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Extremal basis for the p-Bergman space

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Abstract. In Chen and Zhang (2022, *Adv. Math.*, 405, 108516), the authors noted that the 2-Bergman space has an orthonormal basis which can be defined as certain extremal functions. They posed a question of whether similar extremal functions form a Schauder Basis for the p-Bergman space. We provide evidence supporting this conjecture, demonstrating its validity for a selection of complete Reinhardt domains.

1 Introduction

The study of Bergman spaces has long been a cornerstone of complex analysis and functional analysis, with profound connections to operator theory, partial differential equations, and geometric function theory.

For a bounded domain $\Omega \subset \mathbb{C}^n$, the p-Bergman space $\mathcal{A}^p(\Omega)$ consists of holomorphic functions that are $\mathcal{L}^p$ integrable with respect to the Lebesgue measure

$$\mathcal{A}^p(\Omega) = \left\{ f \in \mathcal{O}(\Omega) \text{ such that } \|f\|_p^p := \int_\Omega |f|^p < \infty \right\}.$$

These spaces generalize the classical Bergman space $\mathcal{A}^2(\Omega)$, which carries a natural Hilbert space structure and is a Reproducing Kernel Hilbert Space. Recent years have witnessed significant advances in understanding p-Bergman spaces, particularly regarding the p-Bergman kernel and its properties [CX24, CZ22, CXZ22], the duality of p-Bergman space [Bha24, CEM19], and its role in classifying complex domains [DWZZ20].

In [CZ22], the authors defined the p-Bergman kernel as an extremal function satisfying

$$m_p(z) = \inf \left\{ \|f\|_p : f(z) = 1, f \in \mathcal{A}^p(\Omega) \right\}.$$

The p-Bergman kernel $m_p(\cdot, z)$ is the unique function that optimizes the above extremal problem. This function also mimics the properties of the Bergman kernel in the context of $\mathcal{A}^p(\Omega)$, including the reproducing property of the Bergman kernel.

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In [Ber50], the author observed that a certain set of "extremal functions" form an orthonormal basis for $\mathcal{A}^2(\Omega)$. In this note, we investigate if similar "extremal functions" form a Schauder Basis for $\mathcal{A}^p(\Omega)$. To define these "extremal functions," we first define an ordering on multi-index set $\mathbb{N}_0^n$

$$\mathbb{N}_0^n := \left\{\alpha = (\alpha_1, \ldots, \alpha_n) : \alpha_i \in \mathbb{N} \cup \{0\}\right\}.$$

Definition 1.1 We define a partial order on multi-indices as follows: For $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$, we say $\alpha \prec \beta$ if

a) $|\alpha| < |\beta|$, or
b) $|\alpha| = |\beta|$ and there exists a $k \in \{1, \ldots, n-1\}$ such that $\alpha_j = \beta_j$ for $j < k$ and $\beta_k > \alpha_k$.

Note that this partial order is a well-ordering, and since each multi-index β has finitely many predecessors, we can find multi-indices $\alpha^0 = (0, \ldots, 0), \alpha^1, \ldots, \alpha^i, \alpha^{i+1}, \ldots, \alpha^m = \beta$ such that each α^{i+1} is the immediate successor of α^i.

For $z_0 \in \Omega$, let

$$H_p^{\alpha, z_0} = \{f : \mathcal{A}^p(\Omega) : \partial^\alpha f(z_0) = 1; \partial^\beta f(z_0) = 0 \text{ for } \beta \prec \alpha\}$$

and let

$$m_p^\alpha(z_0) := \inf\left\{\|f\|_p : f \in H_p^{\alpha, z_0}\right\}.$$

By an argument of normality, we see that there exists a unique minimizer function, say $m_p^\alpha(\cdot, z_0)$. An observation by Bergman [Ber50] states that $\{m_2^\alpha(\cdot, z)/\|m_2^\alpha(\cdot, z)\|_2 : \alpha \in \mathbb{N}_0^n\}$ is an orthonormal basis for $\mathcal{A}^2(\Omega)$.

