Sigma - N_{\Sigma'}+m(q+2)}, \tag{10.1.71}$$

*if the contraction $\varphi$ is of type $(i)$, or*

*(ii)*

$$\lambda_k \le \theta_k \le \lambda_{k+q+2}, \tag{10.1.72}$$

*if $\varphi$ is of type $(ii)$,*

*where $\lambda_{N_\Sigma+1} = \ldots = \lambda_{2N_\Sigma - N_{\Sigma'}+m(q+1)} = N$, $\lambda_0 = \lambda_{-1} = \ldots = \lambda_{-n-1} = 0$, and $N$ is the number of vertices of $\Sigma$.*

*Proof* Assume first that $L^q_{up}$ is the combinatorial Laplacian, that is, the weight functions $w_\Sigma$ and $w_{\Sigma'}$ are identically equal to 1. Let the elementary contraction $\varphi$ be given by identification of the facets $F_1$ and $F_1'$ of $\bar{F}_1$. We need to distinguish between the two types of elementary contractions $\varphi$ mentioned earlier. If $\varphi$ is of type $(i)$, let $\bar{F}_2, \bar{F}_2', \ldots, \bar{F}_{m+1}, \bar{F}'_{m+1}$ be the $(q+1)$-simplices of $\Sigma$ that are identified by $\varphi$. Let $\psi : C^q(\Sigma') \to C^q(\Sigma')$ be a map with

$$\psi e_{[G]} = \sum_{G' \in \partial \bar{G}_i} w_\Sigma(\bar{F}_i) \operatorname{sgn}([G], \partial[\bar{G}_i]) \operatorname{sgn}([G'], \partial[\bar{G}_i]) \frac{1}{w_{\Sigma'}(G')} e_{[G']}, \tag{10.1.73}$$

if $\varphi$ is of type $(i)$, $G \in \partial\varphi(\bar{F}_i)$, where $\bar{G}_i = \varphi(\bar{F}_i)$, for all $2 \le i \le m+1$, and

$$\psi e_{[G]} = 0, \tag{10.1.74}$$

for other choices of $G \in \Sigma'$, if $\varphi$ is of type $(ii)$.

According to (10.1.37), by the arguments of Section 10.1.2, we have

$$\begin{aligned} &\varphi^* L^q_{up}(\Sigma) \varphi e_G \\ &\quad = \sum\nolimits_{\substack{\bar{F} \in \Sigma_{q+1}: \\ G \in \partial\varphi(\bar{F})}} w_\Sigma(\bar{F}) \sum_{G' \in \partial \bar{G}} \operatorname{sgn}([G], \partial[\bar{G}]) \operatorname{sgn}([G'], \partial[\bar{G}]) \frac{e_{[G']}}{w_{\Sigma'}(G')}. \end{aligned}$$

Therefore,

$$\begin{aligned} \varphi^* L^q_{up}(\Sigma) \varphi e_{[G]} &= L^q_{up}(\Sigma') e_{[G]} \\ &\quad + w_\Sigma(\bar{F}_i) \sum_{G' \in \partial \bar{G}_i} \operatorname{sgn}([G], \partial[\bar{G}_i]) \operatorname{sgn}([G'], \partial[\bar{G}_i]) \frac{e_{[G']}}{w_{\Sigma'}(G')}, \end{aligned}$$

if $\varphi$ is of type $(i)$ and $G \in \partial\varphi(\bar{F}_i)$, $2 \le i \le m+1$, and

$$\varphi^* L^q_{up}(\Sigma) \varphi e_{[G]} = L^q_{up}(\Sigma') e_{[G]}, \tag{10.1.75}$$

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otherwise. Thus, $L^q_{up}(\Sigma') = \varphi^* L^q_{up}(\Sigma)\varphi - \psi$. We then have the following equalities for $(\varphi g, \varphi g)$.

