Proceedings of the Edinburgh Mathematical Society (2018) **61**, 329–341
doi:10.1017/S0013091517000141

FINITE GROUPS WHOSE COMMON-DIVISOR GRAPH IS REGULAR

MEHDI GHAFFARZADEH[1], MOHSEN GHASEMI[2] AND MARK L. LEWIS[3]*

[1]*Department of Mathematics, Khoy Branch, Islamic Azad University, Khoy, Iran*
(ghaffarzadeh@iaukhoy.ac.ir)

[2]*Department of Mathematics, Urmia University, Urmia 57135, Iran*
(m.ghasemi@urmia.ac.ir)

[3]*Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA*
(lewis@math.kent.edu)

(Received 29 June 2016)

Abstract Let G be a finite group, and write $\mathrm{cd}(G)$ for the set of degrees of irreducible characters of G. The common-divisor graph $\Gamma(G)$ associated with G is the graph whose vertex set is $\mathrm{cd}(G) \setminus \{1\}$ and there is an edge between distinct vertices a and b, if $(a, b) > 1$. In this paper we prove that if $\Gamma(G)$ is a k-regular graph for some $k \geqslant 0$, then for the solvable groups, either $\Gamma(G)$ is a complete graph of order $k + 1$ or $\Gamma(G)$ has two connected components which are complete of the same order and for the non-solvable groups, either $k = 0$ and $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{PSL}_2(2^f))$, where $f \geqslant 2$ or $\Gamma(G)$ is a 4-regular graph with six vertices and $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{Alt}_7)$ or $\mathrm{cd}(\mathrm{Sym}_7)$.

Keywords: Frobenius group; character degrees; regular graph

2010 *Mathematics subject classification:* Primary 20C15
Secondary 05C25

1. Introduction

Given a finite group G, let $\mathrm{Irr}(G)$ be the set of all irreducible complex characters of G and let $\mathrm{cd}(G)$ denote the set $\{\chi(1) \mid \chi \in \mathrm{Irr(G)}\}$. The *common-divisor graph* $\Gamma(G)$ associated with G is the graph whose vertex set is $\mathrm{cd}(G) \setminus \{1\}$ and there is an edge between distinct vertices a and b, if $(a, b) > 1$. We will assume that G is non-abelian throughout the paper, so that $\Gamma(G)$ is not the empty graph. This graph which was introduced by Manz *et al.* in [16] is a useful tool in studying the character degree set $\mathrm{cd}(G)$. It is well known that the character degree set $\mathrm{cd}(G)$ may be used to provide information on the structure of the group G. For example, Manz *et al.* [15, 16] have shown that $\Gamma(G)$ has at most two connected components when G is solvable and it has at most three connected components when G is not solvable. In [2], Berkovich proved that if some prime number p divides every

* Corresponding author.

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degree in $\mathrm{cd}(G) \setminus \{1\}$, then G is solvable. Later Bianchi *et al.* [4] generalized Berkovich's result by proving that if $\Gamma(G)$ is a complete graph, then G is a solvable group.

In this field of study, one of the main questions that arises is which graphs occur as $\Gamma(G)$ for some finite group G? In this paper, we classify all regular graphs which occur as $\Gamma(G)$ of some finite group G. Recall that a graph is called regular if all the vertices have the same degree and we say that the graph is k-regular if the graph is regular and all the vertices have degree k. Also, a graph is complete if there is an edge between every pair of vertices. For solvable groups, we obtain the following.

Theorem 1.1. *Let G be a finite solvable group and $\Gamma(G)$ be a k-regular graph, where $k \geqslant 0$. Then we have one of the following cases.*

(i) *If $\Gamma(G)$ is connected, then $\Gamma(G)$ is a complete graph of order $k+1$.*

(ii) *If $\Gamma(G)$ is not connected, then $\Gamma(G)$ has two connected components which are complete and have the same orders $k+1$.*

It is not difficult to find a solvable group G where $\Gamma(G)$ is a complete graph of order $k+1$. Perhaps the easiest way to obtain such an example is to note that if G is a p-group for some prime p, then $\Gamma(G)$ is a complete graph, and then use the construction in [10] to see for every set of p-powers that includes 1 that there exists a p-group G so that $\mathrm{cd}(G)$ is this set. To obtain a solvable group G where $\Gamma(G)$ is a complete graph of order $k+1$, consider any set of $k+2$ different p-powers which includes 1, and take G to be a p-group whose character degree set is this set of p-powers.

We will use the examples found in § 3 of [17] to show that for every positive integer k there is a solvable group G where $\Gamma(G)$ is isomorphic to $2K_{k+1}$. In particular, consider Example 3.1 of [17]. For our purposes, we take p to be a prime that is greater than $k+1$, take q to be a prime different from p so that p does not divide $q-1$, and take $e = p^{k+1}$ and $n = k+1$. Notice that $(e, q) = (e, n!) = (e, q-1) = 1$. Define G to be the group $PC\Gamma$ from Example 3.1 of [17]. We have from Corollary 3.5 of [17] that $\mathrm{cd}(G)$ is the union of the divisors of $e = p^{k+1}$ and the set $\{(q^e - 1)q^{i(e-1)/2}/(q-1) \mid 0 \leqslant i \leqslant n-1\}$. Note that p^{k+1} has $k+1$ non-trivial divisors, and so, $\Gamma(G)$ will be isomorphic to $2K_{k+1}$.

Finally, for a non-solvable group G, we now show that the only time we have that $\Gamma(G)$ is a regular graph is when G has the same character degree set as either Alt_7, Sym_7 or $\mathrm{PSL}_2(2^f)$, where $f \geqslant 2$.