The question was posed in [CZ22] asking if $\left\{m_p^\alpha(\cdot, z) : \alpha \in \mathbb{N}_0^n\right\}$ forms a Schauder Basis for $\mathcal{A}^p(\Omega)$. We answer this question in the case of Reinhardt domains. Let us now recall the definition of a Schauder Basis.

Definition 1.2 (Schauder Basis) Let X be a Banach space. We say that a sequence of elements $\{f_k\}_{k \in \mathbb{N}}$ is a *Schauder Basis* if for each $f \in X$, there exists a unique sequence of scalars $\{a_k\}$ such that $f = \sum_{k \in \mathbb{N}} a_k(f) f_k$, where the series converges in norm to f.

2 Extremal Schauder Basis for the p-Bergman space

We note the following Bergman inequality for the derivatives.

Lemma 2.1 (Bergman inequality for derivatives) *Let $f \in \mathcal{A}^p(\Omega)$ and let $\alpha \in (\mathbb{N}_0)^n$. For $a \in \Omega$, let $r = \dfrac{\operatorname{dist}(a, b\Omega)}{\sqrt{n}}$. Then*

$$\left|\partial^\alpha f(a)\right| \leq \frac{\alpha!(\alpha_1 + 2)\ldots(\alpha_n + 2)(\operatorname{vol}(\Omega))^{1/q}}{(2\pi)^n r^{2n+|\alpha|}} \|f\|_p \lesssim \frac{(\alpha+2)!}{r^{2n+|\alpha|}} \cdot \|f\|_p.$$

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Proof The proof of this follows from the Cauchy inequalities (see [Ran98, Corollary I.1.7]). Consider the polydisc $P(a,\rho) = \{z \in \mathbb{C}^n : |z_i - (a)_i| < \rho_i \text{ for all } i = 1,2,\dots,n\}$ which is compactly contained in Ω, whenever and for $f \in \mathcal{O}(\Omega)$, $\rho_i < r$,

$$\partial^\alpha f(a) = \frac{\alpha!}{(2\pi i)^n} \int_{b_0P(a,\rho)} \frac{f(\zeta)}{(\zeta - a)^{\alpha+1}} d\zeta_1 d\zeta_2 \dots d\zeta_n,$$

where $b_0P(z_0,\rho) = \left\{z \in \mathbb{C}^n \middle| |z_i - (z_0)_i| = \rho_i \text{ for all } i = 1,2,\dots,n\right\}$ is the distinguished boundary of the polydisc. Let $\mathbb{T} = [0,2\pi]^n$ denote the n-torus. Converting into polar coordinates with $\zeta = \rho e^{i\theta}$

$$\left|\partial^\alpha f(a)\right| \rho^{\alpha+1} \le \frac{\alpha!}{(2\pi)^n} \int_{\mathbb{T}} \left|f(\rho e^{i\theta})\right| \rho_1 \dots \rho_n d\theta_1 \dots d\theta_n.$$

Integrating the above with respect to $\rho_1, \rho_2, \dots, \rho_n$ gives us

$$\begin{aligned}&\left|\partial^\alpha f(a)\right| \cdot \int_0^r \cdots \int_0^r \rho^{\alpha+1} d\rho_1 \dots d\rho_n \\ &\quad \le \frac{\alpha!}{(2\pi)^n} \int_0^r \cdots \int_0^r \int_{[0,2\pi]^n} \left|f(\zeta(\theta))\right| \rho_1 \dots \rho_n d\theta_1 \dots d\theta_n d\rho_1 \dots d\rho_n.\end{aligned}$$

