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ON PARALLEL SYSTEMS WITH HETEROGENEOUS GAMMA COMPONENTS

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In this article, we study ordering properties of lifetimes of parallel systems with two independent heterogeneous gamma components in terms of the likelihood ratio order and the hazard rate order. Let X_1 and X_2 be two independent gamma random variables with X_i having shape parameter $r > 0$ and scale parameter λ_i, $i = 1, 2$, and let X_1^* and X_2^* be another set of independent gamma random variables with X_i^* having shape parameter r and scale parameter λ_i^*, $i = 1, 2$. Denote by $X_{2:2}$ and $X_{2:2}^*$ the corresponding maximum order statistics, respectively. It is proved that, among others, if (λ_1, λ_2) weakly majorize $(\lambda_1^*, \lambda_2^*)$, then $X_{2:2}$ is stochastically greater than $X_{2:2}^*$ in the sense of likelihood ratio order. We also establish, among others, that if $0 < r \leq 1$ and (λ_1, λ_2) is p-larger than $(\lambda_1^*, \lambda_2^*)$, then $X_{2:2}$ is stochastically greater than $X_{2:2}^*$ in the sense of hazard rate order. The results derived here strengthen and generalize some of the results known in the literature.

1. INTRODUCTION

Order statistics play an important role in statistical inference, reliability theory, life testing, operations research, and many other areas. Let $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the order statistics of random variables $X_1, \ldots, X_n$. The kth-order statistic $X_{k:n}$ then corresponds to the lifetime of a $(n - k + 1)$-out-of-n system, which is a very popular structure of redundancy in fault-tolerant systems that have been studied extensively in the literature. In particular, $X_{n:n}$ and $X_{1:n}$ respectively correspond to the lifetimes of parallel and series systems, which are the simplest examples of coherent systems. Much work has been done on various aspects of order statistics when the observations are independent and identically distributed (i.i.d.). The case when observations are non-i.i.d., however, often arises in a natural way in many practical situations. Due

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to the complexity of the distribution theory in this case, not much work is available in the literature; see, for example, David and Nagaraja [7] and Balakrishnan and Rao [2,3] for comprehensive discussions on this topic and the recent review article of Balakrishnan [1] for results on the independent and nonidentically distributed (i.ni.d.) case.

The exponential distribution, due to its nice mathematical form and the unique memoryless property, has widely been utilized in many fields. Pledger and Proschan [20] were the first to compare stochastically the order statistics arising from i.ni.d. exponential random variables. After that, many researchers have devoted themselves on this topic, including Proschan and Sethuraman [21], Kochar and Rojo [12], Dykstra, Kochar and Rojo [8], Khaledi and Kochar [11], Sun and Zhang [23], Bon and Păltănea [6], Kochar and Xu [13], Păltănea [19], Zhao and Balakrishnan [24–26], Zhao, Li and Balakrishnan [27,28], Joo and Mi [9], Mao and Hu [14], and Khaledi, Farsinezhad, and Kochar [10]. It is well known that the gamma distribution is one of the most commonly used distributions in reliability and life testing. Let X be a gamma random variable with shape parameter r and scale parameter λ. Then, in its standard form, X has the probability density function

$$f(x; r, \lambda) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, \qquad x > 0.$$

It is an extremely flexible family of distributions with decreasing, constant, and increasing failure rates when $0 < r < 1$, $r = 1$ and $r > 1$, respectively. In this article, we will focus on the maximum order statistic from two gamma observations (i.e., the lifetime of a parallel system with two independent heterogeneous gamma components). The results established in this article extend the associated ones in the literature from exponential distributions to gamma distributions.

We first review some related results in the literature. Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $X_1^*, \ldots, X_n^*$ be another set of independent exponential random variables with X_i^* having hazard rate λ_i^*. Then, Pledger and Proschan [20] showed, for $1 \le k \le n$, that

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow X_{k:n} \geq_{\mathrm{st}} X_{k:n}^*. \tag{1}$$

Formal definitions of these orderings will be provided in the next section. Proschan and Sethuraman [21] strengthened this result from componentwise stochastic ordering to multivariate stochastic ordering, and Khaledi and Kochar [11] partially improved (1) under a weaker condition as

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{p}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow X_{n:n} \geq_{\mathrm{st}} X_{n:n}^*. \tag{2}$$

Khaledi et al. [10] recently extended the result in (2) to the scale model that actually includes gamma as a special case.

Boland, EL-Neweihi, and Proschan [4] pointed out by a counterexample that (1) cannot be strengthened from usual stochastic order to hazard rate order when $n > 2$;

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however, they showed for the case when $n = 2$ that

$$(\lambda_1, \lambda_2) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\mathrm{hr}} X_{2:2}^*. \tag{3}$$

Dykstra et al. [8] further improved (3) from hazard rate order to likelihood ratio order as

$$(\lambda_1, \lambda_2) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\mathrm{lr}} X_{2:2}^*. \tag{4}$$

Joo and Mi [9] gave some conditions under which the hazard rate order in (3) holds. Especially, they proved, under the condition $0 < \lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$, that

$$(\lambda_1, \lambda_2) \overset{\mathrm{w}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\mathrm{hr}} X_{2:2}^*. \tag{5}$$

Recently, Zhao and Balakrishnan [25] further strengthened the results in (3)–(5) and obtained two characterization results; that is, under the condition $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$, that

$$(\lambda_1, \lambda_2) \overset{\mathrm{w}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longleftrightarrow X_{2:2} \geq_{\mathrm{lr}} X_{2:2}^* \tag{6}$$

and

$$(\lambda_1, \lambda_2) \overset{\mathrm{p}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longleftrightarrow X_{2:2} \geq_{\mathrm{hr}} X_{2:2}^*. \tag{7}$$

In this article the above problems are further studied under the gamma framework. We prove that if X_1 and X_2 are independent gamma random variables with X_i having shape parameter r and scale parameter λ_i, $i = 1, 2$, and X_1^* and X_2^* are another set of independent gamma random variables with X_i^* having shape parameter r and scale parameter λ_i^*, $i = 1, 2$, then

$$(\lambda_1, \lambda_2) \overset{\mathrm{w}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\mathrm{lr}} X_{2:2}^*. \tag{8}$$

It is also proved, under the condition $0 < r \leq 1$ and $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$, that

$$(\lambda_1, \lambda_2) \overset{\mathrm{p}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\mathrm{hr}} X_{2:2}^*. \tag{9}$$

Apparently, these two results in (8) and (9) strengthen and generalize all of those in (3)–(7) established earlier in the literature.

2. DEFINITIONS

In this section we first recall some notions of stochastic orders and majorization and related orders. Throughout this article the term *increasing* is used for *monotone nondecreasing* and *decreasing* is used for *monotone nonincreasing*.

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2.1. Stochastic Orders

DEFINITION 2.1: *For two random variables X and Y with densities f_X and f_Y and distribution functions F_X and F_Y, respectively, let $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y = 1 - F_Y$ be the corresponding survival functions. Then the following hold:*

(i) X is said to be smaller than Y in the likelihood ratio order (denoted by $X \leq_{\mathrm{lr}} Y$) if $f_Y(x)/f_X(x)$ is increasing in x;

(ii) X is said to be smaller than Y in the hazard rate order (denoted by $X \leq_{\mathrm{hr}} Y$) if $\overline{F}_Y(x)/\overline{F}_X(x)$ is increasing in x;

(iii) X is said to be smaller than Y in the stochastic order (denoted by $X \leq_{\mathrm{st}} Y$) if $\overline{F}_Y(x) \geq \overline{F}_X(x)$.

It is well known that likelihood ratio order implies hazard rate order, which, in turn, implies usual stochastic order. For a comprehensive discussion on various stochastic orderings, refer to Shaked and Shanthikumar [22] and Müller and Stoyan [18].

2.2. Majorization and Related Orders

The notion of majorization is quite useful in establishing various inequalities. Let $x_{(1)} \leq \cdots \leq x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, \ldots, x_n)$.

