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Shadowing, thick sets and the Ramsey property

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Abstract. We provide a full characterization of relations between the shadowing property and the thick shadowing property. We prove that they are equivalent properties for non-wandering systems, the thick shadowing property is always a consequence of the shadowing property, and the thick shadowing property on the chain-recurrent set and the thick shadowing property are the same properties. We also provide a full characterization of the cases when for any family $\mathcal{F}$ with the Ramsey property an arbitrary sequence of points can be ε-traced over a set from $\mathcal{F}$.

1. *Introduction*

The notion of the shadowing property has a long tradition in studies of the qualitative behavior of dynamical systems (e.g. see [**2**]). An interesting direction in these studies was initiated in [**8**], where it is proved that topologically mixing surjection with the shadowing property is equivalently characterized by the ergodic shadowing property, which in our terminology is the $(\mathcal{D}, \mathcal{D})$-shadowing property (see the next section for all the definitions). Roughly speaking, if we allow larger 'jumps' in the definition of pseudo orbit (that is, we do not control errors in calculations of approximate trajectories in some iterations) but also adjust demands on tracing, allowing 'mistakes' in shadowing of the given sequence provided that they are sufficiently rare, then this new property is in fact the standard shadowing property with some additional demands on the dynamical properties (i.e. topological mixing).

At the same time, there are some practical consequences of allowing pseudo orbits to be defined over sets in $\mathcal{D}$. To put it simply, if we do not demand control of error in some iterations, then we can combine pseudo orbits calculated separately which gives us more freedom in practical as well as theoretical studies of dynamics of the system. At the same time we need to be able to say something about the relation of these sequences to the genuine orbits in the dynamics (hence we need some notion of shadowing) and additionally not to restrict too much the class of systems in which this property can be

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present. The most desirable situation is to find another, equivalent characterization of the shadowing property, which allows us to use 'non-complete' pseudo orbits (e.g. pseudo orbits defined over sets in the family $\mathcal{D}$, that is, so-called ergodic pseudo orbits).

In [**6**] the authors introduced the thick shadowing property (i.e. $(\mathcal{D}, \mathcal{F}_t)$-shadowing property) which is a weaker form of the ergodic shadowing property. They also mentioned that in chain-transitive systems, the thick shadowing property implies the shadowing property (see [**6**, p. 3764]). Later, an example was provided in [**12**] whereby the thick shadowing property can exist in a non-mixing dynamical system. Finally, [**4**] proved that in (chain-)transitive dynamical systems shadowing and thick shadowing are equivalent notions.

The aim of the present paper is to provide a full characterization of relations between the shadowing property and the thick shadowing property, similar to what was done in [**8**] for the ergodic shadowing property. Strictly speaking, we prove that the following conditions are equivalent (see Theorem 4.5):

(1) $\mathrm{CR}(f) = \Omega(f) = \overline{\mathrm{Rec}(f)}$ and the dynamical system $(\mathrm{CR}(f), f|_{\mathrm{CR}(f)})$ has the shadowing property;

(2) (X, f) has the thick shadowing property.

This completes an interesting chapter in the study of various types of shadowing properties, initiated in [**4**] by Brian *et al*.

2. *Preliminaries*

In the paper (X, ρ) is always a compact metric space. A set which is closed and open at the same time is called *clopen*.

Denote by 2^X the set of all compact subsets of X. For any point $x \in X$ and a non-empty subset A of X, we define the distance from the point x to the set A by $\mathrm{dist}(x, A) = \inf\{\rho(x, y) : y \in A\}$. For any set $A \in 2^X$ and $\varepsilon > 0$, let $D_\rho(A, \varepsilon) = \{x \in X : \mathrm{dist}(x, A) < \varepsilon\}$. We may endow 2^X with the *Hausdorff distance* $\mathscr{H}_\rho$, which is defined as

$$\mathscr{H}_\rho(A, B) = \inf\{\varepsilon > 0 : A \subset D_\rho(B, \varepsilon) \text{ and } B \subset D_\rho(A, \varepsilon)\}$$

provided that $A, B \in 2^X$. It is well known that since X is compact, $(2^X, \mathscr{H}_\rho)$ is also a compact metric space. Hence we call $\mathscr{H}_\rho$ the *Hausdorff metric*.

We define the *lower* and *upper asymptotic density* of a set $J \subset \mathbb{N}$ as

$$d_*(J) = \liminf_{n\to\infty} \frac{1}{n}\#(J \cap [0, n)) \quad \text{and} \quad d^*(J) = \limsup_{n\to\infty} \frac{1}{n}\#(J \cap [0, n)),$$

respectively. The set J is said to have the *asymptotic density* $d(J)$ provided that $d_*(J) = d^*(J) = d(J)$. A set $J \subset \mathbb{N}$ has *positive upper Banach density* if and only if, for some sequence of integers $\langle k_n\rangle_{n=1}^{\infty}$,

$$\limsup_{n\to\infty} \frac{1}{n}\#(J \cap [k_n, k_n + n)) > 0.$$

2.1. *Furstenberg families.*

A *Furstenberg family* or simply a *family* on $\mathbb{N}$ is any set $\mathcal{F} \subset \mathbb{N}$ which is upwards hereditary; that is, if $A \in \mathcal{F}$ and $A \subset B \subset \mathbb{N}$ then $B \in \mathcal{F}$. A *dual family* for $\mathcal{F}$, denoted $\mathcal{F}^*$, consists of sets that meet every element of $\mathcal{F}$, that is, $A \in \mathcal{F}^*$ provided that $\mathbb{N}\backslash A \notin \mathcal{F}$. Clearly, $\mathcal{F}^{**} = \mathcal{F}$ always.

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A family $\mathcal{F}$ has *partition regularity* or the *Ramsey property* if, whenever $F \in \mathcal{F}$ and $F = \bigcup_{i=1}^{n} F_i$ then there is some j such that $F_j \in \mathcal{F}$.

A set $A \subset \mathbb{N}$ is *syndetic* if it has bounded gaps, that is, there is $N > 0$ such that $[i, i+N] \cap A$ for every $i > 0$. The family of syndetic sets is denoted $\mathcal{F}_s$ and its dual is denoted $\mathcal{F}_t = \mathcal{F}_s^*$. A set A is *thick* if it contains arbitrary long block of consecutive integers, that is, $A \in \mathcal{F}_t$. We identify $\mathbb{N}$ with the trivial family $\{\mathbb{N}\}$ and denote by $P(\mathbb{N})$ the family consisting of all subsets of $\mathbb{N}$ (including the empty set). A set A is *piecewise syndetic* if $A = T \cap S$ for some $T \in \mathcal{F}_t$ and $S \in \mathcal{F}_s$. Denote by $\mathcal{F}_{ps}$ the family of piecewise syndetic sets. By [**3**, Theorem 2.3.] the family $\mathcal{F}_{ps}$ has the Ramsey property.

A set F is *thickly syndetic* if $\{n \in \mathbb{N} : n + j \in F \text{ for } 0 \le j \le k\}$ is syndetic for every k. Clearly, a set is thickly syndetic if and only if it belongs to the dual family $\mathcal{F}_{ps}^*$.