Theorem 1.2. *Let G be a finite non-solvable group. Then $\Gamma(G)$ is a k-regular graph for some $k \geqslant 0$ if and only if one of the following holds.*

(i) *$G \cong \mathrm{PSL}_2(2^f) \times A$, where $f \geqslant 2$ and A is an abelian group. In this case, we have $k = 0$ and $\mathrm{cd}(G) = \{1, 2^f - 1, 2^f, 2^f + 1\}$.*

(ii) *$G \cong \mathrm{Alt}_7 \times A$, where A is an abelian group. In this case, we have $k = 4$ and $\mathrm{cd}(G) = \{1, 6, 10, 14, 15, 21, 35\}$.*

(iii) *$G' \cong \mathrm{Alt}_7$, $G/Z(G) \cong \mathrm{Sym}_7$ and $C_G(G') = Z(G)$. In this case, we have $k = 4$ and $\mathrm{cd}(G) = \{1, 6, 14, 15, 20, 21, 35\}$.*

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In the literature, often another graph $\Delta(G)$ has been attached to $\mathrm{cd}(G)$. This graph, which is called the *prime graph* of G, has the primes dividing degrees in $\mathrm{cd}(G)$ as its vertices, and there is an edge between p and q if pq divides some degree $a \in \mathrm{cd}(G)$. The regularity of the prime graph $\Delta(G)$ of a finite group G has been studied in [18, 21]. In [21], it has been shown that $\Delta(G)$ is 3-regular if and only if it is a complete graph with four vertices and then in [18], the question is answered completely in the case of solvable groups. In fact it has been shown that if G is a solvable group with a regular prime graph of order k, then $\Delta(G)$ is either complete or $(k-2)$-regular.

It is not surprising that the prime graph and the common-divisor graph of a finite group G are closely related. Also it is not difficult to show that these graphs have the same number of connected components (see [12, Corollary 3.2]). Thus the various classification results for when $\Delta(G)$ is disconnected will apply when $\Gamma(G)$ is disconnected. For more information we refer the readers to a survey by Lewis [12].

Finally, while we are discussing the literature, we note that often there is a parallelism between results regarding character degrees of groups and conjugacy class sizes. We note that the parallel problem for conjugacy class sizes was studied by Bianchi *et al.* in [3], and one will note the similarity in the ideas of proof with the work here.

2. Common-divisor graphs of groups

In this section, we introduce some graph theoretic terminologies and recall some known results on the common-divisor graphs of finite groups which will be needed in this paper.

Let $\mathcal{G}$ be a graph with vertex set $V(\mathcal{G})$. For a given vertex v of $\mathcal{G}$, denote by $N(v)$, the set of neighbours of v in $\mathcal{G}$. The *closed neighbourhood* is $N[v] = N(v) \cup \{v\}$. We say that two vertices v and w of $\mathcal{G}$ are *partners*, if $N[v] = N[w]$ and in this case we write $v \leftrightarrow w$. (This is similar to Definition 2.1 in [3].) Clearly, this defines an equivalence relation on the vertex set of $\mathcal{G}$. The notation $v \sim w$ will be used to show that two vertices v and w in $\mathcal{G}$ are adjacent and we use $n(\mathcal{G})$, to denote the number of connected components of $\mathcal{G}$.

The following result shows that the number of connected components of the graph $\Gamma(G)$ is at most three and that if the graph is disconnected, then at least one is complete.

Proposition 2.1 ([16], [12, Theorem 7.1]). *Let G be a group. Then $n(\Gamma(G)) \leqslant 3$. Also if $\Gamma(G)$ is disconnected, then one of the following occurs.*

(1) *If G is solvable, then there is a prime dividing all the degrees corresponding to the vertices of one connected component of $\Gamma(G)$, and the other connected component has diameter at most 2.*

(2) *If $n(\Gamma(G)) = 3$, then G is not solvable and each component is an isolated vertex.*

(3) *If G is non-solvable and $n(\Gamma(G)) = 2$, then one connected component is an isolated vertex, and the other component has diameter at most 2.*

Solvable groups with disconnected prime graphs have been completely classified by Lewis, [11]. They fall into six families described in [11, § 2] and in some cases, the set of character degrees are fully determined. Note that for a finite group G,

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$n(\Delta(G)) = n(\Gamma(G))$. Now, by using this classification and Proposition 2.1, we can determine the types of disconnected regular common-divisor graphs of finite groups.

Lemma 2.2. *Let the common-divisor graph $\Gamma(G)$ of a group G be k-regular for some $k \geqslant 0$. If G is solvable, then $\Gamma(G)$ is either connected or a disconnected graph with two complete components of the same orders. Also, if G is non-solvable, then $\Gamma(G)$ is either connected or a disconnected graph with three isolated components.*

Proof. First suppose that G is solvable and $\Gamma(G)$ is not connected. By Proposition 2.1, $\Gamma(G)$ has two connected components, and one of the connected components of $\Gamma(G)$ is complete since there is a prime that divides every degree in that component. By [**11**, Lemmas 3.1–3.6], the other connected component contains a degree a so that every other degree b in that connected component has the property that b divides a. This implies that a is adjacent to every degree in its connected component. As $\Gamma(G)$ is k-regular, this implies that this connected component is also complete. Therefore $\Gamma(G)$ is isomorphic to $2K_{k+1}$.

Now suppose that G is non-solvable. If $\Gamma(G)$ is not connected, then by Proposition 2.1, $\Gamma(G)$ has at most three connected components. Also at least one of the connected components of $\Gamma(G)$ has only one vertex. Hence the regularity of $\Gamma(G)$ implies that each component is an isolated vertex. But the groups with $|\mathrm{cd}(G)| = 3$ are all solvable, so $n(\Gamma(G)) = 3$, as wanted. □

We will need several results before we can prove the main results.