$$(\varphi g, \varphi g) = \sum_{G\in\Sigma'_q} \sum_{\substack{F\in\Sigma_q:\\ \varphi(F)=G}} (\mathrm{sgn}([F],[G]) g(\varphi F))^2 w_\Sigma(F) \tag{10.1.76}$$

$$= \sum_{G\in\Sigma'_q} g([G])^2 \sum_{\substack{F\in\Sigma_q:\\ \varphi(F)=G}} w_\Sigma(F) \tag{10.1.77}$$

$$= (g,g) + (\phi_1 g, g) + (\phi_2 g, g), \tag{10.1.78}$$

where

$$\phi_1 e_{[G]} = \begin{cases} w_\Sigma(F) e_{[G]} & \text{if } \varphi \text{ is of type } (i), \text{ and } \varphi(F) = G \in \partial\varphi(\bar{F}_i), 2 \le i \le m+1, \\ 0 & \text{otherwise,} \end{cases} \tag{10.1.79}$$

and

$$\phi_2 e_{[G]} = \begin{cases} w_\Sigma(F_1) e_{[G]} & \text{if } F_1 \in \varphi^{-1}(G), \text{ and } \varphi \text{ is of type } (ii), \\ 0 & \text{otherwise.} \end{cases} \tag{10.1.80}$$

We then get the interlacing inequalities from the following sequence. Let $\mathcal{Y}$ denote the subspace of $C^q(\Sigma')$ on which $(\psi g, g) = 0$ and $y := \dim \mathcal{Y} \le (q+2)m$ if $\varphi$ is of type $(i)$, and let $y = 0$ if $\varphi$ is an elementary contraction of type $(ii)$. Let also $\mathcal{W}$ be the vector space generated by

$$\{\mathrm{sgn}([F_{ji}],[G_i]) e_{[F_{ji}]} - \mathrm{sgn}([F_{(i+1)j}],[G_i]) e_{[F_{(i+1)j}]} \mid \bigcup_j F_{ji} = \varphi^{-1}(G_i), \text{ and } \bigcup_i G_i = \Sigma'_q\},$$

which has dimension $\dim \mathcal{W} = N_\Sigma - N_{\Sigma'}$. Then,

$$\theta_k = \min_{V_{N_{\Sigma'}-k} \in \mathcal{V}_{N_{\Sigma'}-k}} \max_{g \perp V_{N_{\Sigma'}-k}} \frac{(L_{up}(\Sigma')g, g)}{(g,g)} \tag{10.1.81}$$

$$= \min_{\mathcal{V}_{N_{\Sigma'}-k}} \max_{g\perp V_{N_{\Sigma'}-k}} \frac{(\varphi^* L_{up}(\Sigma)\varphi g, g) - (\psi g, g)}{(g,g)} \tag{10.1.82}$$

$$\ge \min_{\mathcal{V}_{N_{\Sigma'}-k}} \max_{g\perp V_{N_{\Sigma'}-k}, g\perp\mathcal{Y}} \frac{(\varphi^* L_{up}(\Sigma)\varphi g, g)}{(g,g)} \tag{10.1.83}$$

$$= \min_{\mathcal{V}_{N_{\Sigma'}-k}} \max_{g\perp V_{N_{\Sigma'}-k}, g\perp\mathcal{Y}} \frac{(L_{up}(\Sigma)\varphi g, \varphi g)}{(\varphi g, \varphi g) - (\phi_1 g, g) - (\phi_2 g, g)} \tag{10.1.84}$$

$$\ge \min_{\mathcal{V}_{N_{\Sigma'}-k}} \max_{g\perp V_{N_{\Sigma'}-k}, g\perp\mathcal{Y}} \frac{(L_{up}(\Sigma)\varphi g, \varphi g)}{(\varphi g, \varphi g)} \tag{10.1.85}$$

$$\ge \min_{\mathcal{V}_{N_{\Sigma'}-k+y}} \max_{g\perp V_{N_{\Sigma'}-k+y}} \frac{(L_{up}(\Sigma)\varphi g, \varphi g)}{(\varphi g, \varphi g)} \tag{10.1.86}$$