Therefore,

$$\begin{aligned}\left|\partial^\alpha f(a)\right| &\le \frac{\alpha!(\alpha_1+2)\dots(\alpha_n+2)}{(2\pi)^n r^{2n+|\alpha|}} \int_{P(a,r)} \left|f(\zeta)\right| dV_\zeta \le \frac{\alpha!(\alpha_1+2)\dots(\alpha_n+2)}{(2\pi)^n r^{2n+|\alpha|}} \int_\Omega \left|f(\zeta)\right| dV_\zeta \\ &\le \frac{\alpha!(\alpha_1+2)\dots(\alpha_n+2)}{(2\pi)^n r^{2n+|\alpha|}} \left\|f\right\|_p (\mathrm{vol}(\Omega))^{1/q},\end{aligned}$$

where $1/p + 1/q = 1$. The last inequality is obtained by applying Hölder's inequality. ∎

Lemma 2.2 *Let Ω be a bounded domain such that $\Omega \subset B(0,R)$ and $p \in [1,\infty)$. Then there exists a unique minimizer $m_p^\alpha(\cdot, z_0)$ that minimizes*

$$m_p^\alpha(z_0) := \inf\left\{\left\|f\right\|_p : f \in H_p^{\alpha,z_0}\right\}.$$

Further, if $m_p^\alpha(\cdot, z_0)$ is that unique minimizer, it follows that

$$m_p^\alpha(z_0) \le \frac{(2R)^{|\alpha|}}{\alpha!} \cdot (\mathrm{vol}(\Omega))^{1/p} \text{ and } \left|\partial^\beta m_p^\alpha(z_0,z_0)\right| \lesssim \frac{(2R)^{|\alpha|}}{\alpha!} \cdot (\beta+2)! \text{ when } \alpha \prec \beta.$$

Proof The proof of existence and uniqueness is similar to the proof of existence uniqueness of $m_p(\cdot, z_0)$ [CZ22, Section 2.1, 2.2]. We'll prove it here again for the sake of completeness.

i. **Existence**: Fix $z_0 \in \Omega$ and $\alpha \in \mathbb{N}_0^n$. Let $\{\varphi_k\}$ be a sequence of functions in H_p^{α,z_0} such that $\|\varphi_k\|_p \searrow m_p^\alpha(z_0)$ as $k \to \infty$. Applying Lemma 2.1 with $\alpha = 0$ and observing that the $\{\varphi_k\}$ is bounded in norm, we deduce that it is a locally bounded family. Therefore, by Montel's theorem, φ_k converges locally uniformly

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to some holomorphic function $m_p^\alpha(\cdot, z_0)$ and $\partial^\beta m_p(z_0, z_0) = \lim_{k\to\infty} \partial^\beta \varphi_k(z_0)$ for all $\beta \in N_0^n$. Applying Fatou's lemma,

$$\left\| m_p^\alpha(\cdot, z_0) \right\|_p \leq \liminf_{k\to\infty} \|\varphi_k\| = m_p^\alpha(z_0),$$

proving the existence of an extremal function $m_p^\alpha(\cdot, z_0)$ in H_p^{α, z_0}.

ii. **Uniqueness**: Let φ be another holomorphic function in H_p^{α,z_0} such that $\|\varphi\| = \left\| m_p^\alpha(\cdot, z_0) \right\| = m_p^\alpha(z_0)$. Set $f = \dfrac{\varphi + m_p^\alpha(\cdot, z_0)}{2} \in H_p^{\alpha,z_0}$, then by convexity of the function $t \mapsto |t|^p$, we have

$$[m_p^\alpha(z_0)]^p \leq \int_\Omega \left|f(\zeta)\right|^p \, dV_\zeta \leq \int_\Omega \frac{\left|\varphi(\zeta)\right|^p + \left|m_p^\alpha(\zeta, z_0)\right|^p}{2} \, dV_\zeta = [m_p(z_0)]^p,$$

that is, $\|f\| = m_p^\alpha(z_0)$. Now, by strict convexity of $\mathcal{A}^p(\Omega)$ and the fact that $\varphi, m_p(\cdot, z_0)$, we see that $\varphi = m_p(\cdot, z_0)$, proving uniqueness of the minimizer.