Denote by $\mathcal{D}$ the family of sets with density one. It is clear that if $A \subset \mathbb{N}$ is such that $\mathbb{N} \backslash A \in \mathcal{F}_s$ then $d^*(A) \le 1 - d_*(\mathbb{N} \backslash A) < 1$. In particular, $\mathcal{D} \subset \mathcal{F}_t$. We denote by $\mathcal{F}_{\text{pubd}}$ the family of sets with *positive upper Banach density* and by $\mathcal{F}_{cf}$ the family of co-finite sets, that is, sets with finite complements.

For a sequence $\langle p_i \rangle_{i=1}^{\infty}$ and any finite set $\alpha \subset \mathbb{N}$, denote $p_\alpha = \sum_{i \in \alpha} p_i$. The set of finite sums of $\langle p_i \rangle_{i=1}^{\infty}$ is denoted

$$\mathrm{FS}(\langle p_i \rangle_{i=1}^{\infty}) = \{p_\alpha : \alpha \text{ is a non-empty finite subset of } \mathbb{N}\}.$$

We say that $F \subset \mathbb{N}$ is an *IP-set* if there exists a sequence $\langle p_i \rangle_{i=1}^{\infty}$ in $\mathbb{N}$ such that $\mathrm{FS}(\langle p_i \rangle_{i=1}^{\infty}) \subset F$. The family of all IP-sets is denoted $\mathcal{F}_{\text{ip}}$. It is not hard to construct an IP-set in any thick set, hence $\mathcal{F}_t \subset \mathcal{F}_{\text{ip}}$. By Hindman's theorem the family $\mathcal{F}_{\text{ip}}$ has the Ramsey property (see [**7**] for more properties of IP-sets).

2.2. *Topological dynamics.* A *(topological) dynamical system* is a pair (X, f) consisting of a compact metric space (X, ρ) and a continuous map $f : X \to X$. If X is a singleton, then we say that (X, f) is *trivial*. If $K \subset X$ is a non-empty closed subset satisfying $f(K) \subset K$, then we say that (K, f) is a *subsystem* of (X, f), where we identify f with $f|_K$. The *orbit* of a point x is denoted $\mathrm{O}^+(x) = \{x, f(x), f^2(x), \ldots\}$.

A point $x \in X$ is *periodic* if $f^n(x) = x$ for some $n \in \mathbb{N}$ and *recurrent* if $\liminf_{n\to\infty} \rho(f^n(x), x) = 0$. The set consisting of all recurrent points is denoted $\mathrm{Rec}(f)$. A point $x \in X$ is *non-wandering* if there exist a sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ and an increasing sequence $\langle k_n \rangle_{n=1}^{\infty} \subset \mathbb{N}$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} f^{k_n}(x_n) = x$. The set of all non-wandering points is denoted $\Omega(f)$. A dynamical system is *non-wandering* if $X = \Omega(f)$.

A dynamical system (X, f) is *minimal* if $\overline{\mathrm{O}^+(x)} = X$ for every $x \in X$ and *transitive* if there is $x \in X$ such that $\overline{\mathrm{O}^+(f(x))} = X$. A point $x \in X$ is *minimal* if the subsystem $(\overline{\mathrm{O}^+(x)}, f)$ is minimal.

A pair of points $x, y \in X$ is *proximal* if $\liminf_{n\to\infty} \rho(f^n(x), f^n(y)) = 0$ and *syndetically proximal* if for every $\varepsilon > 0$ we have $\{n : \rho(f^n(x), f^n(y)) < \varepsilon\} \in \mathcal{F}_s$. If every pair of points from X is proximal (respectively, syndetically proximal) then we say that (X, f) is *proximal* (respectively, *syndetically proximal*). A dynamical system (X, f) is proximal if and only if it (X, f) has a unique minimal subsystem which is a fixed point (see [**1**]). It is also known that a system is proximal if and only if it is syndetically proximal (see [**5**]).

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2.3. *Shadowing.* Fix $A \subset \mathbb{N}$ and $\delta, \varepsilon > 0$. A sequence $\langle x_i \rangle_{i=0}^{\infty}$ is a *δ-pseudo orbit on A* if $A \subset \{i : \rho(f(x_i), x_{i+1}) < \delta\}$. If ξ is a δ-pseudo orbit on $\mathbb{N}$ then we simply say that ξ is a *δ-pseudo orbit*.

A point $y \in X$ *ε-shadows* (or *ε-traces*) a sequence $\langle y_i \rangle_{i=1}^{\infty}$ on A if $A \subset \{i : d(f^i(y), y_i) < \varepsilon\}$. Again, if $A = \mathbb{N}$ then we simply say that y ε-shadows $\langle y_i \rangle_{i=1}^{\infty}$.

Fix any families $\mathcal{F}, \mathcal{G} \subset \mathbb{N}$. We say that (X, f) has the *$(\mathcal{F}, \mathcal{G})$-shadowing property* if for every $\varepsilon > 0$ there is $\delta > 0$ such that if ξ is a δ-pseudo orbit on a set in $\mathcal{F}$ then there is a point x which ε-shadows ξ on a set in $\mathcal{G}$. We say that f has the *shadowing property* if it has the $(\mathbb{N}, \mathbb{N})$-shadowing property. This definition clearly coincides with the standard definition of the shadowing property (see [**2**]). Another property of this kind is the *thick shadowing property*, which in our terminology is the $(\mathcal{D}, \mathcal{F}_t)$-shadowing property (see [**6**]). Finally, if (X, f) has the $(\mathcal{D}, \mathcal{D})$-shadowing property then we say that (X, f) has the *ergodic shadowing property* (see [**8**]).

A δ-chain (of length n) between x and y is any sequence $\langle x_i \rangle_{i=0}^{n}$ such that $\rho(f(x_i), x_{i+1}) < \delta$ for each $i = 0, \ldots, n-1$ and $x = x_0$, $y = x_n$. Similarly, a δ-chain $\langle x_i \rangle_{i=0}^{n}$ is ε-traced by a point $y \in X$, if $\rho(f^i(y), x_i) < \varepsilon$ for $i = 0, \ldots, n$. It is not hard to see that (X, f) has the shadowing property if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-chain can be ε-traced.

A point x is *chain recurrent* if for every $\delta > 0$ there is a δ-chain from x to x. The set of all chain recurrent points is denoted $\mathrm{CR}(f)$. Using compactness, we easily obtain that $\mathrm{CR}(f)$ is a closed set and for every $y \in \mathrm{CR}(f)$ there is $x \in \mathrm{CR}(f)$ such that $f(x) = y$, that is, $f(\mathrm{CR}(f)) = \mathrm{CR}(f)$. A dynamical system (X, f) is *chain recurrent* if $X = \mathrm{CR}(f)$. If for every $x, y \in X$ and every $\delta > 0$ there exists a δ-chain from x to y then (X, f) is *chain transitive*.