DEFINITION 2.2: *The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$, written as $\mathbf{x} \overset{\mathrm{m}}{\succeq} \mathbf{y}$, if*

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \quad \textit{for } j = 1, \ldots, n-1$$

and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

In addition, the vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ weakly, written as $\mathbf{x} \overset{\mathrm{w}}{\succeq} \mathbf{y}$, if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n.$$

Clearly,

$$\mathbf{x} \overset{\mathrm{m}}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\mathrm{w}}{\succeq} \mathbf{y}.$$

Those functions that preserve the majorization ordering are said to be Schur-convex. For an extensive and comprehensive discussion on the theory and applications of the majorization order, refer to Marshall and Olkin [15]. Bon and Păltănea [5] introduced a preorder on $\Re_+^n$, called p-larger order, which is defined as follows.

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DEFINITION 2.3: *The vector* $\mathbf{x}$ *in* $\Re_+^n$ *is said to be p-larger than another vector* $\mathbf{y}$, *written as* $\mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y}$, *if*

$$\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n.$$

Let $\log(\mathbf{x})$ be the vector of logarithms of the coordinates of $\mathbf{x}$. It is then easy to verify that

$$\mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y} \Longleftrightarrow \log(\mathbf{x}) \overset{\text{w}}{\succeq} \log(\mathbf{y}).$$

Moreover,

$$\mathbf{x} \overset{\text{w}}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y}$$

for $\mathbf{x}, \mathbf{y} \in \Re_+^n$. The converse is, however, not true. For example, we have $(2, 7) \overset{\text{p}}{\succeq} (3, 5)$, but the weak majorization order clearly does not hold between these two vectors.

3. LIKELIHOOD RATIO ORDERING

The following several lemmas will be used to establish our main results. The first one turns out to be a useful tool for showing the monotonicity of a fraction whose numerator and denominator are integrals or summations.

LEMMA 3.1 (Misra and van der Meulen, [16]): *Let* Θ *be a subset of a real line and* U *be a nonnegative random variable having a cumulative distribution function (c.d.f.) belonging to a stochastically ordered family* $\mathcal{P} = \{H(\cdot|\theta), \theta \in \Theta\}$; *that is, for* $\theta_1, \theta_2 \in \Theta$, $H(\cdot|\theta_1) \leq_{\text{st}} (\geq_{\text{st}}) H(\cdot|\theta_2)$ *whenever* $\theta_1 < \theta_2$. *Suppose a real function* $\psi(u, \theta)$ *on* $\mathcal{R} \cdot \Theta$ *is measurable in* u *for each* θ *such that* $\mathsf{E}_\theta[\psi(U, \theta)]$ *exists. Then the following hold:*

(i) $\mathsf{E}_\theta[\psi(U, \theta)]$ *is increasing in* θ *if* $\psi(u, \theta)$ *is increasing in* θ *and increasing (decreasing) in* u;

(ii) $\mathsf{E}_\theta[\psi(U, \theta)]$ *is decreasing in* θ *if* $\psi(u, \theta)$ *is decreasing in* θ *and decreasing (increasing) in* u.

LEMMA 3.2: *If* $(a_1, b_1) \overset{\text{m}}{\succeq} (a_2, b_2)$, *where both* a_i *and* b_i $(i = 1, 2)$ *are nonnegative, then the function*

$$g(y) = \frac{\exp(-a_1 y) + \exp(-b_1 y)}{\exp(-a_2 y) + \exp(-b_2 y)}$$

is increasing in $y \in (0, \infty)$.

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PROOF: Without loss of generality, let us assume that $a_1 \geq a_2 \geq b_2 \geq b_1$ and $a_1 + b_1 = a_2 + b_2$. Taking the derivative with respect to y, we get

$$\begin{aligned}
g'(y) &\stackrel{\text{sgn}}{=} [-a_1 \exp(-a_1 y) - b_1 \exp(-b_1 y)][\exp(-a_2 y) + \exp(-b_2 y)] \\
&\quad - [-a_2 \exp(-a_2 y) - b_2 \exp(-b_2 y)][\exp(-a_1 y) + \exp(-b_1 y)] \\
&= \{-a_1 \exp[-(a_1 + a_2)y] - a_1 \exp[-(a_1 + b_2)y] \\
&\quad - b_1 \exp[-(a_2 + b_1)y] - b_1 \exp[-(b_1 + b_2)y]\} \\
&\quad - \{-a_2 \exp[-(a_1 + a_2)y] - a_2 \exp[-(a_2 + b_1)y] \\
&\quad - b_2 \exp[-(a_1 + b_2)y] - b_2 \exp[-(b_1 + b_2)y]\} \\
&= (a_2 - a_1) \exp[-(a_1 + a_2)y] + (b_2 - a_1) \exp[-(a_1 + b_2)y] \\
&\quad + (a_2 - b_1) \exp[-(a_2 + b_1)y] + (b_2 - b_1) \exp[-(b_1 + b_2)y] \\
&\geq (a_2 - a_1 + b_2 - b_1) \exp[-(a_1 + a_2)y] + (a_2 - b_1 + b_2 - a_1) \\
&\quad \times \exp[-\exp[-(a_1 + b_2)y] \\
&= 0,
\end{aligned}$$

which implies that $g(y)$ is increasing in $y \in (0, \infty)$. ■

LEMMA 3.3: *For $r > 0$ and $y \in \Re_+$, the function*

$$f(x) = x + \frac{y^{r-1} e^{-xy}}{\int_0^y u^{r-1} e^{-xu}\, du}$$

is increasing in $x \in \Re_+$.

PROOF: We will prove the result by distinguishing two cases.

Case 1: $0 < r \leq 1$: Taking the derivative with respect to x, we get

$$\begin{aligned}
f'(x) &= 1 + \frac{-y^r e^{-xy} \int_0^y u^{r-1} e^{-xu}\, du + y^{r-1} e^{-xy} \int_0^y u^r e^{-xu}\, du}{\left[\int_0^y u^{r-1} e^{-xu}\, du\right]^2} \\
&= 1 - \frac{y^r}{\int_0^y u^{r-1} e^{x(y-u)}\, du} + \frac{y^{r-1} e^{-xy} \int_0^y u^r e^{-xu}\, du}{\left[\int_0^y u^{r-1} e^{-xu}\, du\right]^2}. \qquad \textbf{(10)}
\end{aligned}$$

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It is easy to see that the last term on the right-hand side in (10) is nonnegative, and thus it is enough if we could show that

$$1 - \frac{y^r}{\int_0^y u^{r-1} e^{x(y-u)}\, \mathrm{d}u}$$

is also nonnegative, which is true by noting that

$$\begin{aligned} 1 - \frac{y^r}{\int_0^y u^{r-1} e^{x(y-u)}\, \mathrm{d}u} &\geq 1 - \frac{y^r}{\int_0^y u^{r-1}\, \mathrm{d}u} \\ &= 1 - r \\ &\geq 0. \end{aligned}$$

Case 2: $r > 1$: Upon using integration by parts, one gets

$$y^{r-1} e^{-xy} = (r-1) \int_0^y u^{r-2} e^{-xu}\, \mathrm{d}u - x \int_0^y u^{r-1} e^{-xu}\, \mathrm{d}u.$$

Thus, it follows that

$$\begin{aligned} f(x) &= x + \frac{(r-1) \int_0^y u^{r-2} e^{-xu}\, \mathrm{d}u - x \int_0^y u^{r-1} e^{-xu}\, \mathrm{d}u}{\int_0^y u^{r-1} e^{-xu}\, \mathrm{d}u} \\ &= (r-1) \frac{\int_0^y u^{r-2} e^{-xu}\, \mathrm{d}u}{\int_0^y u^{r-1} e^{-xu}\, \mathrm{d}u} \end{aligned}$$

is increasing in x by a simple application of Lemma 3.1. ■

THEOREM 3.4: *Let* (X_1, X_2) *be a vector of independent gamma random variables with common shape parameter* $r > 0$ *and respective scale parameters* λ_1 *and* λ*, and let* (Y_1, Y_2) *be another vector of independent gamma random variables with common shape parameter* r *and respective scale parameters* λ_2 *and* λ*. Suppose* $\lambda \geq \lambda_2 \geq \lambda_1$*. Then, it holds that* $X_{2:2} \geq_{\mathrm{lr}} Y_{2:2}$.