Lemma 2.3. *Let G be a group and suppose that $\Gamma(G)$ is a k-regular connected graph with $k \geqslant 0$. If $\chi(1)$ is a vertex of $\Gamma(G)$ and $\chi(1)$ is a prime power, then $\Gamma(G)$ is a complete graph.*

Proof. Suppose that $\chi(1)$ is a power of the prime number p. Then all neighbours of $\chi(1)$ are divisible by p. Thus the subgraph of $\Gamma(G)$ induced by $N[\chi(1)]$ is complete. If $\Gamma(G)$ is not complete, then the connectedness of $\Gamma(G)$ implies that there exists a vertex of $\Gamma(G)$, say $\psi(1)$, which is not adjacent to $\chi(1)$ but is adjacent to a neighbour of $\chi(1)$. Now the degree of this neighbour is at least $k + 1$, a contradiction. □

If n is a positive integer, we use $\pi(n)$ to denote the set of prime divisors of n.

Lemma 2.4. *Let G be a group and $\Gamma(G)$ be a regular graph. Let χ and ψ be nonlinear irreducible characters of G with $\pi(\chi(1)) \subseteq \pi(\psi(1))$. Then $\chi(1)$ and $\psi(1)$ are partners.*

Proof. Since $\pi(\chi(1)) \subseteq \pi(\psi(1))$, it follows that $N[\chi(1)] \subseteq N[\psi(1)]$. But since $\Gamma(G)$ is regular, we conclude that $N[\chi(1)] = N[\psi(1)]$, as wanted. □

3. Proof of main results

In this section, we give proofs for Theorems 1.1 and 1.2. For non-solvable groups, we use the classification of finite simple groups.

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Proof of Theorem 1.1. First suppose that $\Gamma(G)$ is connected. Also let $K \lhd G$ be maximal such that G/K is non-abelian. By [9, Lemma 12.3], all nonlinear irreducible characters of G/K have equal degree f and one of the following cases holds.

Case 1. G/K is a non-abelian p-group for some prime p.
In this case, G/K has an irreducible character of degree p^r for some integer $r \geqslant 1$. Thus $p^r \in \mathrm{cd}(G)$ and Lemma 2.3 yields that $\Gamma(G)$ is a complete graph.

Case 2. G/K is a Frobenius group with Frobenius kernel $(G/K)'$, an elementary abelian p-group for some prime p, and $f = |G : G'K| \in \mathrm{cd}(G)$.

We suppose $\Gamma(G)$ is not a complete graph, and we find a contradiction. The regularity of $\Gamma(G)$ implies the existence of a nonlinear character $\chi \in \mathrm{Irr}(G)$ such that $(\chi(1), f) = 1$. Set $N = G'K$. Then $(\chi(1), |G : N|) = 1$ and so by [9, Corollary 11.29], $\chi_N \in \mathrm{Irr}(N)$. Now by [9, Theorem 12.4], $f\chi(1) = |G : N|\chi(1) \in \mathrm{cd}(G)$ or $p|\chi(1)$.

If $f\chi(1) \in \mathrm{cd}(G)$, then by Lemma 2.4, $f \leftrightarrow f\chi(1)$. Also $\chi(1)$ is adjacent to $f\chi(1)$ and so $\chi(1)$ is adjacent to f, a contradiction. So $\chi(1)$ is divisible by p. This implies that p divides every degree in $\mathrm{cd}(G) - \{1, f\}$ that is not adjacent to f in $\Gamma(G)$.

Notice that if every neighbour of f is a partner of f, then $N[f]$ would form a connected component of $\Gamma(G)$ and since $\chi(1)$ is not in this set, this would contradict $\Gamma(G)$ connected. Thus, we can find a character $\psi \in \mathrm{Irr}(G)$ so that $\psi(1)$ is a neighbour of f, but not a partner for f. Let ν be an irreducible constituent of ψ_N. By [9, Theorem 12.4], we know that either $f\nu(1) \in \mathrm{cd}(G)$ or $p|\nu(1)$. Applying [9, Corollary 11.29], we have $\psi(1)/\nu(1)$ divides f, so $\psi(1)$ divides $f\nu(1)$. If $f\nu(1) \in \mathrm{cd}(G)$, then by applying Lemma 2.4, we see that both f and $\psi(1)$ are partners with $f\nu(1)$. As the partnership is an equivalence relation on $\mathrm{cd}(G)$, so f and $\psi(1)$ are partners which is a contradiction to the choice of ψ. Thus p divides $\nu(1)$ and so p divides $\psi(1)$. In particular, we can conclude that p divides every degree that is adjacent to f but not a partner to f.

We have shown that $\mathrm{Irr}(G)$ contains a character ψ so that $\psi(1)$ is adjacent to f but not a partner of f. Since $\Gamma(G)$ is not complete, there must exist a degree $e \in \mathrm{cd}(G)$ that is relatively prime to $\psi(1)$. Because p divides $\psi(1)$, we conclude that p does not divide e. This implies that e cannot be relatively prime to f, so e must be adjacent to f. Also, if e is not a partner to f, then p would divide e. Therefore, e must be a partner to f. This is a contradiction since $\psi(1)$ is a neighbour of f, but not a neighbour of e. Thus, we have proved Conclusion (i), that $\Gamma(G)$ is a complete graph of order $k + 1$.

Finally, suppose that $\Gamma(G)$ is disconnected. By Lemma 2.2, $\Gamma(G)$ has two complete components of the same orders. This completes the proof of Conclusion (ii), that $\Gamma(G)$ has two connected components that are complete and have the same orders $k + 1$. □

Let N be a normal subgroup of a group G, and $\nu \in \mathrm{Irr}(N)$. We write $\mathrm{Irr}(G \mid \nu)$ for the set of irreducible constituents of ν^G and $\mathrm{cd}(G \mid \nu) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G \mid \nu)\}$. If ν extends to G, then Gallagher's theorem [9, Corollary 6.17] gives a description of $\mathrm{Irr}(G \mid \nu)$ and $\mathrm{cd}(G \mid \nu)$. In particular, the characters in $\mathrm{Irr}(G \mid \nu)$ are in bijection with the characters in $\mathrm{Irr}(G/N)$. Since we will use the character degree version, we state this as a theorem.