3. *Shadowing implies thick shadowing*

The main result of [**4**] shows that the shadowing property and the thick shadowing property are equivalent under the assumption of chain transitivity (hence, when the system is transitive). The authors also provided an example in which a dynamical system can have the thick shadowing property without the shadowing property. They then asked if the shadowing property suffices to induce the thick shadowing property (see [**4**, Question 3.3]). The aim of this section is to provide an affirmative answer to this question. We will also develop a few auxiliary results, used later in the paper.

Before we start with the proof, we present a simple example which should give the reader an intuition as to why the above-mentioned implication has a chance of holding.

Example 3.1. Let $C = \{0, 1\}^{\mathbb{N}}$ endowed with the metric ρ given by $\rho(x, y) = 2^{-k}$ provided that $x \neq y$, where $k = \min\{j : x(j) \neq y(j)\}$, and $\rho(x, y) = 0$ otherwise. It is not hard to see that (C, ρ) is a compact metric space homeomorphic to the standard Cantor set. Take as $f : C \to C$ the identity map $f(x) = x$.

Note that if we fix any $\varepsilon > 0$ and any $x \in C$ then the set $B_x = \{y : \rho(x, y) < \varepsilon\}$ is clopen and for every $y, z \in C$, if $y \in B_x$ and $\rho(y, z) < \varepsilon$ then $z \in B_x$. Then the sets

$$\mathcal{P} = \{B_x : x \in C\} = \{P_1, \ldots, P_s\}$$

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form a finite clopen partition of C, and, furthermore, if $\langle x_i\rangle_{i=0}^n$ is an ε-chain and $x_0 \in P_j$ for some j then $\langle x_i\rangle_{i=0}^n \subset P_j$. Then any point in P_j is ε-tracing this chain, hence f has the shadowing property (see [**9**] for more details).

Now let ξ be an ε-pseudo orbit on a set in $\mathcal{F}_t$. Elements of ξ can visit any of the sets P_j in complicated order; however, for every n there is a run of n consecutive elements of ξ which is an ε-chain of length $n-1$, and by the same argument as before, it must be contained completely in P_{j_n} for some j_n. Since we have finitely many choices for j_n for different n, infinitely many of the j_n have the same value, say equal to m. But then any element of P_m is ε-tracing all these runs (contained in P_m) and as a result is ε-tracing ξ on a thick set, showing that (C, f) has the thick shadowing property.

In his paper [**14**], Robinson developed an interesting technique to prove that $\mathrm{CR}(f|_{\mathrm{CR}(f)})$ coincides with $\mathrm{CR}(f)$. We will employ this approach in the proof of the next result.

LEMMA 3.2. *For every $\varepsilon > 0$ there is $\delta > 0$ such that if $\langle x_0, \dots, x_n\rangle$ is a δ-chain from x_0 to x_0 then there is an ε-chain $\langle y_0, \dots, y_n\rangle \subset \mathrm{CR}(f)$ from y_0 to y_0 such that $\rho(x_i, y_i) < \varepsilon$ for $i = 0, \dots, n$.*

Proof. First, we claim that for every $\varepsilon > 0$ there is $\delta > 0$ such that if $\langle x_0, \dots, x_n\rangle$ is a δ-chain from x_0 to x_0 then there is an $\varepsilon/3$-chain $\langle y_0, \dots, y_n\rangle \subset \mathrm{CR}(f)$ such that $\rho(x_i, y_i) < \varepsilon/3$ for $i = 0, \dots, n$. Assume to the contrary that there is $\varepsilon > 0$ such that for every $k > 0$ there exist $n_k > 0$ and a $1/k$-chain $\langle x_0^{(k)}, \dots, x_{n_k}^{(k)}\rangle$ from $x_0^{(k)}$ to $x_0^{(k)}$ such that for every $\varepsilon/3$-chain $\langle y_0, \dots, y_{n_k}\rangle \subset \mathrm{CR}(f)$ there is $0 \le i \le n_k$ such that $\rho(x_i^{(k)}, y_i) \ge \varepsilon/3$.

Denote $C_k = \{x_j^{(k)} : j = 0, \dots, n_k\} \in 2^X$. The space $(2^X, \mathscr{H}_\rho)$ is compact, hence without loss of generality we may assume that there is $C \in 2^X$ such that $\lim_{k\to\infty} \mathscr{H}_\rho(C_k, C) = 0$. Note that for every $z \in C$ there is a sequence of points $z_k \in C_k$ such that $\lim_{k\to\infty} \rho(z_k, z) = 0$, therefore $z \in \mathrm{CR}(f)$ and as a result $C \subset \mathrm{CR}(f)$.

There is $\eta \in (0, \varepsilon/9)$ such that if $\rho(p, q) < \eta$ then $\rho(f(p), f(q)) < \varepsilon/9$. Take $k > 0$ such that $1/k < \varepsilon/9$ and $\mathscr{H}_\rho(C_k, C) < \eta$. Then, by the definition of $\mathscr{H}_\rho$, for every $i = 0, \dots, n_k$ we can find $y_i \in C$ such that $\rho(x_i^{(k)}, y_i) < \eta < \varepsilon/3$. But the sequence $\langle y_0, \dots, y_n\rangle \subset C \subset \mathrm{CR}(f)$ is also an $\varepsilon/3$-chain, because

$$\begin{aligned}\rho(f(y_i), y_{i+1}) &\le \rho(f(y_i), f(x_i^{(k)})) + \rho(f(x_i^{(k)}), x_{i+1}^{(k)}) + \rho(x_{i+1}^{(k)}, y_{i+1})\\ &\le \frac{\varepsilon}{9} + \frac{1}{k} + \eta < \frac{\varepsilon}{3}.\end{aligned}$$

This is a contradiction with the choice of the sequence $\langle x_0^{(k)}, \dots, x_{n_k}^{(k)}\rangle$, hence the claim is proved.

To finish the proof, observe that if $\langle x_0, \dots, x_n\rangle$ is a δ-chain from x_0 to x_0, $\langle y_0, \dots, y_n\rangle$ is an $\varepsilon/3$-chain and $\rho(x_i, y_i) < \varepsilon/3$, then

$$\rho(f(y_{n-1}), y_0) \le \rho(f(y_{n-1}), y_n) + \rho(y_n, x_0) + \rho(x_0, y_0) < \varepsilon$$

hence $\langle y_0, \dots, y_{n-1}, y_0\rangle$ is an ε-chain from y_0 to y_0 and $\rho(x_i, y_i) < \varepsilon$ for $i = 0, \dots, n$. The proof is complete. □

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LEMMA 3.3. *For every $\varepsilon > 0$ there is $\delta > 0$ such that for every thick set T and δ-pseudo orbit $\langle x_i\rangle_{i=1}^{\infty}$ on T there exist a thick set $T' \subset T$ and an ε-pseudo orbit $\langle y_i\rangle_{i=1}^{\infty} \subset \mathrm{CR}(f)$ on T' such that $\rho(x_i, y_i) < \varepsilon$ for every $i \in T'$.*

Proof. Let $\eta \in (0, \varepsilon/2)$ be provided for $\varepsilon/2 > 0$ by Lemma 3.2. There is $\delta \in (0, \eta/3)$ such that if $\rho(p, q) < \delta$ then $\rho(f(p), f(q)) < \eta/3$. Let $\mathcal{U} = \{U_1, \ldots, U_s\}$ be a cover of X by open sets U_j with $\mathrm{diam}(U_j) < \delta$. For every j, chose a point $p_j \in U_j$.