PROOF: Let $f_i(g_i)$ and $F_i(G_i)$ be the density and distribution functions of $X_i(Y_i)$, respectively. Denote by $\tilde{r}_{X_{2:2}}$ and $\tilde{r}_{Y_{2:2}}$ the reversed hazard rate functions of $X_{2:2}$ and $Y_{2:2}$, respectively. From Theorem 1.C.4(b) of Shaked and Shanthikumar [22], it is enough

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to show that $X_{2:2} \geq_{\mathrm{rh}} Y_{2:2}$ and the ratio of their reversed hazard rate functions (i.e., $\phi(t) = \tilde{r}_{X_{2:2}}(t)/\tilde{r}_{Y_{2:2}}(t)$) is increasing in $t \in \Re_+$. It is easy to see that

$$\tilde{r}_{X_{2:2}}(t) = \frac{f_1(t)F_2(t) + f_2(t)F_1(t)}{F_1(t)F_2(t)} = \tilde{r}_{X_1}(t) + \tilde{r}_{X_2}(t).$$

Similarly,

$$\tilde{r}_{Y_{2:2}}(t) = \tilde{r}_{Y_1}(t) + \tilde{r}_{Y_2}(t).$$

Observe that $\tilde{r}_{X_2}(t) = \tilde{r}_{Y_2}(t)$ and $\lambda_1 \leq \lambda_2$ implies that $X_1 \geq_{\mathrm{lr}} Y_1$, which, in turn, implies that $X_1 \geq_{\mathrm{rh}} Y_1$ (i.e., $\tilde{r}_{X_1}(t) \geq \tilde{r}_{Y_1}(t)$). Thus, we have $\tilde{r}_{X_{2:2}}(t) \geq \tilde{r}_{Y_{2:2}}(t)$. In what follows we need to show that the function

$$\phi(t) = \left(\frac{e^{-\lambda_1 t}}{\int_0^t u^{r-1}e^{-\lambda_1 u}\,\mathrm{d}u} + \frac{e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right)\left(\frac{e^{-\lambda_2 t}}{\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u} + \frac{e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right)^{-1}$$

is increasing in $t \in \Re_+$. For $t \in \Re_+$, we have

$$\begin{aligned}
&\phi'(t)\left[\frac{e^{-\lambda_2 t}}{\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u} + \frac{e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right]^2 \\
&\quad= \left[-\left(\lambda_1 + \frac{t^{r-1}e^{-\lambda_1 t}}{\int_0^t u^{r-1}e^{-\lambda_1 u}\,\mathrm{d}u}\right)\frac{e^{-\lambda_1 t}}{\int_0^t u^{r-1}e^{-\lambda_1 u}\,\mathrm{d}u}\right. \\
&\qquad\left. -\left(\lambda + \frac{t^{r-1}e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right)\frac{e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right] \\
&\qquad\times\left[\frac{e^{-\lambda_2 t}}{\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u} + \frac{e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right] \\
&\qquad-\left[-\left(\lambda_2 + \frac{t^{r-1}e^{-\lambda_2 t}}{\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u}\right)\frac{e^{-\lambda_2 t}}{\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u}\right. \\
&\qquad\left. -\left(\lambda + \frac{t^{r-1}e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right)\frac{e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right] \\
&\qquad\times\left[\frac{e^{-\lambda_1 t}}{\int_0^t u^{r-1}e^{-\lambda_1 u}\,\mathrm{d}u} + \frac{e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right] \\
&\quad= \left[\left(\lambda_2 + \frac{t^{r-1}e^{-\lambda_2 t}}{\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u} - \lambda_1 - \frac{t^{r-1}e^{-\lambda_1 t}}{\int_0^t u^{r-1}e^{-\lambda_1 u}\,\mathrm{d}u}\right)\right. \\
&\qquad\left.\times\frac{e^{-(\lambda_1+\lambda_2)t}}{\int_0^t u^{r-1}e^{-\lambda_1 u}\,\mathrm{d}u\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u}\right]
\end{aligned}$$

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$$
\begin{aligned}
&+\left[\left(\lambda_2+\frac{t^{r-1}e^{-\lambda_2 t}}{\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u}-\lambda-\frac{t^{r-1}e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right)\right.\\
&\qquad\left.\times\frac{e^{-(\lambda_2+\lambda)t}}{\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right]\\
&+\left[\left(\lambda+\frac{t^{r-1}e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}-\lambda_1-\frac{t^{r-1}e^{-\lambda_1 t}}{\int_0^t u^{r-1}e^{-\lambda_1 u}\,\mathrm{d}u}\right)\right.\\
&\qquad\left.\times\frac{e^{-(\lambda_1+\lambda)t}}{\int_0^t u^{r-1}e^{-\lambda_1 u}\,\mathrm{d}u\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right]\\
&=A+B,\quad\text{say,}
\end{aligned}
$$

where

$$
\begin{aligned}
A=&\left[\left(\lambda_2+\frac{t^{r-1}e^{-\lambda_2 t}}{\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u}-\lambda_1-\frac{t^{r-1}e^{-\lambda_1 t}}{\int_0^t u^{r-1}e^{-\lambda_1 u}\,\mathrm{d}u}\right)\right.\\
&\qquad\left.\times\frac{e^{-(\lambda_1+\lambda_2)t}}{\int_0^t u^{r-1}e^{-\lambda_1 u}\,\mathrm{d}u\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
B=&\left[\left(\lambda_2+\frac{t^{r-1}e^{-\lambda_2 t}}{\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u}-\lambda-\frac{t^{r-1}e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right)\right.\\
&\qquad\left.\times\frac{e^{-(\lambda_2+\lambda)t}}{\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right]\\
&+\left[\left(\lambda+\frac{t^{r-1}e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}-\lambda_1-\frac{t^{r-1}e^{-\lambda_1 t}}{\int_0^t u^{r-1}e^{-\lambda_1 u}\,\mathrm{d}u}\right)\right.\\
&\qquad\left.\times\frac{e^{-(\lambda_1+\lambda)t}}{\int_0^t u^{r-1}e^{-\lambda_1 u}\,\mathrm{d}u\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right].
\end{aligned}
$$

Upon applying Lemma 3.3, we have $A \geq 0$. Next, we will show that $B \geq 0$. It is easy to verify that

$$
\frac{e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}=\frac{1}{\int_0^t u^{r-1}e^{\lambda(t-u)}\,\mathrm{d}u}
$$

is decreasing in $\lambda > 0$, which implies that

$$
\frac{e^{-(\lambda_1+\lambda)t}}{\int_0^t u^{r-1}e^{-\lambda_1 u}\,\mathrm{d}u\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\geq\frac{e^{-(\lambda_2+\lambda)t}}{\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}. \tag{11}
$$

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Moreover, it holds from Lemma 3.3 that

$$\lambda + \frac{t^{r-1}e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u} \geq \lambda_1 + \frac{t^{r-1}e^{-\lambda_1 t}}{\int_0^t u^{r-1}e^{-\lambda_1 u}\,\mathrm{d}u}. \tag{12}$$