Theorem 3.1 (Gallagher [9, Corollary 6.17]). *Let N be a normal subgroup of a group G, and let $\nu \in \mathrm{Irr}(N)$ extend to G. Then $\mathrm{cd}(G \mid \nu) = \{\nu(1)a \mid a \in \mathrm{cd}(G/N)\}$.*

Next, we are going to show that every non-abelian chief factor of non-solvable groups with a connected regular common-divisor graph is isomorphic to Alt_7. Our proof for

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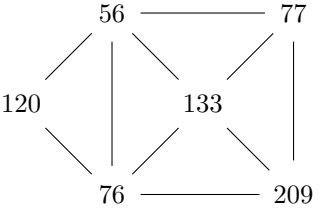
discarding the groups with a chief factor isomorphic to a sporadic group except the Janko group J_1, is the same. So we first focus on J_1.

Lemma 3.2. *Let $N \lhd G$ be groups so that $G/N \cong J_1$, the first Janko group. Then $\Gamma(G)$ is not regular.*

Proof. By using [6], we have that

$$\mathrm{cd}(J_1) = \{1, 56, 76, 77, 120, 133, 209\}.$$

Note that $\Gamma(J_1)$ is not regular.



Suppose to the contrary that $\Gamma(G)$ is regular. Let $\nu \in \mathrm{Irr}(N)$ and fix T as the stabilizer of ν in G. We first assume that ν is G-invariant. As the Schur multiplier of J_1 is trivial, so by Theorem 11.7 of [9], ν extends to G. Thus by Theorem 3.1, $a\nu(1) \in \mathrm{cd}(G)$ for all $a \in \mathrm{cd}(G/N)$. If $\nu(1) > 1$, then $120 \leftrightarrow 120\nu(1) \sim 133\nu(1) \leftrightarrow 133$, which implies $120 \sim 133$, a contradiction. Thus $\nu(1) = 1$ and so $\mathrm{cd}(G \mid \nu) = \mathrm{cd}(G/N)$.

Next, we assume that ν is not G-invariant. Then T/N is contained in a maximal subgroup M/N of G/N. By Atlas [6], the possibilities for $|G : M|$ are 2.7.19, 5.11.19, 7.11.19, 2^2.5.7.11, 2^2.3.7.19, 2.7.11.19, and 2^2.5.11.19. It follows that every degree in $\mathrm{cd}(G \mid \nu)$ is divisible by 7 or 19, and therefore all degrees in $\mathrm{cd}(G \mid \nu)$ are adjacent to 133.

Now, 133 is adjacent to all the vertices in $\Gamma(G)$ except 120, and on the other hand, 120 is not adjacent to any of 77, 133, or 209, so the degree of the vertex associated with 120 will be less than the degree of the vertex for 133. This yields a contradiction to the regularity of $\Gamma(G)$. □

Before proving Theorem 1.2, we must gather some facts about particular character degrees of non-abelian simple groups.

Lemma 3.3. *Let S be a non-abelian simple group with $S \notin \{J_1, \mathrm{Alt}_7\}$. If S is one of the following groups: M_{11}, M_{24}, Co_3, Co_2, a simple group of Lie type, the Tits group, or Alt_n where $n-1$ is a power of some prime number, then there exists a nonlinear character $\sigma \in \mathrm{Irr}(S)$ so that σ extends to $\mathrm{Aut}(S)$ and $\sigma(1)$ is a power of some prime number. If S is one of the remaining sporadic simple groups or Alt_n where $n-1$ is not a prime power, there exist three nonlinear characters $\rho, \sigma, \tau \in \mathrm{Irr}(S)$, that are extendible to $\mathrm{Aut}(S)$ and $\pi(\rho(1)) \subseteq \pi(\sigma(1))$, $(\rho(1), \tau(1)) = 1$ and $(\sigma(1), \tau(1)) > 1$.*

Proof. If S is a simple group of Lie type with characteristic p for some prime p, then the Steinberg character extends to $\mathrm{Aut}(S)$ and its degree is a power of p (see [20] and [5, Theorem 6.4.7]).

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Let $S \cong \mathrm{Alt}_n$, $n \geqslant 5$. Observe that $\mathrm{Alt}_5 \cong \mathrm{PSL}_2(5)$, $\mathrm{Alt}_6 \cong \mathrm{PSL}_2(9)$ and $\mathrm{Alt}_8 \cong \mathrm{PSL}_4(2)$, so we may assume that $n \geqslant 9$. Notice that for $n \neq 6$, $\mathrm{Aut}(\mathrm{Alt}_n) \cong \mathrm{Sym}_n$. We know that the irreducible characters of Sym_n are associated with the partitions of n. First, consider the character σ associated with the partition $(n-1,1)$. Note that σ has degree $n-1$ and for $n \geqslant 2$, it restricts irreducibly to Alt_n. When $n-1$ is a power of a prime, this yields the desired character.

Consider the partitions $(n-2,2)$ and $(n-2,1^2)$ and note that the corresponding characters of Sym_n have coprime degrees $\rho(1) = n(n-3)/2$ and $\tau(1) = (n-1)(n-2)/2$, respectively. For $n \geqslant 6$, the above partitions are not self-associated, so the characters ρ and τ restrict irreducibly to Alt_n. Consider $9 \leqslant n \leqslant 14$. For $n = 9, 10, 12, 14$, we have that $n-1$ is a power of a prime number and so we have the result. This leaves $n = 11$ and 13. For $n = 11$, the characters ρ and τ together with the character σ associated with the partition $(8,3)$ which has degree 110 yield the conclusion. For $n = 13$, we use the characters ρ and τ together with the character labelled χ_{26} in [6] whose degree is $3575 = 5^2 \cdot 11 \cdot 13$. If $n \geqslant 15$ and $n-1$ is not a prime power, then by [1], there exists $\delta \in \mathrm{Irr}(\mathrm{Sym}_n)$ which restricts irreducibly to Alt_n, with the property that its degree is divisible by every prime less than or equal to n. Thus $\pi(\rho(1)) \subseteq \pi(\delta(1))$ and $\pi(\tau(1)) \subseteq \pi(\delta(1))$.