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$$= \min_{\mathcal{V}_{N_{\Sigma'}-k+y}} \max_{f\perp V_{N_{\Sigma'}-k+y}, f\perp\mathcal{W}} \frac{(L_{up}(\Sigma)f,f)}{(f,f)} \tag{10.1.87}$$

$$\geq \min_{\mathcal{V}_{N_{\Sigma'}-k+y}} \max_{f\perp V_{N_{\Sigma}-k+y}} \frac{(L_{up}(\Sigma)f,f)}{(f,f)} \tag{10.1.88}$$

$$\geq \min_{\mathcal{V}_{N_{\Sigma}-k+y}} \max_{f\perp V_{N_{\Sigma}-k+y}} \frac{(L_{up}(\Sigma)f,f)}{(f,f)} \tag{10.1.89}$$

$$\geq \lambda_{k-y}. \tag{10.1.90}$$

We have used that $N_\Sigma - N_{\Sigma'} = q+1$ if $\varphi$ is an elementary collapse of type $(ii)$.

Similarly, for the upper interlacing inequality, we consider the vector space

$$\mathcal{Z} := \{g \in C^q(\Sigma') \mid (\phi_1 g, g) + (\phi_2 g, g) = 0\}$$

of dimension $z := \dim \mathcal{Z} = 1$ if $\varphi$ is of type $(ii)$ and $z = m(q+2)$ for type $(i)$. The upper interlacing inequality then follows from (10.1.55) via

$$\theta_k \leq \max_{V_{k-1}\in\mathcal{V}_{k-1}} \min_{g\perp V_{k-1}} \frac{(L_{up}(\Sigma)\varphi g,\varphi g) - (\psi g, g)}{(\varphi g,\varphi g) - (\phi_1 g, g) - (\phi_2 g, g)} \tag{10.1.91}$$

$$\leq \max_{\mathcal{V}_{k-1}} \min_{g\perp V_{k-1}} \frac{(L_{up}(\Sigma)\varphi g,\varphi g)}{(\varphi g,\varphi g) - (\phi_1 g, g) - (\phi_2 g, g)} \tag{10.1.92}$$

$$\leq \max_{\mathcal{V}_{k-1}} \min_{g\perp V_{k-1}, g\perp\mathcal{Z}} \frac{(L_{up}(\Sigma)\varphi g,\varphi g)}{(\varphi g,\varphi g)} \tag{10.1.93}$$

$$\leq \max_{\mathcal{V}_{k+z-1}} \min_{g\perp V_{k+z-1},} \frac{(L_{up}(\Sigma)\varphi g,\varphi g)}{(\varphi g,\varphi g)} \tag{10.1.94}$$

$$= \max_{\mathcal{V}_{k+z-1}} \min_{f\perp V_{k+z-1}, f\perp\mathcal{W}} \frac{(L_{up}(\Sigma)f,f)}{(f,f)} \tag{10.1.95}$$

$$\leq \max_{\mathcal{V}_{k+z+N_\Sigma-N_L-1}} \min_{f\perp V_{k+z+N_\Sigma-N_L-1}} \frac{(L_{up}(\Sigma)f,f)}{(f,f)} \tag{10.1.96}$$

$$\leq \lambda_{k+N_\Sigma-N_L+z}. \tag{10.1.97}$$

Thus, in case $(i)$, we have

$$\lambda_{k-m(q+2)} \leq \theta_k \leq \lambda_{k+N_\Sigma-N_{\Sigma'}+m(q+2)}, \tag{10.1.98}$$

while in case $(ii)$, we get

$$\lambda_k \leq \theta_k \leq \lambda_{k+N_\Sigma-N_{\Sigma'}+1}. \tag{10.1.99}$$

For the normalized Laplacian, the weight functions $w_\Sigma$ and $w_{\Sigma'}$ have value 1 on all simplices of dimension $q+1$, whereas $w_\Sigma(F) = \deg F$ and $w_{\Sigma'}(G) = \deg G$,