iii. Finally, to obtain the estimates, we note that Ω is a bounded domain, therefore the normalized monomial $\dfrac{(z - z_0)^\alpha}{\alpha!} \in H_p^{\alpha, z_0}$ which shows that

$$m_p^\alpha(z_0) \leq \frac{(2R)^{|\alpha|}}{\alpha!} \cdot (\mathrm{vol}(\Omega))^{1/p} \text{ and } \left|\partial^\beta m_p^\alpha(z_0, z_0)\right| \lesssim \frac{(2R)^{|\alpha|}}{\alpha!} \cdot (\beta + 2)!$$

■

Lemma 2.3 *For $\alpha \in \mathbb{N}_0^n$, there exists a bounded linear functional $a_\alpha : \mathcal{A}^p(\Omega) \to \mathbb{C}$ such that*

$$a_\alpha(m_p^\beta(\cdot, z_0)) = \begin{cases} 0, & \alpha \neq \beta \\ 1 & \alpha = \beta. \end{cases}$$

Proof Fix $z_0 \in \Omega$ and $\alpha \in \mathbb{N}_0^n$ and we can find finite multi-indices $\alpha^0 = (0, \ldots, 0)$, $\alpha^1, \ldots, \alpha^i, \alpha^{i+1}, \ldots, \alpha^k = \alpha$ such that each α^{i+1} is an immediate successor of α^i. Assume that for $f \in \mathcal{A}^p(\Omega)$, we have the formal summation

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha(f) m_p^\alpha(z, z_0).$$

Differentiate the above series with respect to $\partial^{\alpha_0}, \partial^{\alpha_1}, \ldots, \partial^{\alpha_k}$ and substitute $z = z_0$, we get the following set of equations:

$$\begin{aligned} f(z_0) &= a_{\alpha_0}(f) \\ \partial^{\alpha_1} f(z_0) &= \partial^{\alpha_1} m_p^{\alpha_0}(z_0, z_0) \cdot a_{\alpha_0}(f) + a_{\alpha_1}(f) \\ &\vdots \\ \partial^{\alpha_k} f(z_0) &= \partial^{\alpha_k} m_p^{\alpha_0}(z_0, z_0) \cdot a_{\alpha_0}(f) + \cdots + \partial^{\alpha_k} m_p^{\alpha_{k-1}}(z_0, z_0) \cdot a_{\alpha_{k-1}}(f) + a_{\alpha_k}(f). \end{aligned}$$

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Thus, solving for $a_{\alpha_k}(f)$ translates to solving the above system of linear equations, that is, finding the inverse of a lower triangular matrix as below:

$$\begin{bmatrix} f(z_0) \\ \partial^{\alpha_1} f(z_0) \\ \vdots \\ \partial^{\alpha_k} f(z_0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \partial^{\alpha_1} m_p^{\alpha_0}(z_0, z_0) & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \partial^{\alpha_k} m_p^{\alpha_0}(z_0, z_0) & \partial^{\alpha_k} m_p^{\alpha_1}(z_0, z_0) & \partial^{\alpha_k} m_p^{\alpha_2}(z_0, z_0) & \dots & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{\alpha_0}(f) \\ a_{\alpha_1}(f) \\ \vdots \\ a_{\alpha_k}(f) \end{bmatrix}.$$

If we write the above $(k+1) \times (k+1)$ square matrix as $I + N$, where I is the identity matrix and N is the above matrix with diagonal entries replaced by 0. Since, all eigenvalues of N are 0, it follows that $N^{k+1} = 0$ implying

$$(I + N)(I - N + N^2 \dots (-1)^k N^k) = I + (-1)^{k+1} N^{k+1} = I,$$

that is, $I + N$ is invertible, and the inverse is given by

$$(I + N)^{-1} = I - N + N^2 \dots (-1)^k N^k.$$

If each term of $(I + N)^{-1}$ is bounded by M, then

$$\left|a_{\alpha_k}(f)\right| \leq \sum_{j=0}^{k} M \left|\partial^{\alpha_j} f(z_0)\right| \lesssim \|f\|_p \cdot \left(M \sum_{j=0}^{k} (\alpha_j + 2)! \right).$$