Finally for the sporadic groups and Tits group ${}^2F_4(2)'$, we use the Atlas, [6]. Note that, in this case $|\mathrm{Out}(S)| \leqslant 2$. If $|\mathrm{Out}(S)| = 1$, then any irreducible character of S is also viewed as that of $\mathrm{Aut}(S)$, and if $|\mathrm{Out}(S)| = 2$, then by [9, Corollary 6.19], every character of $\mathrm{Irr}(S)$ which is the unique character of its degree will be extendible to $\mathrm{Aut}(S)$. (These are listed in Table 1.) □

In the next two lemmas, we obtain some restrictions on the structure of non-solvable groups whose common-divisor graphs are k-regular for some $k \geqslant 1$.

Lemma 3.4. *Let the common-divisor graph $\Gamma(G)$ of a non-solvable group be k-regular with $k \geqslant 1$. Then every non-abelian chief factor of G is isomorphic to* Alt_7.

Proof. Let M/N be a non-abelian chief factor of G. Then M/N is a direct product $S_1 \times S_2 \times \cdots \times S_t$ of t copies of a non-abelian simple group S. Let $C/N = C_{G/N}(M/N)$. Then G/C has a unique minimal normal subgroup MC/C which is isomorphic to M/N. We have that G/C is isomorphic to a subgroup of $\mathrm{Aut}(M/N) \cong \mathrm{Aut}(S) \wr \mathrm{Sym}_t$. If S has an irreducible character σ of prime power degree which is extendible to $\mathrm{Aut}(S)$, then by [4, Theorem 5], the character $\sigma \times \sigma \times \cdots \times \sigma \in \mathrm{Irr}(M/N)$ has an extension to G/C. So $\sigma(1)^t \in \mathrm{cd}(G)$ and Lemma 2.3 yields that $\Gamma(G)$ is complete which is impossible by [4, Main Theorem].

Thus, we may assume that $\mathrm{Irr}(S)$ does not contain a character of prime power degree that is extendible to $\mathrm{Aut}(S)$. Applying Lemma 3.3, we see that S has irreducible characters ρ and τ that are extendible to $\mathrm{Aut}(S)$ such that $(\rho(1), \tau(1)) = 1$. Consider the following characters of M/N:

$$\eta_1 = \rho \times \rho \times \cdots \times \rho \quad \eta_2 = \tau \times \tau \times \cdots \times \tau,$$

$$\eta_3 = \rho \times 1 \times \cdots \times 1, \quad \eta_4 = \tau \times 1 \times \cdots \times 1.$$

By [4, Lemma 5], the characters η_1 and η_2 extend to G, and so, $\rho(1)^t, \tau(1)^t \in \mathrm{cd}(G)$. Furthermore, G/C acts transitively on $\{S_1, \ldots, S_t\}$ and thus $t = |G/C : N_{G/C}(S_1)|$.

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Table 1. *Extendible character degrees.*

S	Characters	Degrees	$\lvert\mathrm{Out}(S)\rvert$
M_{11}	χ_5	11	1
M_{12}	χ_6	$45 = 3^2.5$	2
	χ_{13}	$120 = 2^3.3.5$	
	χ_{15}	$176 = 2^4.11$	
M_{22}	χ_2	$21 = 3.7$	2
	χ_5	$55 = 5.11$	
	χ_9	$231 = 3.7.11$	
J_2	χ_6	$36 = 2^2.3^2$	2
	χ_{11}	$126 = 2.3^2.7$	
	χ_{13}	$175 = 5^2.7$	
M_{23}	χ_2	$22 = 2.11$	1
	χ_3	$45 = 3^2.5$	
	χ_{15}	$990 = 2.3^2.5.11$	
${}^2F_4(2)'$	χ_4	$27 = 3^3$	2
HS	χ_2	$22 = 2.11$	2
	χ_7	$175 = 5^2.7$	
	χ_{17}	$1386 = 2.3^2.7.11$	
J_3	χ_6	$324 = 2^2.3^4$	2
	χ_9	$816 = 2^4.3.17$	
	χ_{13}	$1615 = 5.17.19$	
M_{24}	χ_2	23	1
M^cL	χ_2	$22 = 2.11$	2
	χ_4	$252 = 2^2.3^2.7$	
	χ_{14}	$5103 = 3^6.7$	
He	χ_9	$1275 = 3.5^2.17$	2
	χ_{15}	$6272 = 2^7.7^2$	
	χ_{26}	$13720 = 2^3.5.7^3$	
Ru	χ_5	$783 = 3^3.29$	1
	χ_7	$3654 = 2.3^2.7.29$	
	χ_{20}	$45500 = 2^2.5^3.7.13$	
Suz	χ_2	$143 = 11.13$	2
	χ_6	$3432 = 2^3.3.11.13$	
	χ_{43}	$248832 = 2^{10}.3^5$	
$O'N$	χ_2	$10944 = 2^6.3^2.19$	2
	χ_{11}	$52668 = 2^2.3^2.7.11.19$	
	χ_{19}	$116963 = 7^3.11.31$	
Co_3	χ_2	23	1
Co_2	χ_2	23	1
Fi_{22}	χ_{56}	$1441792 = 2^{17}.11$	2
	χ_{57}	$1791153 = 3^9.7.13$	
	χ_{62}	$2402400 = 2^5.3.5^2.7.11.13$	