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for all $q$-faces $F$, $G$, of $\Sigma$ and $\Sigma'$, respectively. The maps $\phi_1, \phi_2 \colon C^q(\Sigma') \to C^q(\Sigma')$ are then given by

$$\phi_1 e_{[G]} = \begin{cases} w_\Sigma(\bar{F}_i) e_{[G]} & \text{if } \varphi \text{ is of type } (i), \text{ and } G \in \partial \bar{G}_i, 2 \leq i \leq m+1, \\ 0 & \text{otherwise,} \end{cases} \tag{10.1.100}$$

and

$$\phi_2 e_{[G]} = \begin{cases} 2 w_\Sigma(\bar{F}_1) e_{[G]} & \text{if } G = \varphi(F_1), \\ 0 & \text{otherwise.} \end{cases} \tag{10.1.101}$$

The rest of the proof is the same as for the combinatorial Laplacian, and the same interlacing inequalities hold. □

## 10.2 Spectral Classes

### 10.2.1 Asymptotic Classes of Graphs

In Section 4.1, we have already seen several families of graphs, like the complete graphs $K_N$, the complete bipartite graphs $K_{n,m}$ (including the star graphs $K_{1,n}$), the cycles $C_N$, the wheels $W_N$, the petals $P_\ell$, and the books $B_\ell$. They usually arise from the iteration of some construction, like duplicating a vertex or an edge, and they exhibit corresponding regularities of their spectra that lead to asymptotics as $N \to \infty$. As we have seen and as we want to explore in more detail, those properties can be quite similar for different classes.

For instance, the petals $P_\ell$ and the books $B_\ell$ both arise from the repeated duplication of an edge. In the case of the petals, one starts with the triangle $p_1 = K_3$, while for the books, $B_1$ is a line graph with four vertices.

In the case of the books, we duplicate the middle edge, thereby generating an eigenvalue $3/2$ according to (4.1.86). And since the books are bipartite, by Lemma 4.1.13, we then also get the eigenvalue $1/2$. In fact, the line with four vertices already has the eigenvalues $0, 1/2, 3/2, 2$, and so, by the edge duplications, we just increase the multiplicities of the eigenvalues $1/2$ and $3/2$.

In this context, let us consider another construction of repeated edge duplications, the water wheel graphs $WW_\ell$. Here, we start with the square $WW_1 = C_4$ (Figure 10.5) and we repeatedly duplicate one of the edges.

These graphs $WW_\ell$ are again bipartite (Figure 10.6), and so we have the eigenvalues 0,2, and the edge duplications produce the eigenvalues $1/2$ and $3/2$. The remaining eigenvalues are $\lambda = \frac{1}{2} + \frac{\ell}{\ell+1}$ with an eigenfunction that is $\frac{2\ell}{\ell+1}$ at the central and $-1$ at the peripheral vertices, and $\lambda = \frac{3}{2} - \frac{\ell}{\ell+1}$ with an eigenfunction that is $\pm\frac{2\ell}{\ell+1}$ at the central and $\pm 1$ at the peripheral vertices, according to Lemma 4.1.13. We notice that these eigenvalues converge to $3/2$

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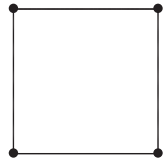


Figure 10.5 The square $WW_1 = C_4$.

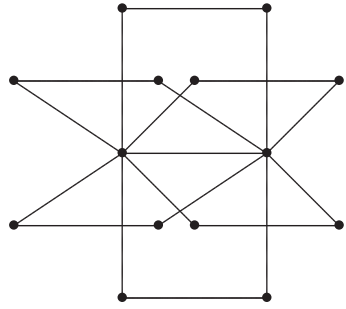


Figure 10.6 The water wheel graphs $WW_\ell$.

and $1/2$ as $\ell \to \infty$. Thus, the spectrum of $WW_\ell$ approaches that of $B_\ell$ for $\ell \to \infty$. This is not surprising as the only difference between the two types of graph is the presence of the connection between the two central vertices in $WW_\ell$ and its absence in $B_\ell$. Also, $P_\ell$ is similar, as it can be seen as the result of merging those two central vertices into a single one.

