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THE DEGREE-WISE EFFECT OF A SECOND STEP FOR A RANDOM WALK ON A GRAPH

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Abstract

In this paper we consider the degree-wise effect of a second step for a random walk on a graph. We prove that under the configuration model, for any fixed degree sequence the probability of exceeding a given degree threshold is smaller after two steps than after one. This builds on recent work of Kramer *et al.* (2016) regarding the friendship paradox under random walks.

Keywords: Friendship paradox; graph; random walk; configuration model

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Secondary 05C07; 60J10; 60C05; 60J15; 91D30; 05C80

1. Introduction

The friendship paradox, introduced by Feld [15], states roughly that, in a network scenario, one's neighbours have (on average) more neighbours than oneself. The result has recently been employed to advantage in epidemic detection and more general sampling scenarios (see [11], [12], [14], [17], [18], [22], [34]), and has received considerable attention from researchers across disciplines. For a recent discussion of societal welfare implications, see [20].

Throughout, we will assume that $G = (V, E)$ is a *multigraph* with a set of vertices or nodes $V = \{v_1, v_2, \ldots, v_n\}$ and a multiset of unordered pairs of elements of V, $E = \{e_1, e_2, \ldots, e_m\}$ with each unordered pair (or *edge*) of the form $\{v, w\}$ having nonnegative multiplicity $\omega(v, w)$=$\omega(w, v)$. We will refer to $\omega\colon V \times V \to \mathbb{Z}^+ \cup \{0\}$ as the *multiplicity function* of the multigraph G.

For $v \in V$, define the *degree* function d from V to $\mathbb{Z}^+$ as

$$d(v) := \sum_{w \in V} \omega(v, w).$$

We will assume throughout that G has no zero-degree vertices, i.e. $d(v) > 0$ for all $v \in V$. If $\omega(v, w) \in \{0, 1\}$ for all $(v, w) \in V \times V$, we will say that the graph G is *simple*.

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For the purposes of neighbour selection, as in [24], it will be convenient to consider a time-homogeneous random walk $\boldsymbol{X} = (X_0, X_1, \dots)$ on the graph G dictated by a transition matrix $\boldsymbol{P} = (P_{i,j})$, i.e.

$$P_{i,j} \coloneqq \mathbb{P}(X_{l+1} = v_j \mid X_l = v_i) \quad \text{for } 1 \le i,\ j \le n,\ l \ge 0.$$

We will assume throughout that X_0 is uniformly selected from V.

Recently, Cao and Ross [9] proved the following version of the friendship paradox (see also [20]).

Theorem 1. *Suppose that $\boldsymbol{X} = (X_0, X_1, \dots)$ is a random walk on a simple graph G. Then, the degree of X_1 is stochastically larger than the degree of X_0, i.e. for all $t \in \mathbb{R}$,*

$$\mathbb{P}(d(X_1) \ge t) \ge \mathbb{P}(d(X_0) \ge t).$$

In addition, Kramer *et al.* [24] proved the following result regarding random walks on simple graphs (see also [35, Theorem 1.2]).

Theorem 2. *Suppose that $\boldsymbol{X} = (X_0, X_1, \dots)$ is a random walk on a simple graph G. Then, for $k \ge 0$, the degree of X_k is no less than the degree of X_0 in expected value, i.e.*

$$\mathbb{E}(d(X_k)) \ge \mathbb{E}(d(X_0)).$$

For related results for weighted and directed networks, see [5].

For further recent work related to the friendship paradox, see, e.g. [13], [16], [19], [21], [23], [25], [32], [33], [36], and for further information on random walks on graphs, see also [1], [8], [26], and [27].

In light of recent applications of the friendship paradox, Theorems 1 and 2 suggest investigation of the relative benefit of additional random walk steps (i.e. friends of friends, friends of friends of friends, and so on, in a social network context). It is, in fact, possible to construct small simple graphs where additional steps can be either beneficial or detrimental, if the goal at hand is to obtain a node of sufficiently high degree. We illustrate this in the following example.

Example. Consider the six-node eight-edge simple graph presented in Figure 1(a). Here, $\mathbb{P}(d(X_1) \ge 3) = \frac{78}{108} < \frac{79}{108} = \mathbb{P}(d(X_2) \ge 3)$. On the other hand, for the six-node seven-edge graph in Figure 1(b) (with the edge connecting nodes 1 and 4 removed), $\mathbb{P}(d(X_1) \ge 3) = \frac{24}{27} > \frac{22}{27} = \mathbb{P}(d(X_2) \ge 3)$. Note that in both instances, $\mathbb{P}(d(X_0) \ge 3) = \frac{2}{3}$.