Fix any thick set T and any δ-pseudo orbit $\langle x_i\rangle_{i=1}^{\infty}$ on T. For every $i \geq 1$ fix an index $j(i)$ such that $U_{j(i)} \ni x_i$ and denote $z_i = p_{j(i)}$. Then $\rho(z_i, x_i) < \eta/3 < \varepsilon/2$ for every $i = 1, 2, \ldots$ and

$$\begin{aligned}\rho(f(z_i), z_{i+1}) &\leq \rho(f(z_i), f(x_i)) + \rho(f(x_i), x_{i+1}) + \rho(x_{i+1}, z_{i+1})\\ &\leq \frac{2\eta}{3} + \rho(f(x_i), x_{i+1}).\end{aligned}$$

In particular, $\langle z_i\rangle_{i=1}^{\infty}$ is an η-pseudo orbit on T and $\{z_i : i \in \mathbb{N}\} \subset \{p_i : 1 \leq i \leq s\}$. Then, there exist an integer $1 \leq l \leq s$ and two increasing sequences of integers $a_k < b_k < a_{k+1} - 1$ such that $\lim_{k\to\infty} b_k - a_k = \infty$ and $\{a_k, a_k + 1, \ldots, b_k\} \subset T$ and $z_{a_k} = z_{b_k} = p_l$ for every k. In other words, each sequence

$$\langle z_{a_k}, z_{a_k+1}, \ldots, z_{b_k}\rangle$$

is an η-chain from p_l to p_l. By Lemma 3.2, for every k there exists an $\varepsilon/2$-chain $\langle y_{a_k}, y_{a_k+1}, \ldots, y_{b_k}\rangle \subset \mathrm{CR}(f)$ such that $\rho(y_i, z_i) < \varepsilon/2$. Put $T' = \mathbb{N} \cap \bigcup_{k=1}^{\infty}[a_k, b_k]$ and fix any point $q \in CR(f)$. For $i \notin T'$ put $y_i = q$. Note that $[a_s, b_s] \cap [a_t, b_t] = \emptyset$ for $s \neq t$ and so the sequence $\langle y_i\rangle_{i=1}^{\infty}$ is an ε-pseudo orbit on T', and for every $i \in T'$ we also have

$$\rho(y_i, x_i) \leq \rho(y_i, z_i) + \rho(z_i, x_i) < \varepsilon/2 + \delta < \varepsilon.$$

The proof is finished. □

The technique used in the proof of Lemma 3.3 can be also used to obtain the following result.

LEMMA 3.4. *For every $\varepsilon > 0$ there is $\delta > 0$ such that for every δ-pseudo orbit $\langle x_i\rangle_{i=1}^{\infty}$ there exist an ε-pseudo orbit $\langle y_i\rangle_{i=1}^{\infty} \subset \mathrm{CR}(f)$ and $N > 0$ such that $\rho(x_i, y_i) < \varepsilon$ for every $i \geq N$.*

Proof. Let $\eta \in (0, \varepsilon/2)$ be provided for $\varepsilon/2 > 0$ by Lemma 3.2. There is $\delta \in (0, \eta/3)$ such that if $\rho(p, q) < \delta$ then $\rho(f(p), f(q)) < \eta/3$. Let $\mathcal{U} = \{U_1, \ldots, U_s\}$ be a cover of X by open sets U_j with $\mathrm{diam}(U_j) < \delta$. For every j, choose a point $p_j \in U_j$.

Fix any δ-pseudo orbit $\langle x_i\rangle_{i=1}^{\infty}$ and, for every i, find $j(i)$ such that $U_{j(i)} \ni x_i$, and denote $z_i = p_{j(i)}$. Then for every $i \geq 1$ we have $\rho(z_i, x_i) < \varepsilon/2$ and $\rho(f(z_i), z_{i+1}) < \eta$. Since $\{z_i : i \in \mathbb{N}\} \subset \{p_i : 1 \leq i \leq s\}$, there exist an integer $1 \leq l \leq s$ and an increasing sequence $\langle a_k\rangle_{k=1}^{\infty}$ such that $z_{a_k} = z_{a_{k+1}} = p_l$ for every $k \geq 1$. In other words, the sequence $\langle x_i\rangle_{i=a_1}^{\infty}$ is a concatenation of η-chains from p_l to p_l. By Lemma 3.2, for every k there exists an $\varepsilon/2$-chain $\langle y_{a_k}, y_{a_k+1}, \ldots, y_{a_{k+1}}\rangle \subset \mathrm{CR}(f)$ such that $\rho(y_i, z_i) < \varepsilon/2$. While it is not directly written in the statement of the Lemma 3.2, we may assume that the starting point of each constructed chain is the same, that is, $y_{a_1} = y_{a_k}$ for every k. We define $\langle y_i\rangle_{i=1}^{a_1}$ to be the

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last a_1 elements in the ε-chain $\langle y_{a_1}, y_{a_1+1}, \ldots, y_{a_{a_1+1}}\rangle$, which clearly has more than a_1 elements, thus obtaining an ε-pseudo orbit $\langle y_i\rangle_{i=1}^{\infty} \subset \mathrm{CR}(f)$ and additionally $\rho(y_i, x_i) < \varepsilon$ for every $i \geq a_1$. It is enough to set $N = a_1$ to complete the proof. □

Fix any $\delta > 0$ and define a relation $\sim_\delta$ on $\mathrm{CR}(f)$ by putting $x \sim_\delta y$ if there is a δ-chain from x to y. The following result is inspired by [**2**, Theorem 3.4.2].

LEMMA 3.5. *For any $\delta > 0$, the relation $\sim_\delta$ is an equivalence relation on* $\mathrm{CR}(f)$ *and there are finitely many equivalence classes which are clopen sets.*

Proof. Clearly $x \sim_\delta x$ for every $x \in \mathrm{CR}(f)$, and directly by the definition we also have that $x \sim_\delta z$ provided that $x \sim_\delta y$ and $y \sim_\delta z$. Next, observe that if $x, y \in \mathrm{CR}(f)$ are such that $\rho(f(x), y) < \delta$ then there is $\eta \in (0, \delta/2)$ such that $\rho(f(x), y) < \delta - 2\eta$. There are also η-chains $\langle x_0, \ldots, x_n\rangle$ and $\langle y_0, \ldots, y_m\rangle$ from x to x and from y to y, respectively. Then the sequence

$$\langle y, y_1, \ldots, y_{m-1}, x_1, \ldots, x_n\rangle$$

is a δ-chain from y to x, because

$$\begin{aligned}\rho(f(y_{m-1}), x_1) &\leq \rho(f(y_{m-1}), y) + \rho(y, f(x)) + \rho(f(x), x_1)\\ &< \eta + (\delta - 2\eta) + \eta = \delta.\end{aligned}$$

By induction, if there is a δ-chain from x to y contained completely in $\mathrm{CR}(f)$ (that is, $x \sim_\delta y$) then there is also a δ-chain contained in $\mathrm{CR}(f)$ from y to x. This shows that $\sim_\delta$ is an equivalence relation on $\mathrm{CR}(f)$.