By using (15) and (14), we get

$$\begin{aligned} B &\geq \left[\left(\lambda_2 + \frac{t^{r-1}e^{-\lambda_2 t}}{\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u} - \lambda - \frac{t^{r-1}e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right)\right. \\ &\quad \left.\times \frac{e^{-(\lambda_2+\lambda)t}}{\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u \int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right] \\ &\quad + \left[\left(\lambda + \frac{t^{r-1}e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u} - \lambda_1 - \frac{t^{r-1}e^{-\lambda_1 t}}{\int_0^t u^{r-1}e^{-\lambda_1 u}\,\mathrm{d}u}\right)\right. \\ &\quad \left.\times \frac{e^{-(\lambda_2+\lambda)t}}{\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u \int_0^t u^{r-1}e^{-\lambda u}\,\mathrm{d}u}\right] \\ &\overset{\text{sgn}}{=} \lambda_2 + \frac{t^{r-1}e^{-\lambda_2 t}}{\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u} - \lambda_1 - \frac{t^{r-1}e^{-\lambda_1 t}}{\int_0^t u^{r-1}e^{-\lambda_1 u}\,\mathrm{d}u} \\ &\geq 0. \end{aligned}$$

Therefore, $\phi'(t) \geq 0$ (i.e., $\phi(t)$ is increasing in $t \in \Re_+$) and this completes the proof. ■

THEOREM 3.5: *Let (X_1, X_2) be a vector of independent gamma random variables with common shape parameter $r > 0$ and respective scale parameters λ_1 and λ_2, and let (X_1^*, X_2^*) be another vector of independent gamma random variables with common shape parameter r and respective scale parameters λ_1^* and λ_2^*. Then*

$$(\lambda_1, \lambda_2) \overset{\text{m}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\text{lr}} X_{2:2}^*.$$

PROOF: Denote by $f_{X_{2:2}}$ and $f_{X_{2:2}^*}$ the density functions of $X_{2:2}$ and $X_{2:2}^*$, respectively. It suffices to prove that $\Delta(t) = f_{X_{2:2}}(t)/f_{X_{2:2}^*}(t)$ is increasing in $t \in \Re_+$. It can be seen that

$$\begin{aligned} \Delta(t) &\propto \left(e^{-\lambda_1 t}\int_0^t u^{r-1}e^{-\lambda_2 u}\,\mathrm{d}u + e^{-\lambda_2 t}\int_0^t u^{r-1}e^{-\lambda_1 u}\,\mathrm{d}u\right) \\ &\quad \times \left(e^{-\lambda_1^* t}\int_0^t u^{r-1}e^{-\lambda_2^* u}\,\mathrm{d}u + e^{-\lambda_2^* t}\int_0^t u^{r-1}e^{-\lambda_1^* u}\,\mathrm{d}u\right)^{-1} \\ &= \left(\int_0^t u^{r-1}[e^{-(\lambda_1 t+\lambda_2 u)} + e^{-(\lambda_2 t+\lambda_1 u)}]\,\mathrm{d}u\right) \\ &\quad \times \left(\int_0^t u^{r-1}[e^{-(\lambda_1^* t+\lambda_2^* u)} + e^{-(\lambda_2^* t+\lambda_1^* u)}]\,\mathrm{d}u\right)^{-1} \end{aligned}$$

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$$
\begin{aligned}
&= \left(\int_0^1 y^{r-1} [e^{-(\lambda_1+\lambda_2 y)t} + e^{-(\lambda_2+\lambda_1 y)t}] \, \mathrm{d}y \right) \\
&\quad \times \left(\int_0^1 y^{r-1} [e^{-(\lambda_1^*+\lambda_2^* y)t} + e^{-(\lambda_2^*+\lambda_1^* y)t}] \, \mathrm{d}y \right)^{-1} \\
&= \mathrm{E}_t \psi(Y, t),
\end{aligned}
$$

where

$$
\psi(y,t) = \frac{e^{-(\lambda_1+\lambda_2 y)t} + e^{-(\lambda_2+\lambda_1 y)t}}{e^{-(\lambda_1^*+\lambda_2^* y)t} + e^{-(\lambda_2^*+\lambda_1^* y)t}}
$$

for $y \in (0,\ 1)$. Here, the distribution function of the random variable Y belongs to the family $\mathcal{P} = \{H(\cdot|t), t \in \Re_+\}$ with densities

$$
h(y|t) = c(t) y^{r-1} [e^{-(\lambda_1^*+\lambda_2^* y)t} + e^{-(\lambda_2^*+\lambda_1^* y)t}]
$$

and a normalizing constant $c(t)$ such that $\int_0^1 h(y|t) \, \mathrm{d}y = 1$. Note that

$$
\psi(y,t) = \frac{e^{-\lambda_1(1-y)t} + e^{-\lambda_2(1-y)t}}{e^{-\lambda_1^*(1-y)t} + e^{-\lambda_2^*(1-y)t}}
$$

is increasing in $t \in \Re_+$ and decreasing in $y \in (0,1)$ from Lemma 3.2. On the other hand, note that, for $t_2 \geq t_1 > 0$,

$$
\begin{aligned}
\Xi(y) &= \frac{h(y|t_2)}{h(y|t_1)} \\
&\propto \frac{e^{-(\lambda_1^*+\lambda_2^* y)t_2} + e^{-(\lambda_2^*+\lambda_1^* y)t_2}}{e^{-(\lambda_1^*+\lambda_2^* y)t_1} + e^{-(\lambda_2^*+\lambda_1^* y)t_1}} \\
&\propto \frac{e^{\lambda_2^*(1-y)t_2} + e^{\lambda_1^*(1-y)t_2}}{e^{\lambda_2^*(1-y)t_1} + e^{\lambda_1^*(1-y)t_1}}
\end{aligned}
$$

is decreasing in $y \in (0,1)$, which can be obtained by checking that

$$
\begin{aligned}
\Xi'(y) &\overset{\text{sgn}}{=} [\lambda_2^* t_1 e^{\lambda_2^*(1-y)(t_1+t_2)} + \lambda_2^* t_1 e^{(\lambda_2^* t_1 + \lambda_1^* t_2)(1-y)} \\
&\quad + \lambda_1^* t_1 e^{(\lambda_1^* t_1 + \lambda_2^* t_2)(1-y)} + \lambda_1^* t_1 e^{\lambda_1^*(1-y)(t_1+t_2)}] \\
&\quad - [\lambda_2^* t_2 e^{\lambda_2^*(1-y)(t_1+t_2)} + \lambda_2^* t_2 e^{(\lambda_2^* t_2 + \lambda_1^* t_1)(1-y)} \\
&\quad + \lambda_1^* t_2 e^{(\lambda_2^* t_1 + \lambda_1^* t_2)(1-y)} + \lambda_1^* t_2 e^{\lambda_1^*(1-y)(t_1+t_2)}] \\
&= \lambda_2^*(t_1 - t_2) e^{\lambda_2^*(1-y)(t_1+t_2)} + \lambda_1^*(t_1 - t_2) e^{\lambda_1^*(1-y)(t_1+t_2)} \\
&\quad + (\lambda_2^* t_1 - \lambda_1^* t_2) e^{(\lambda_2^* t_1 + \lambda_1^* t_2)(1-y)} + (\lambda_1^* t_1 - \lambda_2^* t_2) e^{(\lambda_1^* t_1 + \lambda_2^* t_2)(1-y)} \\
&\leq 0.
\end{aligned}
$$

Thus, we have $H(\cdot|t_1) \geq_{\text{lr}} H(\cdot|t_2)$, which, in turn, implies that $H(\cdot|t_1) \geq_{\text{st}} H(\cdot|t_2)$ whenever $0 < t_1 \leq t_2$. By using Lemma 3.1 now, $\mathrm{E}_t \psi(Y,t)$ is increasing in $t \in (0,\infty)$, which completes the proof. ■

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THEOREM 3.6: *Let (X_1, X_2) be a vector of independent gamma random variables with common shape parameter $r > 0$ and respective scale parameters λ_1 and λ_2, and let (X_1^*, X_2^*) be another vector of independent gamma random variables with common shape parameter r and respective scale parameters λ_1^* and λ_2^*. Suppose*

$$\min(\lambda_1, \lambda_2) \leq \min(\lambda_1^*, \lambda_2^*) \leq \max(\lambda_1^*, \lambda_2^*) \leq \max(\lambda_1, \lambda_2). \tag{13}$$