Here, we will show that one step is more beneficial than two steps, on average, for a given degree sequence, in a certain sense. In particular, consider the degree sequence, $\boldsymbol{d} = (d(v_1), d(v_2), \dots, d(v_n))$. The *configuration* (or *matching*) model provides a well-studied

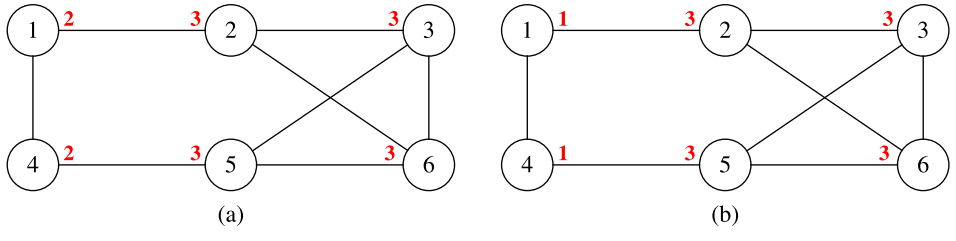


FIGURE 1: Two six-node networks. Degrees are indicated adjacent to the corresponding nodes.

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means to produce a random multigraph with degree sequence $\boldsymbol{d}$; see, e.g. [10], [29], and [35, Chapter 7]. In particular, assign to each node v, $d(v)$ stubs, and randomly pair these stubs to create a (random) graph with degree sequence $\boldsymbol{d}$. For further discussion of configuration models, see, e.g. [3], [30], and [31].

Now, for a given degree sequence $\boldsymbol{d}$, and $t > 0$, define the sets

$$\mathcal{S}^+ = \mathcal{S}_t^+ := \{v : d(v) \geq t\} \quad \text{and} \quad \mathcal{S}^- = \mathcal{S}_t^- := \{v : d(v) < t\}, \tag{1}$$

and the constants

$$D^+ = D_t^+ := \sum_{v \in \mathcal{S}_t^+} d(v), \qquad D^- = D_t^- := \sum_{v \in \mathcal{S}_t^-} d(v), \qquad M := D_t^+ + D_t^-.$$

Here, $\mathcal{S}^+$ and $\mathcal{S}^-$ are the sets of high and low degree nodes, respectively, while D_t^+ and D_t^- are their respective degree sums. Note that total degree sum M must always be even.

We will prove the following result.

Theorem 3. *Suppose that $V = \{v_1, \ldots, v_n\}$ and the degree sequence $\boldsymbol{d}$ is fixed. Consider the ensemble of graphs, $\mathcal{G}$, of all multigraphs with this degree sequence. If $G \in \mathcal{G}$ (with multiplicity function ω) is formed randomly via the configuration model and $\boldsymbol{P} = (P_{i,j})$ is the resulting (random) transition matrix for a random walk on G, i.e.*

$$P_{i,j} = \frac{\omega(v_i, v_j)}{d(v_i)},$$

then for $t \geq 0$,

$$\Delta := \mathbb{P}(d(X_1) \geq t) - \mathbb{P}(d(X_2) \geq t) \geq 0$$

with equality if and only if $\min(|\mathcal{S}^+|, |\mathcal{S}^-|) = 0$.

For further recent work related to random walks on random graphs, see [2], [4], [6], [7], and [28].

2. Proof of Theorem 3

In this section we will prove Theorem 3. The following lemma provides a helpful mean inequality.

Lemma 1. *Suppose that $r, s \geq 1$, and $\boldsymbol{\alpha} = (\alpha_i)_{i \geq 1}$ and $\boldsymbol{\beta} = (\beta_i)_{i \geq 1}$ are two sequences of positive integers satisfying $\alpha_i < \beta_j$ for all $i, j \geq 1$. Then*

$$m_s(\boldsymbol{\beta}) - m_r(\boldsymbol{\alpha}) \geq \tfrac{2}{3}(m_s(\boldsymbol{\beta})A_r(\boldsymbol{\alpha}) - m_r(\boldsymbol{\alpha})A_s(\boldsymbol{\beta})), \tag{2}$$

where for $t \geq 1$ and $\boldsymbol{x} = (x_i)_{i \geq 1}$, $m_t(\boldsymbol{x})$ is the mean of the first t entries in $\boldsymbol{x}$, i.e. $(x_1 + x_2 + \cdots + x_t)/t$, and $A_t(\boldsymbol{x})$ is the mean of the reciprocals of the first t entries, i.e. $(1/x_1 + 1/x_2 + \cdots + 1/x_t)/t$. Equality in (2) holds if and only if $\beta_1 = \beta_2 = \ldots, \beta_s = 2$.