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Table 1. *Continued.*

S	Characters	Degrees	$\lvert\mathrm{Out}(S)\rvert$
HN	χ_{10}	$16929 = 3^4.11.19$	2
	χ_{19}	$270864 = 2^4.3^4.11.19$	
	χ_{45}	$3200000 = 2^{10}.5^5$	
Ly	χ_7	$120064 = 2^8.7.67$	1
	χ_9	$381766 = 2.7.11.37.67$	
	χ_{50}	$53765625 = 3.5^6.31.37$	
Th	χ_2	$248 = 2^3.31$	1
	χ_6	$30628 = 2^2.13.19.31$	
	χ_7	$30875 = 5^3.13.19$	
Fi_{23}	χ_4	$5083 = 13.17.23$	1
	χ_7	$60996 = 2^2.3.13.17.23$	
	χ_{94}	$504627200 = 2^{18}.5^2.7.11$	
Co_1	χ_3	$299 = 13.23$	1
	χ_8	$37674 = 2.3^2.7.13.23$	
	χ_{17}	$673750 = 2.5^4.7^2.11$	
J_4	χ_2	$1333 = 31.43$	1
	χ_6	$887778 = 2.3^2.31.37.43$	
	χ_{11}	$1776888 = 2^3.3^2.23.29.37$	
Fi'_{24}	χ_2	$8671 = 13.23.29$	2
	χ_6	$1603525 = 5^2.7^3.11.17$	
	χ_9	$4864431 = 3.11.13.17.23.29$	
B	χ_2	$4371 = 3.31.47$	1
	χ_6	$9550635 = 3.5.19.23.31.47$	
	χ_{119}	$2642676197359616 = 2^{39}.11.19.23$	
M	χ_2	$196883 = 47.59.71$	1
	χ_7	$293553734298 = 2.3.11.19.29.41.47.59.71$	
	χ_{16}	$8980616927734375 = 5^9.7^6.11^2.17.19$	

By [19, p. 3136], the inertia groups $I_{G/C}(\eta_3)$ and $I_{G/C}(\eta_4)$ are both equal to $N_{G/C}(S_1)$ and the characters η_3 and η_4 are extendible to $N_{G/C}(S_1)$. So by [9, Theorem 6.11], $t\rho(1), t\tau(1) \in \mathrm{cd}(G)$. If $t \geqslant 2$, then $\rho(1)^t \leftrightarrow t\rho(1) \sim t\tau(1) \leftrightarrow \tau(1)^t$ yields that $\rho(1)^t \sim \tau(1)^t$, which is a contradiction. Thus $t = 1$ and so M/N is simple.

Now, G/C has a unique minimal normal subgroup isomorphic to S and S has no irreducible character of a prime power degree which is extendible to $\mathrm{Aut}(S)$. If $S \cong \mathrm{Alt}_7$, then we are done. Hence, we may assume that $S \ncong \mathrm{Alt}_7$. If $S \cong J_1$, then as $|\mathrm{Out(S)}| = 1$, $G/C \cong J_1$; but then Lemma 3.2 yields a contradiction. Hence $S \ncong J_1$. Now by Lemma 3.3, we get characters of G of degrees a, b and c such that $a \leftrightarrow b \sim c$ and $(a, c) = 1$, contradicting the regularity of $\Gamma(G)$. □

Suppose that N is a normal subgroup of G and $\nu \in \mathrm{Irr}(N)$ is G-invariant. When ν does not extend to G, the situation is more complicated than the situation considered by Gallagher's theorem. In this case, to determine the sets $\mathrm{Irr}(G \mid \nu)$ and $\mathrm{cd}(G \mid \nu)$ one needs to use projective representations (see Chapter 11 of [9]). In particular, we can find

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a Schur representation group Γ for G/N. This implies that Γ has a central subgroup A so that $\Gamma/A \cong G/N$, A is isomorphic to the Schur multiplier for G/N, and $A \subseteq \Gamma'$. One can show (see [9, Theorem 11.28]) that there will exist a character $\alpha \in \mathrm{Irr}(A)$ so that the characters in $\mathrm{Irr}(G \mid \nu)$ are in bijection with the characters in $\mathrm{Irr}(\Gamma \mid \alpha)$. In particular, $\mathrm{cd}(G \mid \nu) = \{\nu(1)a \mid a \in \mathrm{cd}(\Gamma \mid \alpha)\}$. We say that the integers $\{a \mid a \in \mathrm{cd}(\Gamma \mid \alpha)\}$ are the projective character degrees with respect to α. The Atlas [6], provides the character tables for the Schur representation groups of the simple groups that it includes. We will make use of these results in the following.

Lemma 3.5. *Let N be a normal subgroup of G with $G/N \cong \mathrm{Alt}_7$ or Sym_7. If $\Gamma(G)$ is a regular graph, then $\mathrm{cd}(G) = \mathrm{cd}(G/N)$.*

Proof. By the Atlas [6], we have that $\mathrm{cd}(\mathrm{Alt}_7) = \{1, 6, 10, 14, 15, 21, 35\}$ and $\mathrm{cd}(\mathrm{Sym}_7) = \{1, 6, 14, 15, 20, 21, 35\}$. So $\Gamma(G/N)$ is a subgraph of $\Gamma(G)$ that is a 4-regular graph with six vertices. Assume that $\mathrm{cd}(G) \neq \mathrm{cd}(G/N)$. Let $\mathcal{A} = \mathrm{cd}(G) \setminus \mathrm{cd}(G/N)$.