Then

$$(\lambda_1, \lambda_2) \overset{w}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{lr} X_{2:2}^*.$$

PROOF: Without loss of generality, let us assume that $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$. Suppose $(\lambda_1, \lambda_2) \overset{w}{\succeq} (\lambda_1^*, \lambda_2^*)$. We then have $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$ and $\lambda_1 + \lambda_2 \leq \lambda_1^* + \lambda_2^*$. The desired result follows for the case when $\lambda_1 + \lambda_2 = \lambda_1^* + \lambda_2^*$ by Theorem 3.5. Assume $\lambda_1 + \lambda_2 < \lambda_1^* + \lambda_2^*$. There then exists some λ such that $\lambda + \lambda_2 = \lambda_1^* + \lambda_2^*$ and $\lambda_1 < \lambda \leq \lambda_1^*$. Let $Y_{2:2}$ be the lifetime of a parallel system consisting of two independent gamma components with common shape parameter r and respective scale parameters λ and λ_2. From Theorem 3.5 once again, we have $Y_{2:2} \geq_{lr} X_{2:2}^*$. Moreover, we have $X_{2:2} \geq_{lr} Y_{2:2}$ from Theorem 3.4, and so we can conclude that $X_{2:2} \geq_{lr} X_{2:2}^*$. ■

In order to illustrate the result obtained in Theorem 3.6, we provide the following numerical example. Let (X_1, X_2) be a vector of heterogeneous gamma random variables with common shape parameter $r = 0.5$ and scale parameter vector $(1, 4)$. Denote by $h(t; 1, 4)$ the corresponding hazard rate function of the maximum order statistic $X_{2:2}$. Let (Y_1, Y_2) be a vector of heterogeneous gamma random variables with common shape parameter $r = 0.5$ and scale parameter vector $(2, 3.5)$, and denote by $h(t; 2, 3.5)$ the corresponding hazard rate function of $Y_{2:2}$. Obviously, it holds that $1 \leq 2 \leq 3.5 \leq 4$ and $(1, 4) \overset{w}{\succeq} (2, 3.5)$. It can be seen from Figure 1 that $h(t; 1, 4) \leq h(t; 2, 3.5)$, which is in accordance with the result of Theorem 3.6. Let (Z_1, Z_2) be a vector of another set of heterogeneous gamma random variables with common shape parameter $r = 0.5$ and scale parameter vector $(2, 2.5)$. Denote by $h(t; 2, 2.5)$ the corresponding hazard rate function of $Z_{2:2}$. Note that condition (13) in Theorem 3.6 does not hold though it holds that $(2, 2.5) \overset{w}{\succeq} (2, 3.5)$, and in this case, we cannot compare the hazard rate functions as shown in Figure 1.

Remark 3.7: It can be seen that Theorem 3.6 considerably strengthens the result of Joo and Mi [9] who proved under the exponential framework that

$$(\lambda_1, \lambda_2) \overset{w}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{hr} X_{2:2}^*.$$

As an immediate consequence of Theorem 3.6, we have the following corollary, which provides a lower bound for the maximum of two gamma random variables.

COROLLARY 3.8: *Let (X_1, X_2) be a vector of independent gamma random variables with common shape parameter $r > 0$ and respective scale parameters λ_1 and λ_2, and*

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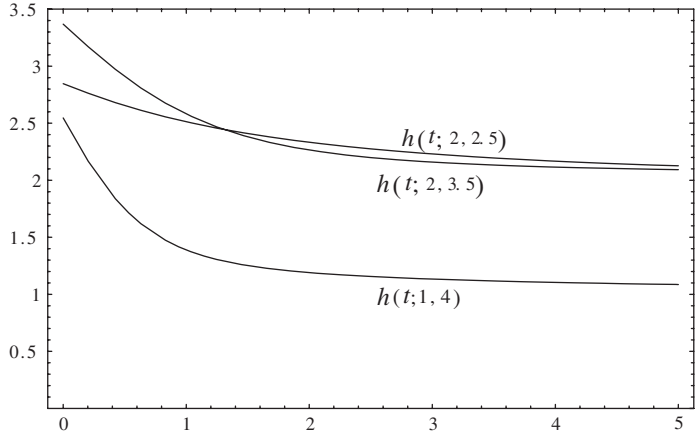


FIGURE 1. Plots of hazard rate functions of $X_{2:2}$ from two gamma distributions with common shape parameter 0.5 and scale parameter vector $(1, 4)$, $Y_{2:2}$ from two gamma distributions with common shape parameter 0.5 and scale parameter vector $(2, 3.5)$, and $Z_{2:2}$ from two gamma distributions with common shape parameter 0.5 and scale parameter vector $(2, 2.5)$

let (X_1^*, X_2^*) be another vector of independent gamma random variables with common shape and scale parameters r and λ. Assume $\lambda \leq \max(\lambda_1, \lambda_2)$. Then

$$\lambda \geq \frac{\lambda_1 + \lambda_2}{2} \Longrightarrow X_{2:2} \geq_{\text{lr}} X_{2:2}^*.$$

Remark 3.9: It is remarkable that the result of Corollary 3.8 is valid without the assumption that $\lambda \leq \max(\lambda_1, \lambda_2)$. Let Z_λ be the second-order statistic of a random sample of size 2 from a gamma distribution with shape parameter r and scale parameter λ. Then, it is known that Z_λ is decreasing in λ according to likelihood ratio order from Theorem 1.C.33 of Shaked and Shanthikumar [22]. Based on this fact, we can conclude that the result of Corollary 3.8 is also valid for the case when $\lambda > \max(\lambda_1, \lambda_2)$.

4. HAZARD RATE ORDERING

In this section we establish some comparison results similar to those presented in the preceding section for the hazard rate order. We first introduce several lemmas that will be used for proving the main result.

LEMMA 4.1 (Marshall and Olkin [15], p. 57): *Let $I \subset \Re$ be an open interval and let $\phi : I^n \to \Re$ be continuously differentiable. Then ϕ is Schur-convex [Schur-concave]*

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on I^n if and only if ϕ is symmetric on I^n and, for all $i \neq j$,

$$(z_i - z_j)\left[\frac{\partial}{\partial z_i}\phi(\mathbf{z}) - \frac{\partial}{\partial z_j}\phi(\mathbf{z})\right] \geq [\leq]\, 0 \quad \textit{for all } \mathbf{z} \in I^n,$$

where $(\partial/\partial z_i)\ \phi(\mathbf{z})$ denotes the partial derivative of $\phi(\mathbf{z})$ with respect to its ith argument.

LEMMA 4.2 (Mitrinović and Vasić [17]): *For real numbers α_i, β_i, $i = 1, \ldots, n$, if $\beta_i > 0$, then*

$$\min_{1\leq i\leq n}\left\{\frac{\alpha_i}{\beta_i}\right\} \leq \frac{\sum_{i=1}^n \alpha_i}{\sum_{i=1}^n \beta_i}.$$

Equalities hold if and only if $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ are proportional.