Proof. Let W and Z be independent random variables uniformly distributed on the index sets $\{1, 2, \ldots, r\}$ and $\{1, 2, \ldots, s\}$, respectively, $X = \alpha_W$ and $Y = \beta_Z$. Then

$$m_r(\boldsymbol{\alpha}) = \mathbb{E}(X), \qquad m_s(\boldsymbol{\beta}) = \mathbb{E}(Y), \qquad A_r(\boldsymbol{\alpha}) = \mathbb{E}\left(\frac{1}{X}\right), \qquad A_s(\boldsymbol{\beta}) = \mathbb{E}\left(\frac{1}{Y}\right).$$

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Setting $B \coloneqq m_s(\boldsymbol{\beta}) - m_r(\boldsymbol{\alpha}) - \frac{2}{3}(m_s(\boldsymbol{\beta})A_r(\boldsymbol{\alpha}) - m_r(\boldsymbol{\alpha})A_s(\boldsymbol{\beta}))$, we have

$$\begin{aligned} B &= \mathbb{E}(Y) - \mathbb{E}(X) - \frac{2}{3}\left(\mathbb{E}(Y)\mathbb{E}\left(\frac{1}{X}\right) - \mathbb{E}(X)\mathbb{E}\left(\frac{1}{Y}\right)\right) \\ &= \mathbb{E}\left((Y-X) - \frac{2}{3}\left(\frac{Y^2 - X^2}{XY}\right)\right) \\ &= \mathbb{E}\left((Y-X)\left(1 - \frac{2}{3}\left(\frac{1}{X} + \frac{1}{Y}\right)\right)\right) \\ &\geq 0, \end{aligned}$$

since $\mathbb{P}(Y > X \geq 1) = 1$. If $\mathbb{P}(Y = 2) = 1$ then $\mathbb{P}(X = 1) = 1$ and $B = 0$, otherwise $B > 0$ and the result follows. □

Proof of Theorem 3. For convenience, suppose that $v_i = i$ for $1 \leq i \leq n$, and write d_i for $d(v_i)$.

Note that, employing the notation in (1), we have

$$\begin{aligned} \Delta &= \mathbb{P}(X_1 \in \mathcal{S}^+) - \mathbb{P}(X_2 \in \mathcal{S}^+) \\ &= (\mathbb{P}(X_1 \in \mathcal{S}^+, X_2 \in \mathcal{S}^-) + \mathbb{P}(X_1 \in \mathcal{S}^+, X_2 \in \mathcal{S}^+)) \\ &\quad - (\mathbb{P}(X_1 \in \mathcal{S}^-, X_2 \in \mathcal{S}^+) + \mathbb{P}(X_1 \in \mathcal{S}^+, X_2 \in \mathcal{S}^+)) \\ &= \mathbb{P}(X_1 \in \mathcal{S}^+, X_2 \in \mathcal{S}^-) - \mathbb{P}(X_1 \in \mathcal{S}^-, X_2 \in \mathcal{S}^+). \end{aligned} \tag{3}$$

If $\min(|\mathcal{S}^+|, |\mathcal{S}^-|) = 0$ then $\Delta = 0$. So assume $\mathcal{S}^+$ and $\mathcal{S}^-$ are both nonempty. Note that $D^- \geq 1$ and $D^+ \geq 2$, so $M \geq 4$ since M is even.