We now look at another construction, *graph doubling*. This is to be distinguished from the duplications of motifs that we have considered earlier. Here, we take a graph $\Gamma_0$ and an isomorphic copy $\Gamma_0'$. The vertex $v$ of $\Gamma_0$ corresponds to the vertex $v'$ of $\Gamma_0'$ and the graph $\Gamma$ then is formed from the union of $\Gamma_0$ and $\Gamma_0'$ by also connecting each $v$ with its twin $v'$. Thus, the degree $\deg_\Gamma v$ of each vertex $w$ in $\Gamma$ equals $\deg_{\Gamma_0} v + 1$ and the same of course for each $v'$. We now assume that $\Gamma_0$ is $k$-regular, that is

$$\deg_{\Gamma_0} v = k \quad \text{for all } v. \tag{10.2.1}$$

Let then $\lambda$ be an eigenvalue of $\Gamma_0$ with eigenfunction $u$, that is,

$$u(v) - \frac{1}{k} \sum_{v' \sim_{\Gamma_0} v} u(v') = \lambda u(v). \tag{10.2.2}$$

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We extend $u$ to $\Gamma$ by putting $u(v') = u(v)$ for all $v$. The corresponding quantity in $\Gamma$ then is

$$u(v) - \frac{1}{k+1}\sum_{v'\sim_\Gamma v} u(v') = u(v) - \frac{1}{k+1}u(v) - \frac{1}{k+1}\sum_{v'\sim_{\Gamma_0} v} u(v') = \frac{k}{k+1}\lambda u(v) \tag{10.2.3}$$

by (10.2.2), and so, $u$ then becomes an eigenfunction on $\Gamma$ for the eigenvalue $\frac{k}{k+1}\lambda$.

If we instead extend $u$ to $\Gamma$ by putting $u(v') = -u(v)$ for all $v$, we obtain

$$\begin{aligned} u(v) - \frac{1}{k+1}\sum_{v'\sim_\Gamma v} u(v') &= u(v) + \frac{1}{k+1}u(v) - \frac{1}{k+1}\sum_{v'\sim_{\Gamma_0} v} u(v') \\ &= \left(\frac{k}{k+1}\lambda + \frac{2}{k+1}\right)u(v), \end{aligned} \tag{10.2.4}$$

that is, the eigenvalue $\frac{k}{k+1}\lambda + \frac{2}{k+1}$.

We record the preceding as

**Lemma 10.2.1.** *Let $\Gamma$ be the graph obtained from doubling a $k$-regular graph $\Gamma_0$. Then, $\Gamma$ is $(k+1)$-regular, and it has the eigenvalues*

$$\frac{k}{k+1}\lambda \quad \text{and} \quad \frac{k}{k+1}\lambda + \frac{2}{k+1}, \tag{10.2.5}$$

*where the $\lambda$'s are the eigenvalues of $\Gamma_0$.*

□

In the same manner, we can then also triple, quadruple, ..., graphs, and if the starting graph is regular, determine their spectra.

From Lemma 10.2.1, we conclude

**Corollary 10.2.1.** *The cube graph $Q_n$ on $2^n$ vertices has the eigenvalues $\frac{2j}{n}$ with multiplicities $\binom{n}{j}$ for $j = 0, \ldots, n$.*

*Proof* $Q_n$ arises by iterated doubling of the line $K_2$, which has the eigenvalues 0 and 2. The next graph $Q_2$ (which is the cycle $C_4$) then acquires the eigenvalue 1 with multiplicity 2, once coming from the eigenvalue 0 via (10.2.4), and once coming from 2 via (10.2.3). At the next step, from (10.2.3) and (10.2.4), we get the eigenvalues $2/3$ and $4/3$ with multiplicity 3, with two of each coming from the two eigenvalues 1. Iterating this procedure proves the result. □

We note that the spectra of the cubes $Q_n$ are quite different from, and in particular behave asymptotically very different than, those of the other graph families we have discussed. In the next section, we want to develop an approach that can quantify that difference.