LEMMA 4.3: *For $0 < r \leq 1$ and $y_i \in \Re_+$ $(i = 1, 2)$, we have*

$$1 - \prod_{i=1}^{2}\int_0^1 \frac{y_i^r}{\Gamma(r)} u^{r-1} e^{-y_i u}\,\mathrm{d}u \leq \sum_{i=1}^{2} \frac{e^{-y_i}}{y_i \displaystyle\int_0^1 u^{r-1} e^{-y_i u}\,\mathrm{d}u + e^{-y_i}}.$$

PROOF: It is easy to see that

$$1 - \prod_{i=1}^{2}\int_0^1 \frac{y_i^r}{\Gamma(r)} u^{r-1} e^{-y_i u}\,\mathrm{d}u \leq \sum_{i=1}^{2}\left(1 - \int_0^1 \frac{y_i^r}{\Gamma(r)} u^{r-1} e^{-y_i u}\,\mathrm{d}u\right).$$

So it suffices to prove that, for each $y_i \in \Re_+ (i = 1, 2)$,

$$1 - \int_0^1 \frac{y_i^r}{\Gamma(r)} u^{r-1} e^{-y_i u}\,\mathrm{d}u \leq \frac{e^{-y_i}}{y_i \displaystyle\int_0^1 u^{r-1} e^{-y_i u}\,\mathrm{d}u + e^{-y_i}},$$

this is equivalent to showing

$$\int_0^1 \frac{y_i^r}{\Gamma(r)} u^{r-1} e^{-y_i u}\,\mathrm{d}u + \frac{y_i^{r-1}}{\Gamma(r)} \geq 1,$$

which is actually true by observing that

$$\int_0^1 \frac{y_i^r}{\Gamma(r)} u^{r-1} e^{-y_i u}\,\mathrm{d}u + \frac{y_i^{r-1}}{\Gamma(r)} e^{-y_i} \geq \int_0^1 \frac{y_i^r}{\Gamma(r)} u^{r-1} e^{-y_i u}\,\mathrm{d}u + \int_1^\infty \frac{y_i^r}{\Gamma(r)} u^{r-1} e^{-y_i u}\,\mathrm{d}u = 1.$$

■

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LEMMA 4.4: *For* $0 < r \leq 1$ *and* $y_i \in \Re_+$ $(i = 1, 2)$, *we have*

$$\left(\sum_{i=1}^{2} \frac{e^{-y_i}}{\int_0^1 u^{r-1} e^{-y_i u}\, du}\right)\left(1 - \prod_{i=1}^{2} \int_0^1 (y_i^r / \Gamma(r)) u^{r-1} e^{-y_i u}\, du\right)^{-1}$$
$$\geq y_1 + \frac{e^{-y_1}}{\int_0^1 u^{r-1} e^{-y_1 u}\, du},$$

where $y_1 < y_2$.

PROOF: From Lemma 4.3, it holds that

$$\left(\sum_{i=1}^{2} \frac{e^{-y_i}}{\int_0^1 u^{r-1} e^{-y_i u}\, du}\right)\left(1 - \prod_{i=1}^{2} \int_0^1 \frac{y_i^r}{\Gamma(r)} u^{r-1} e^{-y_i u}\, du\right)^{-1}$$
$$\geq \left(\sum_{i=1}^{2} \frac{e^{-y_i}}{\int_0^1 u^{r-1} e^{-y_i u}\, du}\right)\left(\sum_{i=1}^{2} \frac{e^{-y_i}}{y_i \int_0^1 u^{r-1} e^{-y_i u}\, du + e^{-y_i}}\right)^{-1}. \quad \textbf{(14)}$$

From Lemma 3.3, it follows that the function

$$y + \frac{e^{-y}}{\int_0^1 u^{r-1} e^{-yu}\, du}$$

is increasing in $y \in \Re_+$. Upon using this and Lemma 4.2, we have

$$\left(\sum_{i=1}^{2} \frac{e^{-y_i}}{\int_0^1 u^{r-1} e^{-y_i u}\, du}\right)\left(\sum_{i=1}^{2} \frac{e^{-y_i}}{y_i \int_0^1 u^{r-1} e^{-y_i u}\, du + e^{-y_i}}\right)^{-1}$$
$$\geq y_1 + \frac{e^{-y_1}}{\int_0^1 u^{r-1} e^{-y_1 u}\, du}. \quad \textbf{(15)}$$

Combining (14) and (15), we now get the required result. ■

Now, we are ready to present the main result of this section.

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THEOREM 4.5: *Let (X_1, X_2) be a vector of independent gamma random variables with common shape parameter $0 < r \leq 1$ and respective scale parameters λ_1 and λ_2, and let (X_1^*, X_2^*) be another vector of independent gamma random variables with common shape parameter r and respective scale parameters λ_1^* and λ_2^*. If $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$ and $\lambda_1\lambda_2 = \lambda_1^*\lambda_2^*$, Then*

$$X_{2:2} \geq_{\text{hr}} X_{2:2}^*.$$

PROOF: The hazard rate function of $X_{2:2}$ can be written as

$$\begin{aligned} r_{X_{2:2}}(t) &= \frac{f_{X_{2:2}}(t)}{\overline{F}_{X_{2:2}}(t)} \\ &= \left(\prod_{i=1}^{2}\int_0^t \frac{\lambda_i^r}{\Gamma(r)}u^{r-1}e^{-\lambda_i u}\,\mathrm{d}u\right)\left(1-\prod_{i=1}^{2}\int_0^t \frac{\lambda_i^r}{\Gamma(r)}u^{r-1}e^{-\lambda_i u}\,\mathrm{d}u\right)^{-1} \\ &\quad \times \sum_{i=1}^{2}\frac{t^{r-1}e^{-\lambda_i t}}{\displaystyle\int_0^t u^{r-1}e^{-\lambda_i u}\,\mathrm{d}u} \\ &= \frac{1}{t}\left(\prod_{i=1}^{2}\int_0^1 \frac{(\lambda_i t)^r}{\Gamma(r)}u^{r-1}e^{-\lambda_i tu}\,\mathrm{d}u\right)\left(1-\prod_{i=1}^{2}\int_0^1 \frac{(\lambda_i t)^r}{\Gamma(r)}u^{r-1}e^{-\lambda_i tu}\,\mathrm{d}u\right)^{-1} \\ &\quad \times \sum_{i=1}^{2}\frac{e^{-\lambda_i t}}{\displaystyle\int_0^1 u^{r-1}e^{-\lambda_i tu}\,\mathrm{d}u}. \end{aligned}$$

To obtain the required result, we now need to show, under the conditions $\lambda_1 \leq \lambda_1^*$ and $\lambda_1\lambda_2 = \lambda_1^*\lambda_2^*$, that

$$\begin{aligned} &\frac{1}{t}\left(\prod_{i=1}^{2}\int_0^1 \frac{(\lambda_i t)^r}{\Gamma(r)}u^{r-1}e^{-\lambda_i tu}\,\mathrm{d}u\right)\left(1-\prod_{i=1}^{2}\int_0^1 \frac{(\lambda_i t)^r}{\Gamma(r)}u^{r-1}e^{-\lambda_i tu}\,\mathrm{d}u\right)^{-1} \\ &\quad \times \sum_{i=1}^{2}\frac{e^{-\lambda_i t}}{\displaystyle\int_0^1 u^{r-1}e^{-\lambda_i tu}\,\mathrm{d}u} \\ &\leq \frac{1}{t}\left(\prod_{i=1}^{2}\int_0^1 \frac{(\lambda_i^* t)^r}{\Gamma(r)}u^{r-1}e^{-\lambda_i^* tu}\,\mathrm{d}u\right)\left(1-\prod_{i=1}^{2}\int_0^1 \frac{(\lambda_i^* t)^r}{\Gamma(r)}u^{r-1}e^{-\lambda_i^* tu}\,\mathrm{d}u\right)^{-1} \\ &\quad \times \sum_{i=1}^{2}\frac{e^{-\lambda_i^* t}}{\displaystyle\int_0^1 u^{r-1}e^{-\lambda_i^* tu}\,\mathrm{d}u} \end{aligned}$$