Suppose that the stubs outgoing from each node v have been ordered, and for $(i, j) \in V \times V$, $1 \leq k_1 \leq d_i$, and $1 \leq k_2 \leq d_j$, define $\mathbf{1}_{\{i,j,k_1,k_2\}}$ to be the indicator of the event that the k_1th stub outgoing from v_i attaches to the k_2th stub outgoing from v_j. In addition, notationally, for $z \in \mathbb{Z}^+$, let

$$[z] = \{1, 2, \ldots, z\}.$$

Now, note that

$$\begin{aligned} &\mathbb{P}(X_1 \in \mathcal{S}^+, X_2 \in \mathcal{S}^-) \\ &\quad = \mathbb{P}(X_0 \in \mathcal{S}^-, X_1 \in \mathcal{S}^+, X_2 \in \mathcal{S}^-) + \mathbb{P}(X_0 \in \mathcal{S}^+, X_1 \in \mathcal{S}^+, X_2 \in \mathcal{S}^-). \end{aligned}$$

First, consider $\mathbb{P}(X_0 \in \mathcal{S}^+, X_1 \in \mathcal{S}^+, X_2 \in \mathcal{S}^-)$ and recall that X_0 is assumed to be uniformly selected from V. We have (see Figure 2(a))

$$\mathbb{P}(X_0 \in \mathcal{S}^+, X_1 \in \mathcal{S}^+, X_2 \in \mathcal{S}^-) = n^{-1}\mathbb{E}(A + B),$$

where

$$A = \sum_{(i,j) \in \mathcal{S}^+ \times \mathcal{S}^-} \left(\sum_{(k_1,k_2,k_3,k_4) \in [d_i] \times [d_i] \times [d_i] \times [d_j],\ k_1,k_2,k_3 \text{ distinct}} \frac{1}{d_i}\frac{1}{d_i}\, \mathbf{1}_{\{i,i,k_1,k_2\}}\, \mathbf{1}_{\{i,j,k_3,k_4\}} \right) \tag{4}$$

and

$$B = \sum_{(i,j,l) \in \mathcal{S}^+ \times \mathcal{S}^+ \times \mathcal{S}^-,\ i \neq j} \left(\sum_{(k_1,k_2,k_3,k_4) \in [d_i] \times [d_j] \times [d_j] \times [d_l],\ k_2 \neq k_3} \frac{1}{d_i}\frac{1}{d_j}\, \mathbf{1}_{\{i,j,k_1,k_2\}}\, \mathbf{1}_{\{j,l,k_3,k_4\}} \right). \tag{5}$$

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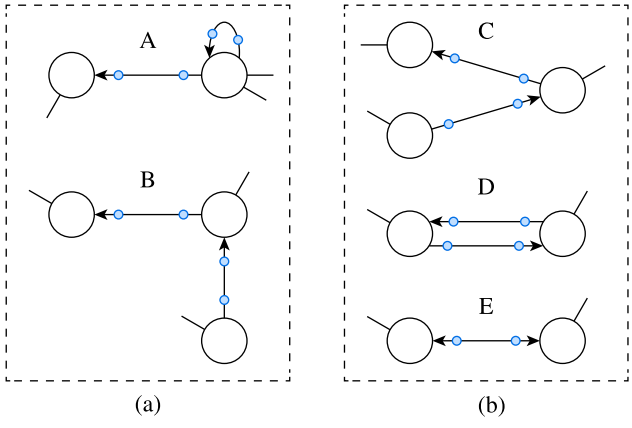


FIGURE 2: (a) The configuration types leading to a positive value for $\mathbb{P}(X_0 \in \mathcal{S}^+, X_1 \in \mathcal{S}^+, X_2 \in \mathcal{S}^-)$. Nodes on the left and right are assumed to be in $\mathcal{S}^-$ and $\mathcal{S}^+$, respectively. (b) The configuration types leading to a positive value for $\mathbb{P}(X_0 \in \mathcal{S}^-, X_1 \in \mathcal{S}^+, X_2 \in \mathcal{S}^-)$. The configurations corresponding to (4), (5), (9), (10), and (11) are given in A, B, C, D, and E, respectively.

For a given node $i \in \mathcal{S}^+$, there are $d_i(d_i-1)(d_i-2)$ distinct triples of stubs outgoing from the node. For each of these, the probability that the first of these stubs connects to the second and the third connects to a stub of a node in $\mathcal{S}^-$ is $(1/(M-1))(D^-/(M-3))$. Thus,