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There are also systematic constructions of classes of graphs that are not regular. The most important ones are the *random graphs* introduced by Erdős and Rényi [91]. Here, one has a collection of $N$ vertices and some $0 \leq p \leq 1$, and one connects any two vertices with probability $p$. These connections, that is, the edges of the graph, are randomly drawn independently of each other. For $p = 0$, we get of course a completely disconnected graph without any edges, whereas for $p = 1$, we get the complete graph $K_N$. For other values of $p$, the graph is usually not connected, but when $p$ is above a certain threshold (in fact, the threshold is $\frac{1}{N-1}$, as already shown by Erdős and Rényi), then with probability 1, there is a large component containing most of the vertices.

### 10.2.2 Spectral Asymptotics

We now want to compare the spectra of different graphs of possibly different size, and we want to define a distance between any two such spectra. For that purpose, we shall describe the constructions of [118, 119]. Let $\sigma(\Gamma) = \{\lambda_i, i = 1, \ldots, N\}$ denote the spectrum, that is, the collection of eigenvalues of the (possibly weighted, but finite and simple) graph $\Gamma$ with $N$ nodes. We consider here the vertex Laplacian $\Delta = L^0$, but obviously, an analogous theory can be developed for any (normalized) Laplacian on a simplicial complex or a hypergraph. And, one can also develop a corresponding theory for possibly infinite graphs, simplicial complexes and hypergraphs, but this is not our concern.

We define the measure

$$\mu_{\sigma(\Gamma)} := \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda_i), \tag{10.2.6}$$

where $\delta(\lambda)$ denotes the Dirac measure supported at $\lambda$. Thus, for any function $\phi\colon [0,2] \to \mathbb{R}$,

$$\mu_{\sigma(\Gamma)}(\phi) = \frac{1}{N} \sum_{i=1}^{N} \phi(\lambda_i). \tag{10.2.7}$$

Because of the normalizing factor $\frac{1}{N}$ and since the eigenvalues satisfy $0 \leq \lambda_i \leq 2$, $\mu_{\sigma(\Gamma)}$ is a probability measure on $[0,2]$. We can therefore invoke the theory of probability measures on compact metric spaces $(X,d)$. On the space $\mathcal{P}(X)$ of probability measures on $X$, we have the $p$-Wasserstein distance defined by

$$d_p(\mu,\nu) := \left( \inf_{\pi \in \Pi(\mu,\nu)} \int_{X\times X} d(x,y)^p \, d\pi(x,y) \right)^{1/p}, \tag{10.2.8}$$

where $\Pi(\mu,\nu)$ is the collection of all measures on $X \times X$ whose first marginal is $\mu$ and whose second marginal is $\nu$. That means that for all Borel subsets $A, B$

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of $X$, $\pi(A \times X) = \mu(A), \pi(X \times B) = \nu(B)$. This amounts to a *coupling*, and we shall describe it here for the case where $\mu$ and $\nu$, as in (10.2.6), are sums of rescaled Dirac measures. Thus, let

$$\mu = \sum_{i=1}^{N} a_i \delta(\lambda_i), \quad \nu = \sum_{j=1}^{M} b_j \delta(\sigma_j) \tag{10.2.9}$$

with $0 \leq \lambda_i, \sigma_j \leq 2, a_i, b_j \geq 0$ and $\sum a_i = 1 = \sum b_j$. Then, $\pi$ should be a measure of the form

$$\pi = \sum c_{ij} \delta(\lambda_i, \sigma_j) \tag{10.2.10}$$

with

$$c_{ij} \geq 0, \sum_{i,j} c_{ij} = 1, \sum_k c_{ik} = a_i, \sum_\ell c_{\ell j} = b_j \quad \text{for all } i, j.$$