Further, from the above matrix, we note that

$$a_\alpha(m_p^\beta(\cdot, z_0)) = \begin{cases} 0, & \alpha \neq \beta \\ 1, & \alpha = \beta. \end{cases}$$

■

2.1 Reinhardt domain

A domain $\Omega \subset \mathbb{C}^n$ is said to be a *Reinhardt domain* if

$$z = (z_1, z_2, \dots, z_n) \in \Omega \implies (e^{i\tau_1} z_1, e^{i\tau_2} z_2, \dots, e^{i\tau_n} z_n) \in \Omega \text{ for all } \tau_1, \tau_2, \dots, \tau_n \in \mathbb{R}.$$

A Reinhardt domain Ω is said to be a *complete Reinhardt domain* if

$$z = (z_1, z_2, \dots, z_n) \in \Omega \implies (r_1 e^{i\tau_1} z_1, r_2 e^{i\tau_2} z_2, \dots, r_n e^{i\tau_n} z_n) \in \Omega \text{ for all } \tau_i \in \mathbb{R},\ 0 \leq r_i \leq 1.$$

For a bounded Reinhardt domain Ω, the *shadow of* Ω, denoted $|\Omega|$, is

$$|\Omega| = \left\{ (r_1, r_2, \dots, r_n) \in (\mathbb{R}_{\geq 0})^n : (r_1, r_2, \dots, r_n) \in \Omega \right\}.$$

When Ω is a Reinhardt domain, the normalized monomial $e_\alpha(\zeta) := \dfrac{\zeta^\alpha}{\alpha!}$ functions play an important role in the study of p-Bergman spaces. Chakrabarti and Edholm [CE24, Theorem 2.16] show that these normalized monomials form a Schauder Basis for $\mathcal{A}^p(\Omega)$. More precisely, let $S(\Omega, \mathcal{A}^p)$ denote the set of all $\mathcal{A}^p$ allowable multi-indices, that is,

$$S(\Omega, \mathcal{A}^p) = \{\alpha \in \mathbb{Z}^n : e_\alpha(z) \in \mathcal{A}^p(\Omega)\}$$

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then $\left\{e_\alpha : \alpha \in S(\Omega, \mathcal{A}^p)\right\}$ is a Schauder Basis for $\mathcal{A}^p(\Omega)$. Note that for a bounded complete Reinhardt domain Ω, $S(\Omega, \mathcal{A}^p) = \mathbb{N}_0^n$ and thus $\left\{e_\alpha : \alpha \in \mathbb{N}_0^n\right\}$ is a Schauder Basis for $\mathcal{A}^p(\Omega)$.

Lemma 2.4 *Let Ω be a bounded complete Reinhardt domain and $z_0 = 0$, then*

$$m_p^\alpha(w, z_0) = \frac{1}{\alpha!} w^\alpha = e_\alpha(w) \quad \textit{and thus} \quad \left\{m_p^\alpha(\cdot, z_0) : \alpha \in \mathbb{N}_0^n\right\}$$

forms a Schauder Basis for $\mathcal{A}^p(\Omega)$.

Proof Recall that $H_p^{\alpha,0} = \left\{f \in \mathcal{A}^p(\Omega) : \partial^\alpha f(0) = 1,\ \partial^\beta f(0) = 0 \text{ for all } \beta \prec \alpha\right\}$. Since Ω is a complete Reinhardt domain, any function $f \in H_p^{\alpha,0}$ can be written via its power series:

$$f(w) = \frac{w^\alpha}{\alpha!} + \sum_{\alpha \prec \eta} c_\eta w^\eta = e_\alpha + h,$$

where $e_\alpha(w) = \frac{w^\alpha}{\alpha!}$, and h is holomorphic with $\partial^\beta h(0) = 0$ for all $\beta \preceq \alpha$.