Fix any $x \in \mathrm{CR}(f)$ and any δ-chain $\langle x_i\rangle_{i=1}^{n}$ from x to x. There is $\eta > 0$ such that if $\rho(z, x_1) < \eta$ then $\rho(x_2, f(z)) < \delta$ and $\rho(f(x_{n-1}), z) < \delta$. In other words, for every $x \in \mathrm{CR}(f)$ there is an open set U such that $x \sim_\delta z$ for every $z \in U \cap \mathrm{CR}(f)$. This shows that equivalence classes $[x]_{\sim_\delta}$ are open, and hence by compactness there are finitely many of them. But two equivalence classes are either disjoint or identical, hence the result follows. □

The proof of the following fact follows the same lines as the proof of [**13**, Lemma 7]. We present it for completeness.

LEMMA 3.6. *Fix $\delta > 0$ and let $A = [z]_{\sim_\delta}$ for some $z \in \mathrm{CR}(f)$. Then there exists $k_A \geq 1$ such that for every $x \in A$ the number k_A is the greatest common divisor of the lengths of all δ-chains from x to itself contained in* $\mathrm{CR}(f)$.

Proof. Fix any $x \in A$ and let k be the greatest common divisor of the lengths of all δ-chains from x to itself contained in $\mathrm{CR}(f)$. Take any $y \in A$ and any δ-chain $\langle y_0, \ldots, y_n\rangle$ from y to itself. By the definition of A we have $x \sim_\delta y$, hence there exists a δ-chain $\langle x_0, \ldots, x_m\rangle$ from x to itself such that $x_i = y$ for some $1 < i < m$. But then the sequence

$$\langle x_0, \ldots, x_i, y_1, \ldots, y_n, x_{i+1}, \ldots, x_m\rangle$$

is a δ-chain from x to x with length $m + n$. Since k divides m and $m + n$ it also divides n, hence it divides the greatest common divisor of the lengths of all δ-chains from y to itself contained in $\mathrm{CR}(f)$. But the situation is symmetric, so k is uniquely determined for A (i.e. does not depend on the choice of x). Putting $k_A = k$ completes the proof. □

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The following result is inspired by [**4**, Lemma 2.3] (see also the proof of [**13**, Theorem 6]).

LEMMA 3.7. *Let $\varepsilon, \delta > 0$ be such that every δ-pseudo orbit contained in $\mathrm{CR}(f)$ can be ε-traced. If T is a thick set and $\langle x_i \rangle_{i=1}^{\infty} \subset \mathrm{CR}(f)$ is a δ-pseudo orbit on T then there exist a thick set $T' \subset T$ and z such that $\rho(f^j(z), x_j) < \varepsilon$ for every $j \in T'$.*

Proof. Let $\langle x_i \rangle_{i=1}^{\infty} \subset \mathrm{CR}(f)$ and $T \subset \mathbb{N}$ be as in the assumptions of the theorem. There are increasing sequences $k_n < t_n - n^3 < t_n < k_{n+1}$ such that $[k_n, t_n] \cap \mathbb{N} \subset T$, so, in particular, $\rho(f(x_i), x_{i+1}) < \delta$ for every $i \in [k_n, t_n) \cap \mathbb{N}$. If we put $A = [x_{k_n}]_{\sim_\delta}$ then clearly $\langle x_i \rangle_{i=k_n}^{t_n} \subset A$. Since there are only finitely many different equivalence classes under the relation $\sim_\delta$, we may assume that $[x_{k_1}]_{\sim_\delta} = [x_{k_n}]_{\sim_\delta}$ for every n, that is, A is independent of the choice of n.

Let $k = k_A$ be provided by Lemma 3.6 for the class A. Define a relation $\approx$ on A by putting $x \approx y$ provided that there is a δ-chain from x to y and of length which is a multiple of k. Clearly $x \approx x$ and $x \approx z$ when $x \approx y$ and $y \approx z$. If $x \approx y$ then there is a δ-chain ξ from x to y with length divisible by k, and since both points $x, y \in A$ there is also a δ-chain ψ from y to x. The chain ψ combined with ξ gives a δ-chain from x to x whose length, by the definition of k, is divisible by k. Since ξ has also length divisible by k, the length of ψ must have this property too. This shows that $y \approx x$, hence $\approx$ is an equivalence relation on A. Repeating the same argument as in the case of $\sim_\delta$, we obtain that there are finitely many distinct classes $[x]_\approx$ in A which are clopen.

Fix any $q \in A$. By the definition of k, there must exist $s \geq 2$ and δ-chains from q to q of lengths p_i, where $i = 1, \ldots, s$, such that the greatest common divisor of $p_1, \ldots, p_s$ is k. It is not hard to see that there are integers a_i such that $k = \sum_{i=1}^{s} a_i p_i$ (see [**10**, Excercise 4.5.5.]). Combining these chains properly, we obtain a number $r = -\sum_{a_i<0} a_i p_i / k$ such that there are chains ξ, ψ from q to q of length rk and $(r+1)k$, respectively.

Using the compactness of A, we easily obtain that there is $M > 0$ such that for every $x \in [q]_\approx$ there is a δ-chain from x to q and from q to x, both of length bounded by M and divisible by k. But then, using appropriate concatenations of chains ξ, ψ, it is not hard to verify that there is $N > 0$ such that for every $n > N$ and every $x, y \in [q]_\approx$ there is a δ-chain from x to y of length nk. We can repeat this argument for any $q \in A$ and its equivalence class $[q]_\approx$, but since there are only finitely many distinct classes, we may assume that the same value of N is good for each class.

Going to a subsequence of $\langle k_n \rangle_{n=1}^{\infty}$ if necessary, we may assume that $[x_{k_n}]_\approx = [q]_\approx$ for some $q \in A$ and every n, since there are only finitely many classes with respect to the relation $\approx$ in A. Finally, we may also assume that there is $0 \leq r < k$ such that $k_n = r \pmod k$ for every n, where $k = k_A$ is provided by Lemma 3.6 to A. Passing to a subsequence the last time (for example, dropping a few initial elements of the sequence), we may assume that $C_n = \{k_n, \ldots, k_n + kn\} \subset T$. Taking $n \geq 2(k + N)$, we obtain that $k_{n+1} - k_n - kn > kN$. Note that $\langle x_{k_n}, \ldots, x_{k_n+kn} \rangle \subset A$ is a δ-chain of length kn, hence $x_{k_n} \approx x_{k_n+kn}$; in particular, both these points belong to the class $[q]_\approx$.

