Barycentre

Viète's formula tells us that $-\frac{b}{a}$ is the sum of the roots. If we divide this identity by 3 (or more generally by the degree), we obtain the barycentre, or centre of mass, $\mu = -\frac{b}{3a}$ of the roots. From this perspective, what we seek when depressing a polynomial is simply to translate the roots so as to centre them around the origin. This can be done readily by translating each root by μ, i.e. through operation (1).

Example: The roots of the polynomial $x^3 - 3x^2 + x + 5$, are -1 and $2 \pm i$. Their barycentre is 1 and thus by applying the change of coordinates $x \to x - 1$ we obtain the depressed cubic $y^3 - 2y + 4$ whose roots are the latter but shifted so as to be balanced around the origin.

References

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109.40 On the Euler sequence

As shown in [1], the well-known Euler-Mascheroni constant $\gamma = 0.5772\ldots$ is the limit as $n \to \infty$ of the sequence $\{a_n\}$ with general term given by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n. \tag{1}$$

In this Note, we will examine the related sequence $\{b_n\}$ defined by

$$b_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \frac{4}{3}\log\left(n + \frac{1}{2}\right) + \frac{1}{6}\log\left(n(n+1)\right) \tag{2}$$

which also tends to γ,

Now, from (1), (2), we have the initial terms

$$a_1 = 1 \text{ and } b_1 = 1 - \frac{4}{3}\log\frac{3}{2} + \frac{1}{6}\log 2 = 0.5749\ldots \tag{3}$$

and the differences of consecutive terms (for $n > 1$)

$$\begin{aligned} a_n - a_{n-1} &= \frac{1}{n} - \log n + \log(n-1) \\ &= \frac{1}{n} + \log\left(1 - \frac{1}{n}\right) = -\frac{1}{2n^2} + O\left(\frac{1}{n^3}\right), \end{aligned} \tag{4}$$

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$$\text{and } b_n - b_{n-1} = \frac{1}{n} - \frac{4}{3}\log\left(\frac{n+\frac{1}{2}}{n-\frac{1}{2}}\right) + \frac{1}{6}\log\left(\frac{n+1}{n-1}\right)$$
$$= \frac{1}{n} - \frac{4}{3}\log\left(\frac{1+\frac{1}{2n}}{1-\frac{1}{2n}}\right) + \frac{1}{6}\log\left(\frac{1+\frac{1}{n}}{1-\frac{1}{n}}\right)$$
$$= \frac{1}{n} - \frac{8}{3}\left(\frac{1}{2n} + \frac{1}{3(2n)^3} + \frac{1}{5(2n)^5} + O\left(\frac{1}{n^7}\right)\right)$$
$$+ \frac{1}{3}\left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5}\right) + O\left(\frac{1}{n^7}\right) = \frac{1}{20n^5} + O\left(\frac{1}{n^7}\right). \quad (5)$$

Hence not only the initial terms (3) but also the consecutive term differences (4), (5) would seem to indicate that the new increasing monotone sequence $\{b_n\}$ converges to γ more rapidly than the original decreasing monotone sequence $\{a_n\}$.

Note finally that, while $a_{20} > 0.6$, we have $b_{20} = 0.57721559\ldots$ which compares well with $\gamma = 0.5772156649\ldots$ gleaned from the Internet.

Reference

1. J. C. Burkill, *A first course in mathematical analysis*, CUP (1970) pp. 141-142.

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109.41 Elementary proofs of some trigonometric inequalities

In this Note, we present some results concerning inequalities using both geometric and trigonometric methods.

Geometrical proofs

In order to prove the inequality

$$\frac{\tan x + \tan y}{2} \geqslant \tan\left(\frac{x+y}{2}\right),\ x \in \left[0, \tfrac{1}{2}\pi\right),\ y \in \left[0, \tfrac{1}{2}\pi\right), \quad (1)$$

we are using Figure 1, where the semicircle is a part of the circle of radius $OA = 1$, and the segment EA is tangent to a given circle at the point A.

Let $x = \angle AOB$, $y = \angle AOE$. According to Figure 1, if C_1 is the midpoint of arc $\overset{\frown}{B_1E_1}$, then $\overset{\frown}{B_1C_1} = \overset{\frown}{C_1E_1} = \frac{1}{2}\overset{\frown}{B_1E_1} = \frac{1}{2}(y - x)$.

The point D is the midpoint of the segment BE, OC represents an angle-bisector and OD a median in the triangle BOE. Using the result that a median is at least as long as the corresponding angle-bisector, [1], then $OD \geqslant OC$ and after that $AD \geqslant AC$, equivalent to

$$AD = AB + BD = AB + \frac{AE - AB}{2} = \frac{AB + AE}{2} = \frac{\tan x + \tan y}{2},$$
$$AC = \tan\angle AOC_1 = \tan\left(x + \frac{y-x}{2}\right) = \tan\left(\frac{x+y}{2}\right).$$

The inequality (1) is validated.

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