Thus, from the perspective of $\mu$, by (10.2.9), each $\lambda_i$ is assigned a couple of $\sigma_j$'s so that the total weight remains $a_i$. The relation (10.2.8) then becomes

$$d_p(\mu, \nu) = \left( \inf_{(c_{ij})} \sum c_{ij} |\lambda_i - \sigma_j|^p \right)^{1/p}. \tag{10.2.11}$$

For $1 \leq p < \infty$, $d_p$ defines a metric on $\mathcal{P}(X)$, turning it into a compact metric space and inducing the topology of weak convergence of measures. Every $\mu \in \mathcal{P}(X)$ is a convex combination of Dirac measures supported at points of $X$. Therefore, in particular, the diameter of $(\mathcal{P}(X), d_p)$ equals the diameter of $(X, d)$. In our case, $X = [0,2]$ with the Euclidean metric ($d(x,y) = |x - y|$), and so the diameter of $(\mathcal{P}[0,2], d_p)$ is 2. We note, however,

**Lemma 10.2.2.** *When the average of $\mu$ is $= 1$, that is, $\int_{[0,2]} x d\mu(x) = 1$, then*

$$d_1(\mu, \nu) \leq 1 \tag{10.2.12}$$

*for any $\nu \in \mathcal{P}[0,2]$.*

*Proof* Since any probability measure is a convex combination of Dirac measures, it can be approximated by finite such combinations, and so, we may assume that $\mu, \nu$ are as in (10.2.9). The condition on the average of $\mu$ then becomes

$$\sum_{i=1}^{N} a_i \lambda_i = 1. \tag{10.2.13}$$

(10.2.13) and $\sum a_i = 1$ imply that when $0 \leq \sigma \leq 2$,

$$\sum_{i=1}^{N} a_i |\lambda_i - \sigma| \leq 1. \tag{10.2.14}$$

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Since also $\sum b_j = 1$, therefore also

$$\sum_{i,j} a_i b_j |\lambda_i - \sigma_j| \le 1. \tag{10.2.15}$$

Thus, putting $c_{ij} = a_i b_j$, (10.2.12) follows from (10.2.11). □

As in [118],

**Definition 10.2.1.** The *spectral p-distance* between two graphs $\Gamma_1, \Gamma_2$ is

$$d_p(\Gamma_1, \Gamma_2) := d_p(\mu_{\sigma(\Gamma_1)}, \mu_{\sigma(\Gamma_2)}). \tag{10.2.16}$$

We shall usually take $p = 1$ and in this case simply speak of the *spectral distance* between $\Gamma_1$ and $\Gamma_2$.

We note that this does not define a metric on the set of finite graphs, but only a pseudo-metric, because isospectral graphs have vanishing distance with respect to $d_p$.

We then have (with a simpler proof) an inequality of [118].

**Lemma 10.2.3.** *For any two (possibly weighted) graphs* $\Gamma_1, \Gamma_2$*,*

$$d_1(\Gamma_1, \Gamma_2) \le 1. \tag{10.2.17}$$

*Proof* When both graphs consist of isolated vertices only, with no edges, then their spectral measures both equal $\delta(0)$, and so, their spectral distance vanishes. Slightly expanding this argument, we may assume that one of them is connected and has more than one vertex. But then, its eigenvalues satisfy

$$\sum_{i=1}^{N} \lambda_i = N, \tag{10.2.18}$$

and hence, the spectral measure (10.2.6) satisfies (10.2.13) (with $a_i = \frac{1}{N}$ for all $i$). Lemma 10.2.2 then implies the result. □

We note that the result is not completely trivial because the diameter of $\mathcal{P}[0,2]$ is 2 and not 1. Therefore, we need (10.2.18).