By the choice of N and the fact that k divides $k_{n+1} - k_n - kn$, for every $n > 2(k + N)$ we can find a δ-chain in A of length $k_{n+1} - k_n - kn$ between x_{k_n+kn} and $x_{k_{n+1}}$. We can

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also find a δ-pseudo orbit in A of length $k_{2(k+N)}$ which ends in $x_{2(k+N)}$ (simply take a sufficiently long δ-chain from $x_{k_{2(k+N)}}$ to $x_{k_{2(k+N)}}$ and keep the last $k_{2(k+N)}$ positions). Therefore, if we put $T' = \bigcup_{n \geq 2(k+N)} C_n$ then there exists a δ-pseudo orbit $\langle y_i \rangle_{i=1}^{\infty} \subset \mathrm{CR}(f)$ such that $y_j = x_j$ for each $j \in T'$. By the choice of ε and δ, there is $z \in X$ such that $\rho(f^j(z), y_j) < \varepsilon$. This concludes the proof. □

We are now in a position to answer [**4**, Question 3.3].

THEOREM 3.8. *If (X, f) has the shadowing property then it has the $(\mathcal{F}_t, \mathcal{F}_t)$-shadowing property.*

Proof. Fix any $\varepsilon > 0$ and let $\delta \in (0, \varepsilon/2)$ be such that every δ-pseudo orbit can be $\varepsilon/2$-traced. Let $\eta > 0$ provided by Lemma 3.3 to δ.

Now take any η-pseudo orbit $\langle x_i \rangle_{i=1}^{\infty}$ on a set $T \in \mathcal{F}_t$ and use Lemma 3.3 to find a thick set $T' \subset T$ and a δ-pseudo orbit $\langle y_i \rangle_{i=1}^{\infty} \subset \mathrm{CR}(f)$ on T' such that $\rho(x_j, y_j) < \delta < \varepsilon/2$ for every $j \in T'$. Finally, use Lemma 3.7 to find a thick set $T'' \subset T'$ and a point $z \in X$ such that $\rho(f^j(z), y_j) < \varepsilon/2$ for every $j \in T''$. Since $T'' \in \mathcal{F}_t$ and

$$\rho(f^j(z), x_j) \leq \rho(f^j(z), y_j) + \rho(y_j, x_j) < \varepsilon$$

for every $j \in T''$, the result follows. □

4. *Characterization of thick shadowing*

Recently it was proved in [**4**] that in transitive dynamical systems the shadowing property and the thick shadowing property are equivalent. The aim of this section is to provide a full characterization of relations between shadowing and thick shadowing. We will prove that (X, f) has the thick shadowing property if and only if its restriction to the chain-recurrent set is a non-wandering dynamical system with the shadowing property. In particular, in every non-wandering dynamical system the shadowing property and the thick shadowing property are equivalent.

We will need the following fact which is a straightforward consequence of the definition of the $(\mathcal{F}, \mathcal{G})$-shadowing property.

LEMMA 4.1. *Suppose that $\mathcal{F}_1 \subset \mathcal{F}_0$ and $\mathcal{G}_0 \subset G_1$. If (X, f) has the $(\mathcal{F}_0, \mathcal{G}_0)$-shadowing property then it also has the $(\mathcal{F}_1, \mathcal{G}_1)$-shadowing property.*

We will also need the following simple observation.

Remark 4.2. Note that if (X, f) is a chain-recurrent dynamical system with the shadowing property, then it is non-wandering and $X = CR(f) = \Omega(f) = \overline{\mathrm{Rec}(f)}$.

THEOREM 4.3. *For every chain-recurrent dynamical system (X, f) the following properties are equivalent:*

(1) *the shadowing property;*
(2) *the thick shadowing property;*
(3) *the $(\mathcal{F}_t, \mathcal{F}_t)$-shadowing property;*
(4) *the $(\mathbb{N}, \mathcal{F}_t)$-shadowing property.*

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Proof. Since $\mathbb{N} \subset \mathcal{D} \subset F_t$ we obtain by Lemma 4.1 that (3) $\Longrightarrow$ (2) $\Longrightarrow$ (4). By Theorem 3.8 we obtain (1) $\Longrightarrow$ (3).

It remains to show (4) $\Longrightarrow$ (1). Fix any $\varepsilon > 0$ and let $\delta > 0$ be provided by the $(\mathbb{N}, \mathcal{F}_t)$-shadowing property. It is enough to show that any δ-chain can be ε-traced. Fix any δ-chain $\langle x_0, \ldots, x_n\rangle$. By Lemma 3.5 the relation $\sim_\delta$ is an equivalence relation on $\mathrm{CR}(f) = X$, hence $\langle x_i\rangle_{i=0}^n \subset [x_0]_{\sim_\delta}$ and also there exists a δ-chain $\langle y_0, \ldots, y_m\rangle$ from x_n to x_0. Then the periodic sequence

$$\xi = \langle \xi_i\rangle_{i=0}^\infty = \langle x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}, x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}, x_0, \ldots\rangle$$

is a δ-pseudo orbit and hence by the $(\mathbb{N}, \mathcal{F}_t)$-shadowing property there exist a point z and a thick set $T \in \mathcal{F}_t$ such that $\rho(f^j(z), \xi_j) < \varepsilon$. Since T contains arbitrarily long blocks of consecutive integers, there is k such that $[k(n+m), k(n+m)+n] \cap \mathbb{N} \subset T$. But then if we put $N = k(n+m)$ and $x = f^N(z)$ then for $j = 0, \ldots, n$ we have

$$\rho(f^j(x), x_j) = \rho(f^{N+j}(z), \xi_{N+j}) < \varepsilon.$$

We have just proved that x is ε-tracing the δ-chain $\langle x_i\rangle_{i=0}^n$, which ends the proof. □

The next result allows us to perform partial shadowing with the help of minimal points.

THEOREM 4.4. *If (X, f) has the $(\mathbb{N}, \mathcal{F}_t)$-shadowing property then for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\langle x_i\rangle_{i=1}^\infty$ is a δ-pseudo orbit then there exists a minimal point z such that*

$$\{j : \rho(x_j, f^j(z)) < \varepsilon\} \in \mathcal{F}_t.$$

Proof. Let $\delta > 0$ be provided by the $(\mathbb{N}, \mathcal{F}_t)$-shadowing property to $\varepsilon/6$. Fix any δ-pseudo orbit $\langle x_i\rangle_{i=1}^\infty$. There exist a thick set T and a point x such that $\rho(x_j, f^j(x)) < \varepsilon/6$ for all $j \in T$. Let j_n be an increasing sequence such that $\{j_n, \ldots, j_n + n\} \subset T$ for each n. Going to a subsequence if necessary, we may assume that there is $q \in X$ such that $\lim_{n\to\infty} f^{j_n}(x) = q$.

By the Auslander–Ellis theorem [7] there exist a minimal set $M \subset X$ and a point $y \in M$ such that $\liminf_{j\to\infty} \rho(f^j(q), f^j(y)) = 0$. But then we can find a sequence $\langle t_n\rangle_{n=1}^\infty$ such that $\lim_{n\to\infty} \mathrm{dist}(f^{t_n}(x), M) = 0$ and the set $T' = \mathbb{N} \cap \bigcup_{n=1}^\infty [t_n, t_n + n]$ satisfies $T' \subset T$. Using induction, it is not hard to construct an increasing sequence $\langle p_n\rangle_{n=1}^\infty$ such that $\{p_n, \ldots, \sum_{i=1}^n p_i + n\} \subset T'$ and for every m and any finite $\alpha \subset [m, \infty) \cap \mathbb{N}$ we have $\rho(f^{p_\alpha}(x), M) < 1/m$, where as usual $p_\alpha = \sum_{i\in\alpha} p_i$.