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for all $t \in \Re_+$, which is actually equivalent to showing that

$$\left(\prod_{i=1}^{2}\int_0^1 \frac{x_i^r}{\Gamma(r)}u^{r-1}e^{-x_iu}\,\mathrm{d}u\right)^{-1}\left(1-\prod_{i=1}^{2}\int_0^1 \frac{x_i^r}{\Gamma(r)}u^{r-1}e^{-x_iu}\,\mathrm{d}u\right)^{-1}$$
$$\times\sum_{i=1}^{2}\frac{e^{-x_i}}{\displaystyle\int_0^1 u^{r-1}e^{-x_iu}\,\mathrm{d}u}$$
$$\le\left(\prod_{i=1}^{2}\int_0^1 \frac{(x_i^*)^r}{\Gamma(r)}u^{r-1}e^{-x_i^*u}\,\mathrm{d}u\right)\left(1-\prod_{i=1}^{2}\int_0^1 \frac{(x_i^*)^r}{\Gamma(r)}u^{r-1}e^{-x_i^*u}\,\mathrm{d}u\right)^{-1}$$
$$\times\sum_{i=1}^{2}\frac{e^{-x_i^*}}{\displaystyle\int_0^1 u^{r-1}e^{-x_i^*u}\,\mathrm{d}u}$$

under the conditions $x_1 \le x_1^*$ and $x_1x_2 = x_1^*x_2^*$. Denote $y_1 = \log x_1$, $y_2 = \log x_2$, $y_1^* = \log x_1^*$, and $y_2^* = \log x_2^*$. We then have the following relation:

$$(y_1, y_2) \overset{\mathrm{m}}{\succeq} (y_1^*, y_2^*).$$

It can be seen that it suffices to show that the symmetrical differentiable function $H : (-\infty, \infty)^2 \to (0, \infty)$ given by

$$H(y_1, y_2) = \left(\prod_{i=1}^{2}\int_0^1 \frac{e^{ry_i}}{\Gamma(r)}u^{r-1}e^{-e^{y_i}u}\,\mathrm{d}u\right)\left(1-\prod_{i=1}^{2}\int_0^1 \frac{e^{ry_i}}{\Gamma(r)}u^{r-1}e^{-e^{y_i}u}\,\mathrm{d}u\right)^{-1}$$
$$\times\sum_{i=1}^{2}\frac{e^{-e^{y_i}}}{\displaystyle\int_0^1 u^{r-1}e^{-e^{y_i}u}\,\mathrm{d}u}$$

is Schur-concave in (y_1, y_2). We have

$$\frac{\partial H}{\partial y_1}(y_1, y_2)\left[1-\prod_{i=1}^{2}\int_0^1 \frac{e^{ry_i}}{\Gamma(r)}u^{r-1}e^{-e^{y_i}u}\,\mathrm{d}u\right]^2\left(\prod_{i=1}^{2}\int_0^1 \frac{e^{ry_i}}{\Gamma(r)}u^{r-1}e^{-e^{y_i}u}\,\mathrm{d}u\right)^{-1}$$
$$=\frac{e^{ey_1}e^{-e^{ey_1}}\left[\displaystyle\int_0^1 u^r e^{-e^{ey_1}u}\,\mathrm{d}u-\int_0^1 u^{r-1}e^{-e^{ey_1}u}\,\mathrm{d}u\right]}{\left[\displaystyle\int_0^1 u^{r-1}e^{-e^{ey_1}u}\,\mathrm{d}u\right]^2}$$
$$\times\left[1-\prod_{i=1}^{2}\int_0^1 \frac{e^{ry_i}}{\Gamma(r)}u^{r-1}e^{-e^{ey_i}u}\,\mathrm{d}u\right]+\sum_{i=1}^{2}\frac{e^{-e^{ey_i}}}{\displaystyle\int_0^1 u^{r-1}e^{-e^{y_i}u}\,\mathrm{d}u}\frac{e^{-e^{y_1}}}{\displaystyle\int_0^1 u^{r-1}e^{-e^{y_1}u}\,\mathrm{d}u}.$$

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By using integration by parts, we get

$$y\int_0^1 u^r e^{-yu}\,\mathrm{d}u = r\int_0^1 u^{r-1}e^{-yu}\,\mathrm{d}u - e^{-y}.$$

Thus, we then have

$$\begin{aligned}
&\frac{\partial H}{\partial y_1}(y_1,y_2) - \frac{\partial H}{\partial y_2}(y_1,y_2)\\
&\overset{\text{sgn}}{=} r\left[\frac{e^{-e^{y_1}}}{\int_0^1 u^{r-1}e^{-e^{y_1}u}\,\mathrm{d}u} - \frac{e^{-e^{y_2}}}{\int_0^1 u^{r-1}e^{-e^{y_2}u}\,\mathrm{d}u}\right]\left[1-\prod_{i=1}^{2}\int_0^1 \frac{e^{ry_i}}{\Gamma(r)}u^{r-1}e^{-e^{y_i}u}\,\mathrm{d}u\right]\\
&\quad + \frac{e^{-e^{y_1}}}{\int_0^1 u^{r-1}e^{-e^{y_1}u}\,\mathrm{d}u}\left[\sum_{i=1}^{2}\frac{e^{-e^{y_i}}}{\int_0^1 u^{r-1}e^{-e^{y_i}u}\,\mathrm{d}u} - \left(e^{y_1}+\frac{e^{-e^{y_1}}}{\int_0^1 u^{r-1}e^{-e^{y_1}u}\,\mathrm{d}u}\right)\right.\\
&\qquad\qquad \left.\times\left(1-\prod_{i=1}^{2}\int_0^1\frac{e^{ry_i}}{\Gamma(r)}u^{r-1}e^{-e^{y_i}u}\,\mathrm{d}u\right)\right]\\
&\quad - \frac{e^{-e^{y_2}}}{\int_0^1 u^{r-1}e^{-e^{y_2}u}\,\mathrm{d}u}\left[\sum_{i=1}^{2}\frac{e^{-e^{y_i}}}{\int_0^1 u^{r-1}e^{-e^{y_i}u}\,\mathrm{d}u} - \left(e^{y_2}+\frac{e^{-e^{y_2}}}{\int_0^1 u^{r-1}e^{-e^{y_2}u}\,\mathrm{d}u}\right)\right.\\
&\qquad\qquad \left.\times\left(1-\prod_{i=1}^{2}\int_0^1\frac{e^{ry_i}}{\Gamma(r)}u^{r-1}e^{-e^{y_i}u}\,\mathrm{d}u\right)\right]\\
&= \alpha_1+\alpha_2, \quad \text{say,}
\end{aligned}$$

where

$$\alpha_1 = r\left[\frac{e^{-e^{y_1}}}{\int_0^1 u^{r-1}e^{-e^{y_1}u}\,\mathrm{d}u} - \frac{e^{-e^{y_2}}}{\int_0^1 u^{r-1}e^{-e^{y_2}u}\,\mathrm{d}u}\right]\left[1-\prod_{i=1}^{2}\int_0^1 \frac{e^{ry_i}}{\Gamma(r)}u^{r-1}e^{-e^{y_i}u}\,\mathrm{d}u\right]$$

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and

$$\alpha_2 = \frac{e^{-e^{y_1}}}{\int_0^1 u^{r-1}e^{-e^{y_1}u}\,du}\left[\sum_{i=1}^{2}\frac{e^{-e^{y_i}}}{\int_0^1 u^{r-1}e^{-e^{y_i}u}\,du} - \left(e^{y_1} + \frac{e^{-e^{y_1}}}{\int_0^1 u^{r-1}e^{-e^{y_1}u}\,du}\right)\right.$$
$$\left.\times\left(1 - \prod_{i=1}^{2}\int_0^1 \frac{e^{ry_i}}{\Gamma(r)}u^{r-1}e^{-e^{y_i}u}\,du\right)\right]$$
$$- \frac{e^{-e^{y_2}}}{\int_0^1 u^{r-1}e^{-e^{y_2}u}\,du}\left[\sum_{i=1}^{2}\frac{e^{-e^{y_i}}}{\int_0^1 u^{r-1}e^{-e^{y_i}u}\,du} - \left(e^{y_2} + \frac{e^{-e^{y_2}}}{\int_0^1 u^{r-1}e^{-e^{y_2}u}\,du}\right)\right.$$
$$\left.\times\left(1 - \prod_{i=1}^{2}\int_0^1 \frac{e^{ry_i}}{\Gamma(r)}u^{r-1}e^{-e^{y_i}u}\,du\right)\right].$$