$$\begin{aligned}\mathbb{E}(A) &= \sum_{i\in\mathcal{S}^+} \frac{1}{d_i}\frac{1}{d_i}\frac{1}{M-1}\frac{D^-}{M-3} d_i(d_i-1)(d_i-2)\\ &= \frac{1}{M-1}\frac{D^-}{M-3}\sum_{i\in\mathcal{S}^+}\left(d_i - 3 + \frac{2}{d_i}\right)\\ &= \frac{1}{M-1}\frac{D^-}{M-3}\left(D^+ - 3|\mathcal{S}^+| + 2\sum_{i\in\mathcal{S}^+}\frac{1}{d_i}\right). \qquad (6)\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbb{E}(B) &= \sum_{i\in\mathcal{S}^+}\sum_{j\in\mathcal{S}^+,\, i\neq j} \frac{1}{d_i}\frac{1}{d_j}\frac{1}{M-1}\frac{D^-}{M-3} d_i d_j(d_j-1) \qquad (7)\\ &= \frac{D^-}{(M-1)(M-3)}\sum_{i\in\mathcal{S}^+}\sum_{j\in\mathcal{S}^+,\, i\neq j}(d_j-1)\\ &= \frac{D^-}{(M-1)(M-3)}\sum_{i\in\mathcal{S}^+}((D^+-d_i)-(|\mathcal{S}^+|-1))\\ &= \frac{D^-}{(M-1)(M-3)}(|\mathcal{S}^+|D^+ - D^+ - |\mathcal{S}^+|^2 + |\mathcal{S}^+|)\\ &= \frac{D^-}{(M-1)(M-3)}(D^+ - |\mathcal{S}^+|)(|\mathcal{S}^+|-1). \qquad (8)\end{aligned}$$

For $\mathbb{P}(X_0 \in \mathcal{S}^-, X_1 \in \mathcal{S}^+, X_2 \in \mathcal{S}^-)$, we have (see Figure 2(b))

$$\mathbb{P}(X_0 \in \mathcal{S}^-, X_1 \in \mathcal{S}^+, X_2 \in \mathcal{S}^-) = n^{-1}\mathbb{E}(C+D+E),$$

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where

$$C = \sum_{(i,j,l)\in\mathcal{S}^-\times\mathcal{S}^+\times\mathcal{S}^-,\, i\neq l} \left(\sum_{(k_1,k_2,k_3,k_4)\in[d_i]\times[d_j]\times[d_j]\times[d_l],\, k_2\neq k_3} \frac{1}{d_i}\frac{1}{d_j}\mathbf{1}_{\{i,j,k_1,k_2\}}\mathbf{1}_{\{j,l,k_3,k_4\}} \right), \tag{9}$$

$$D = \sum_{(i,j)\in\mathcal{S}^-\times\mathcal{S}^+} \left(\sum_{(k_1,k_2,k_3,k_4)\in[d_i]\times[d_j]\times[d_j]\times[d_i],\, k_1\neq k_4,\, k_2\neq k_3} \frac{1}{d_i}\frac{1}{d_j}\mathbf{1}_{\{i,j,k_1,k_2\}}\mathbf{1}_{\{j,i,k_3,k_4\}} \right), \tag{10}$$

$$E = \sum_{(i,j)\in\mathcal{S}^-\times\mathcal{S}^+} \left(\sum_{(k_1,k_2)\in[d_i]\times[d_j]} \frac{1}{d_i}\frac{1}{d_j}\mathbf{1}_{\{i,j,k_1,k_2\}} \right). \tag{11}$$

We then have

$$\begin{aligned}
\mathbb{E}(C) &= \sum_{i\in\mathcal{S}^-}\sum_{j\in\mathcal{S}^+} \frac{1}{d_i}\frac{1}{d_j}\frac{1}{M-1}\frac{D^- - d_i}{M-3} d_i d_j (d_j - 1) \\
&= \frac{1}{(M-1)(M-3)} \sum_{i\in\mathcal{S}^-} (D^- - d_i) \sum_{j\in\mathcal{S}^+} (d_j - 1) \\
&= \frac{1}{(M-1)(M-3)} (D^+ - |\mathcal{S}^+|) \sum_{i\in\mathcal{S}^-} (D^- - d_i) \\
&= \frac{D^-}{(M-1)(M-3)} (D^+ - |\mathcal{S}^+|)(|\mathcal{S}^-| - 1),
\end{aligned} \tag{12}$$

$$\begin{aligned}
\mathbb{E}(D) &= \sum_{i\in\mathcal{S}^-}\sum_{j\in\mathcal{S}^+} \frac{1}{d_i}\frac{1}{d_j}\frac{1}{M-1}\frac{d_i - 1}{M-3} d_i d_j (d_j - 1) \\
&= \frac{1}{(M-1)(M-3)} \sum_{i\in\mathcal{S}^-} (d_i - 1) \sum_{j\in\mathcal{S}^+} (d_j - 1) \\
&= \frac{1}{(M-1)(M-3)} (D^+ - |\mathcal{S}^+|)(D^- - |\mathcal{S}^-|),
\end{aligned} \tag{13}$$