There is a trivial example showing that the bound (10.2.17) is sharp. We let $\Gamma_1$ consist of an isolated vertex, hence $\mu_{\sigma(\Gamma_1)} = \delta(0)$, and $\Gamma_2$ has two vertices connected by an edge, hence $\mu_{\sigma(\Gamma_2)} = \frac{1}{2}(\delta(0) + \delta(2))$, and the spectral distance is 1. For a slightly less trivial example, let $\Gamma_1$ be a complete graph $K_N$ or a complete bipartite graph $K_{m,n}$. For $N \to \infty$ or $m + n \to \infty$, the spectral measure of $\Gamma_1$ then converges to $\delta(1)$, and the spectral distance from the two vertex graph $\Gamma_2$ converges to 1. In fact, inspection of the proof of Lemma 10.2.3 shows that such an extreme case can occur only if one of the two spectral

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measures is supported at 0 and 2, and this is possible only for graphs that have vertices only of degree 0 or 1. We can thus formulate

**Corollary 10.2.2.** *When one of the finite graphs is connected and has at least three vertices, then*

$$d_1(\Gamma_1, \Gamma_2) < 1. \tag{10.2.19}$$

□

We may now introduce a concept.

**Definition 10.2.2.** Let $(\Gamma_N)$ be some family of graphs, with $N$ denoting the number of vertices. When

$$\lim_{N\to\infty} \mu_{\sigma(\Gamma_N)} =: \mu_{\sigma(\Gamma_N)_{N\to\infty}} \tag{10.2.20}$$

exists as a weak limit of measures, we call it the *spectral class* of the family.

**Examples** can be obtained from the families discussed in Section 10.2.1:

- The families $K_N$ of complete and $K_{m,n}$ $(N = m + n)$ of complete bipartite graphs both belong to the spectral class $\delta(1)$.
- The petals $P_\ell$, books $B_\ell$, and water wheels $WW_\ell$ $(\ell \to \infty)$ belong to the spectral class $\frac{1}{2}\left(\delta\left(\frac{1}{2}\right) + \delta\left(\frac{3}{2}\right)\right)$.
- In contrast, by Corollary 10.2.1, the spectral class of the cubes $Q_\ell$ is not given by a finite sum of delta measures. The same holds for paths, cycles, or wheels whose spectra we have analyzed in Section 4.1.

We can also define and determine spectral classes for all geometric objects that possess Laplacian spectra, for instance, for families of oriented hypergraphs. For an oriented hypergraph $\Gamma$ on $N$ nodes, the corresponding *spectral measure* can be defined by (10.2.6). While for graphs, this is a probability measure on $[0,2]$, in the more general case, it is a probability measure on $[0,N]$. Similarly, also the *spectral class* of a family of hypergraphs can be defined by (10.2.20), as investigated in [215].

For example, if we consider the signless Laplacian $\Delta^+$ from Definition 4.9.5,

- For any fixed $c \geq 2$, the family of $c$-complete hypergraphs belongs to the spectral class $\delta(1)$, as a consequence of Lemma 4.9.3.
- For any fixed $t, M \in \mathbb{N}$, the family $\Gamma_N$ of hyperflowers with $t$ twins on $N$ nodes and $M$ edges belongs to the spectral class $\delta(0)$, as a consequence of Proposition 4.9.2.
- Instead of considering a family of hyperflowers in which the number of central nodes grows, as in the previous case, we can also consider a family of hyperflowers in which the number of hyperedges (and therefore the number

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of peripheral vertices) grows. In particular, for any fixed $t, c \in \mathbb{N}$, we can consider the family $\Gamma_{c+tM}$ of hyperflowers with $t$ twins on $c+tM$ nodes and $M$ edges. This belongs to the spectral class $\frac{t-1}{t} \cdot \delta(0) + \frac{1}{t} \cdot \delta(t)$, again by Proposition 4.9.2.

*Remark* 10.2.1. Several general notions of the convergence of families of graphs have been developed in the literature, and several of these also imply convergence properties of spectra. In this book, however, we do not go into this subject; relevant references include [2, 190, 280] and many more.

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