Let $h \in \mathcal{A}^p(\Omega)$ satisfy $\partial^\beta h(0) = 0$ for all $\beta \preceq \alpha$. Consider the one-parameter family $\{e_\alpha + th\}_{t \in \mathbb{R}} \subset H_p^{\alpha,0}$. Computing the derivative of the $\mathcal{L}^p$ norm to the p-th power at $t = 0$ gives

$$\frac{d}{dt} \|e_\alpha + th\|_p^p \Big|_{t=0} = p \int_\Omega \left|e_\alpha(\zeta)\right|^{p-2} \operatorname{Re}\left(e_\alpha(\zeta)\overline{h(\zeta)}\right) dV_\zeta = 0,$$

since h is orthogonal to e_α by hypothesis on h. In fact,

$$\int_\Omega \left|e_\alpha(\zeta)\right|^{p-2} e_\alpha(\zeta)\overline{h(\zeta)} dV_\zeta = \int_{|\Omega|} \left|r^\alpha\right|^{p-1} \int_{\mathbb{T}} e^{i\alpha\cdot\tau} \cdot \overline{\sum_{\alpha \prec \eta} c_\eta r^\eta e^{i\eta\cdot\tau}}\, d\tau\, dr,$$

where $\zeta_i = r_i e^{i\tau_i}$ are the polar coordinates. Now we note that $\int_{\mathbb{T}^n} e^{i\alpha\cdot\tau} \cdot \overline{e^{i\eta\cdot\tau}}\, d\tau = 0$, whenever $\alpha \neq \eta$, proving orthogonality between h and e_α.

Therefore, for any $f \in H_p^{\alpha,0}$, the convex function $t \mapsto \|e_\alpha + th\|_p$ attains its minimum at $t = 0$. By the uniqueness of the extremal function, we conclude that $m_p^\alpha(w, 0) = e_\alpha(w)$.

Finally, the set $\{m_p^\alpha(\cdot, 0) : \alpha \in \mathbb{N}_0^n\}$ forms a Schauder Basis for $\mathcal{A}^p(\Omega)$; that is, for any $f \in \mathcal{A}^p(\Omega)$,

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} a_{\alpha,0}(f) m_p^\alpha(z, 0),$$

where the convergence is in $\mathcal{L}^p(\Omega)$, and $a_{\alpha,0}$ is a bounded linear functional. ∎

Corollary 2.5 *For a Unit Ball $\mathbb{B}^n$, $\left\{m_p^\alpha(\cdot, 0) : \alpha \in \mathbb{N}_0^n\right\}$ is a Schauder Basis for $\mathcal{A}^p(\mathbb{B}^n)$.*

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Theorem 2.6 *Let $\Omega = \mathbb{B}^n$ denote the Unit Ball and $z_0 \in \Omega$. Let ϕ_0 be an automorphism of the Unit Ball that maps 0 to z_0 and set $c_0 = J_{\phi_0}(0)^{-2/p}$. Then*

$$m_p^\alpha(w, z_0) = \frac{1}{c_0} e_\alpha(\phi_0^{-1}(w)) \cdot J_{\phi_0}(\phi_0^{-1}(w))^{-2/p} \quad \text{and} \quad \left\{ m_p^\alpha(\cdot, z_0) : \alpha \in \mathbb{N}_0^+ \right\}$$

forms a Schauder Basis for $\mathcal{A}^p(\Omega)$.

Proof Let $f \in H_p^{\alpha, z_0}$, then it follows that $T_f := c_0 f \circ \phi_0 \cdot J_{\phi_0}^{2/p} \in H_p^{\alpha, 0}$, where $c_0 = J_{\phi_0}(0)^{-2/p}$ and by change of variables formula,

$$\min \left\{ \int_\Omega |T_f|^p : f \in H_p^{\alpha, z_0} \right\} = |c_0|^p \min \left\{ \int_\Omega |f|^p : f \in H_p^{\alpha, z_0} \right\}.$$