Consider a degree $x \in \mathcal{A}$. Suppose there is a character $\chi \in \mathrm{Irr}(G)$ with $x = \chi(1)$ and $\chi_N \in \mathrm{Irr}(N)$. By Theorem 3.1 and Lemma 2.4, we have $6 \leftrightarrow 6x \sim 35x \leftrightarrow 35$. This yields that $6 \sim 35$, which is a contradiction. Thus, the restriction to N of every irreducible character of G having degree in $\mathcal{A}$ is not irreducible. It follows that $(x, |G : N|) > 1$ for all $x \in \mathcal{A}$. Hence, every vertex in $\mathcal{A}$ is adjacent to some vertices in $\Gamma(G/N)$.

If $x \in \mathcal{A}$ is divisible by at least three of the primes in $\pi = \{2, 3, 5, 7\}$, then x is adjacent to all vertices of $\Gamma(G/N)$. Without loss of generality we may suppose that $2 \cdot 3 \cdot 5 \mid x$. So $6 \leftrightarrow x$ and $x \sim 35$. It follows that $6 \sim 35$, a contradiction. Therefore every degree $x \in \mathcal{A}$ is divisible by at most two primes in the set π.

For distinct primes p and q in π, we define $\mathcal{A}_{p,q}$ to be the degrees in $\mathcal{A}$ that are divisible by pq. For each prime p in π, we define $\mathcal{A}_p$ to be the degrees in $\mathcal{A}$ that are divisible by p and are coprime to the other primes in π. We claim that $\mathcal{A}_p$ is empty when p is odd and that $\mathcal{A}_{5,7}$ is empty.

Choose a character $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \in \mathcal{A}$ and let $\nu \in \mathrm{Irr}(N)$ be a constituent of χ_N. Let T be the stabilizer of ν in G, and suppose that $T < G$. Notice that $|G : T|$ can be divisible by at most two primes from π. From the lists of maximal subgroups of Alt_7 and Sym_7 that are found in [6], we see that the only possibilities this allows for $|G : T|$ are 7, 15, 21, and 35 if $G/N \cong \mathrm{Alt}_7$ and 2, 7, 14, 21, and 35 if $G/N \cong \mathrm{Sym}_7$.

If $|G : T| = 7$ or 14, then there exists $A \leqslant T$ so that $A/N \cong \mathrm{Alt}_6$ and $|T : A|$ divides 2. Using the possible projective degrees found in the atlas [6] (including the case were ν extends to A), we see that $\mathrm{cd}(A \mid \nu)$ must contain one of $10\nu(1)$, $12\nu(1)$, or $15\nu(1)$, and using Clifford's theorem, we obtain a degree in $\mathrm{cd}(G \mid \nu)$ that is divisible by one of 70, 84, or 105. In particular, we obtain a degree in $\mathcal{A}$ that is divisible by three primes in π, a contradiction.

If $|G : T| = 15$, then $T/N \cong \mathrm{PSL}_2(7)$. Again, using the possible projective degrees from [6], we see that $6\nu(1) \in \mathrm{cd}(T \mid \nu)$ whether or not ν extends to T. Applying Clifford's theorem, this implies that $90\nu(1) \in \mathrm{cd}(G \mid \nu)$ and this is a contradiction since 90 is divisible by three distinct primes in π.

If $|G : T| = 21$, then $T/N \cong \mathrm{Sym}_5$ or $T/N \cong \mathrm{Sym}_5 \times Z_2$. Hence, there is a subgroup A of T so that $A/N \cong \mathrm{Alt}_5$ and $|T : A| = 2$ or $|T : A| = 4$. Again, using the possible projective degrees from [6], we have $4\nu(1) \in \mathrm{cd}(A \mid \nu)$, and so, by Clifford's theorem

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$84\nu(1)$ will divide some degree in $\mathrm{cd}(G \mid \nu)$. Since 84 is divisible by three distinct primes in π, this will yield a contradiction.

If $|G:T| = 35$, then $T/N \cong (\mathrm{Alt}_4 \times Z_3) : Z_2$ or $T/N \cong \mathrm{Sym}_4 \times \mathrm{Sym}_3$. In particular, T/N is a non-abelian $\{2,3\}$-group. It follows that either $2\nu(1)$ or $3\nu(1)$ divides some degree in $\mathrm{cd}(T \mid \nu)$. Applying Clifford's theorem, we obtain either $70\nu(1)$ or $105\nu(1)$ divides some degree in $\mathrm{cd}(G \mid \nu)$. In both cases, we have a degree in $\mathcal{A}$ that is divisible by at least three distinct primes in π, a contradiction.

If $|G:T| = 2$, then $T/N \cong \mathrm{Alt}_7$. Observe by applying Clifford's theorem that every degree in $\mathrm{cd}(G \mid \nu)$ will be divisible by 2. Thus, if $\chi(1)$ is divisible by only one prime in π, then the prime would be 2. Since $\chi(1)$ can be divisible by at most two primes in π and one of those primes has to be 2, the two primes cannot be 5 and 7. It follows that $\chi(1)$ is not in $\mathcal{A}_p$ when p is odd and $\chi(1)$ is not in $\mathcal{A}_{5,7}$ in this case.

We now have that ν is G-invariant. If ν is nonlinear, then as mentioned above, we see that ν cannot extend to G. If ν were linear and extended to G, then $\mathrm{cd}(G \mid \nu)$ would be either $\mathrm{cd}(\mathrm{Alt}_7)$ or $\mathrm{cd}(\mathrm{Sym}_7)$, and since $\chi(1) \in \mathrm{cd}(G \mid \nu)$, this is a contradiction. Thus, we conclude that ν does not extend to G.