By [7, Lemma 8.15] it is possible to find a subsequence $\langle q_n\rangle_{n=1}^\infty \subset \langle p_n\rangle_{n=1}^\infty$ and a point $z \in M$ such that for every $m > 0$ and every finite set $\alpha \subset [m, \infty) \cap \mathbb{N}$ we have $\rho(f^{q_\alpha}(x), z) < 1/m$ and $\rho(f^{q_\alpha}(z), z) < 1/m$. Fix any n and take $m = m(n)$ sufficiently large, so that if $\rho(a, b) < 1/m$ then $\rho(f^i(a), f^i(b)) < \varepsilon/6$ for $i = 0, \ldots, n$. Since $\{q_m, \ldots, q_m + n\} \subset T'$, for every $j = 0, \ldots, n$, we have

$$\begin{aligned}\rho(x_{q_m+j}, f^{q_m+j}(z)) &\le \rho(x_{q_m+j}, f^{q_m+j}(x)) + \rho(f^{q_m+j}(x), f^j(z))\\ &\quad + \rho(f^j(z), f^{q_m+j}(z))\\ &\le \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} < \varepsilon.\end{aligned}$$

The proof is complete. □

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We are now ready to provide a full characterization of the thick shadowing property.

THEOREM 4.5. *For every dynamical system* (X, f) *the following conditions are equivalent:*

(1) $\mathrm{CR}(f) = \Omega(f) = \overline{\mathrm{Rec}(f)}$ *and the dynamical system* $(\mathrm{CR}(f), f|_{\mathrm{CR}(f)})$ *has the shadowing property;*
(2) (X, f) *has the thick shadowing property;*
(3) (X, f) *has the* $(\mathcal{F}_t, \mathcal{F}_t)$*-shadowing property;*
(4) (X, f) *has the* $(\mathbb{N}, \mathcal{F}_t)$*-shadowing property;*
(5) (X, f) *has the* $(\mathbb{N}, \mathcal{F}_{cf})$*-shadowing property.*

Proof. By Lemma 4.1 we obtain that (3) $\Longrightarrow$ (2) $\Longrightarrow$ (4) and (5) $\Longrightarrow$ (4).

We will show that (4) $\Longrightarrow$ (1). Fix any $x \in \mathrm{CR}(f)$ and any $\varepsilon > 0$. Let $\delta > 0$ be provided to $\varepsilon > 0$ by Theorem 4.4. Take any δ-chain $\langle x_0, \ldots, x_n\rangle$ from x to x and extend it to a periodic δ-pseudo orbit

$$\langle z_i\rangle_{i=1}^{\infty} = \langle x_0, x_1, \ldots, x_{n-1}, x_0, x_1, \ldots, x_{n-1}, x_0, \ldots\rangle.$$

There exist a minimal point $z \in X$ and a thick set $T \in \mathcal{F}_t$ such that $\rho(z_j, f^j(z)) < \varepsilon$ for each $j \in T$. In particular, there exists $i > 0$ such that $\mathbb{N} \cap [j, j+n] \subset T$ and hence there exists $j \le k < j+n$ such that $\rho(x, f^k(z)) = \rho(z_k, f^k(z)) < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we can present x as a limit of minimal points in X. This shows that $\mathrm{CR}(f) \subset \overline{\mathrm{Rec}(f)} \subset \Omega(f) \subset \mathrm{CR}(f)$ and hence $\Omega(f) = \mathrm{CR}(f) = \overline{\mathrm{Rec}(f)}$.

Another consequence of Theorem 4.4 is that every δ-pseudo orbit in $\mathrm{CR}(f)$ can be ε-traced on a thick set by a point in $\mathrm{Rec}(f) \subset \mathrm{CR}(f)$, hence $(\mathrm{CR}(f), f|_{\mathrm{CR}(f)})$ has the $(\mathbb{N}, \mathcal{F}_t)$-shadowing property. But we have already proved that $\mathrm{CR}(f) = \overline{\mathrm{Rec}(f)}$, hence $(\mathrm{CR}(f), f|_{\mathrm{CR}(f)})$ is a non-wandering system. Applying Theorem 4.3, we obtain that $(\mathrm{CR}(f), f|_{\mathrm{CR}(f)})$ has the shadowing property. Indeed (4) $\Longrightarrow$ (1).

The implication (1) $\Longrightarrow$ (5) is a simple consequence of Lemma 3.4. To see this, fix any $\varepsilon > 0$, let $\delta \in (0, \varepsilon/2)$ be provided by shadowing (on $\mathrm{CR}(f)$) to $\varepsilon/2$ and let $\eta > 0$ be provided to δ by Lemma 3.4. Fix any η-pseudo orbit $\langle x_i\rangle_{i=1}^{\infty}$ and use Lemma 3.4 to obtain a δ-pseudo orbit $\langle y_i\rangle_{i=1}^{\infty} \subset \mathrm{CR}(f)$ and $N > 0$ such that $\rho(x_i, y_i) < \delta < \varepsilon/2$ for every $i \ge N$. By shadowing of $(\mathrm{CR}(f), f|_{\mathrm{CR}(f)})$ we obtain an $\varepsilon/2$-tracing point z for $\langle y_i\rangle_{i=1}^{\infty}$ and then, for every $i \ge N$, we obtain

$$\rho(f^i(z), x_i) \le \rho(f^i(z), y_i) + \rho(y_i, x_i) < \varepsilon.$$

In other words, z is ε-tracing the η-pseudo orbit $\langle x_i\rangle_{i=1}^{\infty}$ on a set in $\mathcal{F}_{cf}$, proving $(\mathbb{N}, \mathcal{F}_{cf})$-shadowing of (X, f).

It remains to show (1) $\Longrightarrow$ (3). Let $\delta \in (0, \varepsilon/2)$ be provided to $\varepsilon > 0$ by the shadowing property of $(\mathrm{CR}(f), f|_{\mathrm{CR}(f)})$. Let $\eta > 0$ be provided to δ by Lemma 3.3. Fix any thick set T and any η-pseudo orbit $\langle x_i\rangle_{i=1}^{\infty}$ on T. By Lemma 3.3 we obtain a thick set $T' \subset T$ and a δ-pseudo orbit $\langle y_i\rangle_{i=1}^{\infty} \subset \mathrm{CR}(f)$ on T' such that $\rho(x_i, y_i) < \delta < \varepsilon/2$. By Lemma 3.7 there exist a thick set $T'' \subset T'$ and $z \in X$ such that $\rho(f^i(z), y_i) < \varepsilon/2$ for every $i \in T''$,

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in particular $\rho(f^i(z), x_i) < \varepsilon$ for every $i \in T''$. We have just proved that (X, f) has the $(\mathcal{F}_t, \mathcal{F}_t)$-shadowing property, hence the proof is complete. □

COROLLARY 4.6. *A dynamical system (X, f) has the thick shadowing property if and only if $(\mathrm{CR}(f), f|_{\mathrm{CR}(f)})$ has the thick shadowing property.*

Proof. It is enough to note that $\mathrm{CR}(f) = \mathrm{CR}(f|_{\mathrm{CR}(f)})$. The result follows by Theorem 4.5. □

Remark 4.7. It was proved in [**8**] that the $(\mathcal{D}, \mathcal{D})$-shadowing property is equivalent to the shadowing property with topological mixing. Hence the properties of $(\mathbb{N}, \mathcal{D})$-shadowing and $(\mathcal{D}, \mathcal{F}_t)$-shadowing (implicitly embedded in Theorem 4.5) are in some sense 'optimal' choices of families to force shadowing (on a chain-recurrent set) without chain transitivity.