Observe that the function

$$\frac{e^{-y}}{\int_0^1 u^{r-1}e^{-yu}\,du} = \frac{1}{\int_0^1 u^{r-1}e^{(1-u)y}\,du}$$

is decreasing in $y \in \Re_+$ which implies that $\alpha_1 \geq 0$ for $y_1 < y_2$. Next, we will prove α_2 is also nonnegative for $y_1 < y_2$. From Lemma 4.4, it is known that

$$\sum_{i=1}^{2}\frac{e^{-e^{y_i}}}{\int_0^1 u^{r-1}e^{-e^{y_i}u}\,du} - \left(e^{y_1} + \frac{e^{-e^{y_1}}}{\int_0^1 u^{r-1}e^{-e^{y_1}u}\,du}\right)$$
$$\times\left(1 - \prod_{i=1}^{2}\int_0^1 \frac{e^{ry_i}}{\Gamma(r)}u^{r-1}e^{-e^{y_i}u}\,du\right) \geq 0$$

for $0 < r \leq 1$ and $y_1 < y_2$. Thus, we have

$$\alpha_2 \geq \frac{e^{-e^{y_2}}}{\int_0^1 u^{r-1}e^{-e^{y_2}u}\,du}\left[\left(e^{y_2} + \frac{e^{-e^{y_2}}}{\int_0^1 u^{r-1}e^{-e^{y_2}u}\,du}\right) - \left(e^{y_1} + \frac{e^{-e^{y_1}}}{\int_0^1 u^{r-1}e^{-e^{y_1}u}\,du}\right)\right]$$
$$\times\left[1 - \prod_{i=1}^{2}\int_0^1 \frac{e^{ry_i}}{\Gamma(r)}u^{r-1}e^{-e^{y_i}u}\,du\right]$$
$$\geq 0,$$

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where the last inequality can be obtained by using Lemma 3.3. Now, we can get

$$(y_1 - y_2)\left(\frac{\partial H}{\partial y_1}(y_1, y_2) - \frac{\partial H}{\partial y_2}(y_1, y_2)\right) \leq 0,$$

and the desired result can be derived from Lemma 4.1. We finish the entire proof. ■

THEOREM 4.6: *Let* (X_1, X_2) *be a vector of independent gamma random variables with common shape parameter* $0 < r \leq 1$ *and respective scale parameters* λ_1 *and* λ_2*, and let* (X_1^*, X_2^*) *be another vector of independent gamma random variables with common shape parameter r and respective scale parameters* λ_1^* *and* λ_2^**. Suppose*

$$\min(\lambda_1, \lambda_2) \leq \min(\lambda_1^*, \lambda_2^*) \leq \max(\lambda_1^*, \lambda_2^*) \leq \max(\lambda_1, \lambda_2).$$

Then

$$(\lambda_1, \lambda_2) \overset{\text{p}}{\geq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\text{hr}} X_{2:2}^*.$$

PROOF: Without loss of generality, let us assume that $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$. Suppose $(\lambda_1, \lambda_2) \overset{\text{p}}{\geq} (\lambda_1^*, \lambda_2^*)$. Clearly, we have $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$ and $\lambda_1\lambda_2 \leq \lambda_1^*\lambda_2^*$. The result for the case when $\lambda_1\lambda_2 = \lambda_1^*\lambda_2^*$ follows from Theorem 4.5. Next, let us assume that $\lambda_1\lambda_2 < \lambda_1^*\lambda_2^*$. Let $\lambda' = \lambda_1^*\lambda_2^*/\lambda_2$. We then have $\lambda'\lambda_2 = \lambda_1^*\lambda_2^*$ and $\lambda_1 < \lambda' \leq \lambda_1^*$. Let $Z_{2:2}$ be the lifetime of a parallel system consisting of two independent gamma components with shape parameter r and respective scale parameters λ' and λ_2. From Theorem 4.5, it follows that $Z_{2:2} \geq_{\text{hr}} X_{2:2}^*$. As likelihood ratio order implies hazard rate order, we also have $X_{2:2} \geq_{\text{hr}} Z_{2:2}$ from Theorem 3.4 and, hence, we obtain the desired result that $X_{2:2} \geq_{\text{hr}} X_{2:2}^*$. ■

Remark 4.7: It can be seen that the result in Theorem 4.6 strengthens that in Zhao and Balakrishnan [25] from exponential to gamma framework.

Remark 4.8: Under the exponential framework, Boland et al. [4] pointed out with the help of a counterexample that the result in Theorem 4.6 does not hold even for the case when $n = 3$. We therefore conclude, in general, that Theorem 4.6 cannot be extended to the general case when $n \geq 3$.

The following corollary is a direct consequence of Theorem 4.6, which provides a upper bound for the hazard rate function of maximum of two gamma random variables.

COROLLARY 4.9: *Let* (X_1, X_2) *be a vector of independent gamma random variables with common shape parameter* $0 < r \leq 1$ *and respective scale parameters* λ_1 *and* λ_2*, and let* (X_1^*, X_2^*) *be another vector of independent gamma random variables with common shape and scale parameters r and* λ*. Assume* $\lambda \leq \max(\lambda_1, \lambda_2)$*. Then*

$$\lambda \geq \sqrt{\lambda_1\lambda_2} \Longrightarrow X_{2:2} \geq_{\text{hr}} X_{2:2}^*.$$

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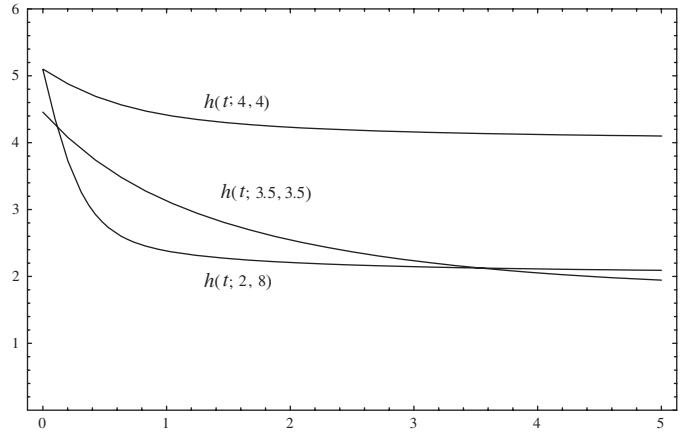


FIGURE 2. *Plots of hazard rate functions of* $X_{2:2}$ *from two independent gamma distributions with common shape parameter* 0.5 *and scale parameter vector* $(2, 8)$*,* $Y_{2:2}$ *from two gamma distributions with common shape parameter* 0.5 *and scale parameter vector* $(4, 4)$*, and* $Z_{2:2}$ *from two gamma distributions with common shape parameter* 0.5 *and scale parameter vector* $(3.5, 3.5)$.

Remark 4.10: Following an argument similar to that of Remark 3.9, we can claim that the result of Corollary 4.9 is also valid without the assumption that $\lambda \leq \max(\lambda_1, \lambda_2)$.

Let (X_1, X_2) be a vector of two independent gamma random variables with common shape parameter $r = 0.5$ and scale parameter vector $(2, 8)$. Denote by $h(t; 2, 8)$ the corresponding hazard rate function of the maximum order statistic $X_{2:2}$. Let (Y_1, Y_2) be a vector of two independent gamma random variables with common shape parameter $r = 0.5$ and scale parameter vector $(4, 4)$ and denote by $h(t; 4, 4)$ the corresponding hazard rate function of $Y_{2:2}$. Figure 2 presents the hazard rate functions of $X_{2:2}$ and $Y_{2:2}$. It can be seen that $h(t; 2, 8) \leq h(t; 4, 4)$ and this is in accordance with the result derived in Corollary 4.9. Let (Z_1, Z_2) be a vector of two independent gamma random variables with common shape parameter $r = 0.5$ and scale parameter vector $(3.5, 3.5)$ and denote by $h(t; 3.5, 3.5)$ the corresponding hazard rate function of $Z_{2:2}$. It is clear that the hazard rate order does not hold between $X_{2:2}$ and $Z_{2:2}$, as shown in Figure 2.

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