From the projective degree found in [6], we see when $G/N \cong \mathrm{Alt}_7$ that $\mathrm{cd}(G \mid \nu)$ is one of the following sets: $\{\nu(1)a \mid a \in \{4,14,20,36\}\}$, $\{\nu(1)a \mid a \in \{6,15,21,24\}\}$, and $\{\nu(1)a \mid a \in \{6,24,36\}\}$. Looking at these sets, we see that the only possible degree which might be divisible by only one prime in π is $4\nu(1)$, so $\chi(1)$ cannot lie in $\mathcal{A}_p$ when p is odd in this case. Also, $4\nu(1)$ cannot be divisible by both 5 and 7, and the remaining degrees are all divisible by two primes from π and in no case are the two primes 5 and 7. Thus, $\chi(1)$ cannot lie in $\mathcal{A}_{5,7}$ in this case.

Finally when $G/N \cong \mathrm{Sym}_7$ we use the projective degrees from [6] to see since ν does not extend to G that we must have $\mathrm{cd}(G \mid \nu) = \{\nu(1)a \in \{8,20,28,36\}\}$. Note that the only degree that could possibly be divisible by a single prime in π is $8\nu(1)$, so in this case, we cannot have $\chi(1)$ in $\mathcal{A}_p$ for an odd prime p. Also, 5 and 7 cannot divide any degree, so $\chi(1)$ is not in $\mathcal{A}_{5,7}$ in this case. Since this exhausts all possible cases, we conclude that $\mathcal{A}_p$ is empty when p is odd and $\mathcal{A}_{5,7}$ is empty. This proves the claim.

Set $|\mathcal{A}_2| = n_2$ and $|\mathcal{A}_{p,q}| = n_{p,q}$ for all distinct primes $p < q \in \pi$. By the claim, we have that $n_{5,7} = 0$.

Suppose that $\mathcal{A}_2$ is not empty and fix a degree $a \in \mathcal{A}_2$. Observe that a will be adjacent to 6, 10, 14 and all the elements in the sets $\mathcal{A}_2 - \{a\}$, $\mathcal{A}_{2,3}$, $\mathcal{A}_{2,5}$, and $\mathcal{A}_{2,7}$. We claim that a is not adjacent to any element in $\mathcal{A}_{3,5}$. If $\mathcal{A}_{3,5}$ is empty, then this is obvious. Thus, we may assume that there exists $b \in \mathcal{A}_{3,5}$. We know that 15 is a vertex in $\Gamma(G)$. Since 15 divides b, we may use Lemma 2.4 to see that 15 and b are partners. Since 15 is not adjacent to any element in $\mathcal{A}_2$, we conclude that b is not adjacent to any element in $\mathcal{A}_2$, and in particular, b is not adjacent to a. We deduce that a is not adjacent to any element in $\mathcal{A}_{3,5}$. In a similar manner, we can prove that a is not adjacent to any element in $\mathcal{A}_{3,7}$. We see that $\deg(a) = 3 + (n_2 - 1) + n_{2,3} + n_{2,5} + n_{2,7} = 2 + n_2 + n_{2,3} + n_{2,5} + n_{2,7}$.

Since $\Gamma(G)$ is regular and 14 is a vertex in $\Gamma(G)$, we conclude that $\deg(a) = \deg(14)$. We see that $\deg(14) = 4 + n_2 + n_{2,3} + n_{2,5} + n_{2,7} + n_{3,7}$. This yields $2 + n_2 + n_{2,3} + n_{2,5} + n_{2,7} = 4 + n_2 + n_{2,3} + n_{2,5} + n_{2,7} + n_{3,7}$, and we obtain $n_{3,7} = -2$, a contradiction. We deduce that $\mathcal{A}_2$ is empty.

Thus, every element of $\mathcal{A}$ is divisible by exactly two prime factors of π. We have $\deg(6) = 4 + n_{2,3} + n_{2,5} + n_{2,7} + n_{3,5} + n_{3,7}$ and $\deg(35) = 4 + n_{2,5} + n_{2,7} + n_{3,5} + n_{3,7}$.

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Since these are equal, we deduce that $n_{2,3} = 0$. Now, we have $\deg(21) = 4 + n_{2,7} + n_{3,5} + n_{3,7}$, and since this equals the degree of 6, we deduce that $n_{2,5} = 0$. Next, we determine that $\deg(15) = 4 + n_{3,5} + n_{3,7}$. Since this must equal $\deg(6)$, we obtain $n_{2,7} = 0$. We see that $\deg(14) = 4 + n_{3,7}$, and since this must equal $\deg(21)$, we have $n_{3,5} = 0$. Finally, let c be the vertex for either 10 or 20, whichever appears in $\mathrm{cd}(G/N)$; we see that $\deg(c) = 4$, and as this degree equals $\deg(14)$, we have $n_{3,7} = 0$. We have now shown that $\mathcal{A}$ is empty, and the result is proved. □

Proof of Theorem 1.2. First suppose that $\Gamma(G)$ is disconnected. By Lemma 2.2, $k = 0$ and $n(\Gamma(G)) = 3$. So [**13**, Theorem 4.1] yields that $G \cong \mathrm{PSL}_2(2^f) \times A$, where $f \geqslant 2$ and A is an abelian group. Note that $\mathrm{cd}(G)$ can be easily derived from $\mathrm{cd}(\mathrm{SL}_2(2^f))$ found in [**7**, Theorem 38.2].

Now suppose that $\Gamma(G)$ is connected. Let M/N be a chief factor of G. By Lemma 3.4, $M/N \cong \mathrm{Alt}_7$. Let $C/N = C_{G/N}(M/N)$. Then G/C has a unique minimal normal subgroup CM/C isomorphic to M/N. As $\mathrm{Aut}(\mathrm{Alt}_7) = \mathrm{Sym}_7$, G/C is isomorphic to either Alt_7 or Sym_7. In both cases, $\Gamma(G/C)$ is a subgraph of $\Gamma(G)$ and $\Gamma(G/C)$ is a 4-regular graph with six vertices. By Lemma 3.5, $\mathrm{cd}(G) = \mathrm{cd}(G/C)$. Now the result follows from [**8**, Theorem 2.8] and [**14**, Theorem 4.3]. □

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