In [**4**, Example 3.1] the authors considered the map $f(x) = x + \frac{1}{4}|\sin(\pi x)|$ on $[0, 2]$ and proved that this map has the $(\mathcal{F}_t, \mathcal{F}_t)$-shadowing property but not the shadowing property. This is a simple consequence of the following fact.

COROLLARY 4.8. *Let f be a homeomorphism on $[0, 1]$ with finitely many fixed points and $f(0) = 0$. Then $([0, 1], f)$ has the $(\mathcal{F}_t, \mathcal{F}_t)$-shadowing property. If there exist a fixed point $p \in (0, 1)$ and its neighborhood U such that either $x \le f(x)$ or $x \ge f(x)$ for all x in U then f does not have the shadowing property.*

Proof. Clearly $\Omega(f)$ is finite and $\mathrm{CR}(f) = \Omega(f) = \mathrm{Rec}(f)$, so $(\mathrm{CR}(f), f|_{\mathrm{CR}(f)})$ has the shadowing property and hence $([0, 1], f)$ has the $(\mathcal{F}_t, \mathcal{F}_t)$-shadowing property by Theorem 4.5. But if for all $x \in U$ we have $x \le f(x)$ (or symmetrically $x \ge f(x)$) then for every $\delta > 0$ there is a δ-pseudo orbit ξ which visits both connected components of $[0, 1]\backslash U$. But f is a homeomorphism with $f(0) = 0$, hence the set $[0, p]$ and as a consequence also $[p, 1]$ is invariant. In particular, ξ cannot be ε-traced by any point in $[0, 1]$, provided that $\varepsilon < \mathrm{dist}(p, [0, 1]\backslash U)/2$. □

Example 4.9. Consider the map $f(x) = x + \frac{1}{4}|\sin(\pi x)|$ on $[0, 2]$ and identify it with a map on the unit circle by collapsing $\{0, 2\}$ to a single point. Then the resulting map (call it F) becomes chain recurrent without the shadowing property, but $\Omega(F)$ is still finite, hence $F|_{\Omega(F)}$ has the shadowing property (and the thick shadowing property as well). This shows that the condition $\mathrm{CR}(f) = \Omega(f)$ in Theorem 4.5(1) cannot be removed.

5. *Shadowing and families with the Ramsey property*

In [**4**], the authors proved that if a (chain-)transitive map has the shadowing property and $\mathcal{F}$ is a family with the Ramsey property then any sequence of points can be ε-shadowed (for any fixed $\varepsilon > 0$) on a set in $\mathcal{F}$. The aim of this section is to provide a full characterization of cases when this kind of the shadowing property is possible. As we will see, neither the shadowing property nor transitivity are essential requirements for this result to hold. Before we are able to present the main result of this section, we need the following auxiliary fact.

LEMMA 5.1. *If minimal points are not dense in X then (X, f) does not have the $(P(\mathbb{N}), \mathcal{F}_{ps})$-shadowing property.*

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Proof. If we denote by M the closure of all minimal points, then $M \neq X$. Denote by (Y, g) a dynamical system obtained by collapsing the closed invariant set M to a single point and let $\pi : (X, f) \to (Y, g)$ be the respective factor map. Let $p = \pi(M)$ be the unique minimal subset of Y (a fixed point) and take any non-dense open neighborhood $p \in U$. Denote $W = \pi^{-1}(U) \subset X$. The dynamical system (Y, g) is proximal, hence syndetically proximal (see [**11**, Theorem 1]; it can be also derived from a more general result in [**5**]). It is also an easy observation (see [**11**, Proposition 4]) that for every $y \in Y$ the set $\{n : f^n(y) \in U\}$ is thickly syndetic.

By the choice of U we have $\overline{W} \neq X$, hence there exist $x \in X \setminus W$ and $\varepsilon > 0$ such that $B(x, \varepsilon) \cap W = \emptyset$. Now let us suppose that there exist a point $z \in X$ and a set $F \in \mathcal{F}_{ps}$ such that $F = \{n : f^n(z) \in B(x, \varepsilon)\}$, that is, it is possible to ε-trace the constant sequence $\langle x, x, \ldots \rangle$ on a set in $\mathcal{F}_{ps}$. Putting $y = \pi(z)$, we obtain that

$$F \subset \{f^n(z) \notin W\} = \{n : f^n(y) \notin U\} \notin \mathcal{F}_{ps}$$

as a complement of a thickly syndetic set, which is a contradiction. □

The following result shows that the shadowing property and chain transitivity in [**4**, Theorem 2.5] are much too strong assumptions.

THEOREM 5.2. *The following conditions are equivalent:*

(1) *minimal points are dense in X;*
(2) *(X, f) has the $(P(\mathbb{N}), \mathcal{F})$-shadowing property for every family $\mathcal{F}$ with the Ramsey property;*
(3) *(X, f) has the $(P(\mathbb{N}), \mathcal{F}_{ps})$-shadowing property.*

Proof. The implication (2) $\Longrightarrow$ (3) is obvious (recall that by [**3**, Theorem 2.3] the family $\mathcal{F}_{ps}$ has the Ramsey property) and (3) $\Longrightarrow$ (1) is a consequence of Lemma 5.1.

The implication (1) $\Longrightarrow$ (2) a simple consequence of the Ramsey property. To see this, fix any family $\mathcal{F}$ with the Ramsey property, any sequence $\langle x_i \rangle_{i=1}^{\infty} \subset X$ and any $\varepsilon > 0$. Let $\{U_i\}_{i=1}^{s}$ be a finite cover of X by sets with $\operatorname{diam}(U_j) < \varepsilon$ for every j. Fix any $F \in \mathcal{F}$. Since each element $x_i \in U_{j(i)}$ for some $j(i)$ and there are finitely many choices of $j(i)$, without loss of generality (by the Ramsey property) we may assume that there is $U \in \{U_i : i \leq s\}$ such that $x_i \in U$ for every $i \in F$. Take any minimal point $y \in U$. Since the set $N(y, U)$ is syndetic, there is k such that $\bigcup_{r=0}^{k} N(f^r(y), U) = \mathbb{N}$. Hence there is $l \leq k$ such that $F' = F \cap N(f^l(y), U) \in \mathcal{F}$. Putting $z = f^l(y)$ completes the proof, since $d(f^j(z), x_j) \leq \operatorname{diam}(U) < \varepsilon$ for every $j \in F'$. □

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