Thus,

$$T_{m_p^\alpha(\cdot, z_0)}(\zeta) = c_0 m_p^\alpha(\phi_0(\zeta), z_0) \cdot J_{\phi_0}^{2/p}(\zeta) = e_\alpha(\zeta) = m_p(\zeta, 0),$$

that is,

$$\begin{aligned} m_p^\alpha(w, z_0) &= \frac{1}{c_0} m_p^\alpha(\phi_0^{-1}(w), 0) \cdot J_{\phi_0}(\phi_0^{-1}(w))^{-2/p} \\ &= \frac{1}{c_0} e_\alpha(\phi_0^{-1}(w)) \cdot J_{\phi_0}(\phi_0^{-1}(w))^{-2/p}. \end{aligned}$$

Further, by change of variables, $f \in \mathcal{A}^p(\Omega)$ if and only if $f(\phi_0) \cdot J_{\phi_0}^{2/p} \in \mathcal{A}^p(\Omega)$ and by the previous result,

$$\begin{aligned} f(\phi_0) \cdot J_{\phi_0}^{2/p} &= \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha^0 (f(\phi_0) \cdot J_{\phi_0}^{2/p}) \cdot e_\alpha \\ \iff f &= \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha^0 (f(\phi_0) \cdot J_{\phi_0}^{2/p}) \cdot e_\alpha(\phi_0^{-1}) J_{\phi_0}^{2/p}(\phi_0^{-1}) \\ \iff f &= \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha^0 (c_0 f(\phi_0) \cdot J_{\phi_0}^{2/p}) \cdot m_p^\alpha(\cdot, z_0) \\ \iff f &= \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha^0 (T_f) \cdot m_p^\alpha(\cdot, z_0), \end{aligned}$$

which shows that $a_\alpha^{z_0}(f) = a_\alpha^0(T_f)$:

$$\begin{aligned} S_N^{z_0}(f)(w) &= \sum_{|a_\alpha|_\infty \le N} a_\alpha^{z_0}(f) m_p^\alpha(w, z_0) \\ &= \sum_{|a_\alpha|_\infty \le N} a_\alpha^0(T_f) \cdot \frac{1}{c_0} e_\alpha(\phi_0^{-1}(w)) \cdot J_{\phi_0}(\phi_0^{-1}(w))^{-2/p} \\ &= J_{\phi_0}(\phi_0^{-1}(w))^{-2/p} \cdot S_N^0(T_f)(\phi_0^{-1}(w)) \cdot \frac{1}{c_0}. \end{aligned}$$

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By change of variables,

$$\left\| T_f - S_n^0(T_f) \right\| = \left\| c_0 f \circ \phi_0 \cdot J_{\phi_0}^{2/p} - c_0 \cdot S_N^{z_0}(f) \circ \phi_0 \cdot J_{\phi_0}^{2/p} \right\| = |c_0| \left\| f - S_N^{z_0}(f) \right\|$$

proving $\left\{ m_p^\alpha(\cdot, z_0) : \alpha \in \mathbb{N}_0^+ \right\}$ forms a Schauder Basis for $\mathcal{A}^p(\Omega)$. ∎

Corollary 2.7 *Similarly, if Ω is a simply-connected complete Reinhardt domain which has a transitive automorphism group, then for all $z_0 \in \Omega$, $\left\{ m_p^\alpha(\cdot, z_0) : \alpha \in \mathbb{N}_0^n \right\}$ forms a Schauder Basis for $\mathcal{A}^p(\Omega)$.*

Also, we have proved the following result.

Corollary 2.8 *Let Ω be a bounded domain and let $z, w \in \Omega$ such that there exists an $\mathcal{A}^p$ preserving biholomorphism (see [Bha23, Definition 5.1]) $F : \Omega \to \Omega$ such that $F(z) = w$. Then $\left\{ m_p^\alpha(\cdot, z) : \alpha \in \mathbb{N}_0^n \right\}$ is a Schauder Basis for $\mathcal{A}^p(\Omega)$ if and only if $\left\{ m_p^\alpha(\cdot, w) : \alpha \in \mathbb{N}_0^n \right\}$ is a Schauder Basis for $\mathcal{A}^p(\Omega)$.*

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