and, finally,

$$\mathbb{E}(E) = \sum_{i\in\mathcal{S}^-}\sum_{j\in\mathcal{S}^+} \frac{1}{d_i}\frac{1}{d_j}\frac{1}{M-1} d_i d_j = \frac{1}{M-1}\sum_{i\in\mathcal{S}^-}\sum_{j\in\mathcal{S}^+} 1 = \frac{1}{M-1}|\mathcal{S}^+||\mathcal{S}^-|. \tag{14}$$

Thus,

$$\mathbb{P}(X_1 \in \mathcal{S}^+, X_2 \in \mathcal{S}^-) = n^{-1}\mathbb{E}(A + B + C + D + E), \tag{15}$$

where A, B, C, D, and E are as in (6), (8), (12), (13), and (14). In a similar fashion, we have

$$\mathbb{P}(X_1 \in \mathcal{S}^-, X_2 \in \mathcal{S}^+) = n^{-1}\mathbb{E}(A_2 + B_2 + C_2 + D_2 + E_2), \tag{16}$$

where A_2, B_2, C_2, D_2, and E_2 are formed from A, B, C, D, and E by swapping signs for all occurrences of D^+, D^-, $\mathcal{S}^+$, and $\mathcal{S}^-$ in (6), (8), (12), (13), and (14), respectively.

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Let $\Delta^* = n(M-1)(M-3)\Delta$. Combining (3), (15), and (16) and noting that $D_2 = D$ and $E_2 = E$, we have

$$\begin{aligned}
\Delta^* &= (M-1)(M-3)\mathbb{E}((A+B+C)-(A_2+B_2+C_2)) \\
&= D^-\Bigg(D^+ - 3|\mathcal{S}^+| + 2\sum_{i\in\mathcal{S}^+}\frac{1}{d_i} + (D^+ - |\mathcal{S}^+|)(|\mathcal{S}^+|-1) \\
&\qquad + (D^+ - |\mathcal{S}^+|)(|\mathcal{S}^-|-1)\Bigg) \\
&\quad - D^+\Bigg(D^- - 3|\mathcal{S}^-| + 2\sum_{i\in\mathcal{S}^-}\frac{1}{d_i} + (D^- - |\mathcal{S}^-|)(|\mathcal{S}^-|-1) \\
&\qquad + (D^- - |\mathcal{S}^-|)(|\mathcal{S}^+|-1)\Bigg) \\
&= D^-(D^+ - |\mathcal{S}^+|)(|\mathcal{S}^-| + |\mathcal{S}^+| - 2) - D^+(D^- - |\mathcal{S}^-|)(|\mathcal{S}^+| + |\mathcal{S}^-| - 2) \\
&\quad - 3(|\mathcal{S}^+|D^- - |\mathcal{S}^-|D^+) + 2D^-\sum_{i\in\mathcal{S}^+}\frac{1}{d_i} - 2D^+\sum_{i\in\mathcal{S}^-}\frac{1}{d_i} \\
&= (|\mathcal{S}^+| + |\mathcal{S}^-| - 2 + 3)(|\mathcal{S}^-|D^+ - |\mathcal{S}^+|D^-) + 2D^-\sum_{i\in\mathcal{S}^+}\frac{1}{d_i} - 2D^+\sum_{i\in\mathcal{S}^-}\frac{1}{d_i}.
\end{aligned}$$

Since $|\mathcal{S}^-| + |\mathcal{S}^+| = n$, we have

$$\begin{aligned}
\Delta^* &= (n+1)(|\mathcal{S}^-|D^+ - |\mathcal{S}^+|D^-) + 2D^-\sum_{i\in\mathcal{S}^+}\frac{1}{d_i} - 2D^+\sum_{i\in\mathcal{S}^-}\frac{1}{d_i} \\
&= (n+1)|\mathcal{S}^+||\mathcal{S}^-|\Bigg(\left(\frac{D^+}{|\mathcal{S}^+|} - \frac{D^-}{|\mathcal{S}^-|}\right) \\
&\qquad - \frac{2}{n+1}\left(\frac{D^+}{|\mathcal{S}^+|}\frac{\sum_{i\in\mathcal{S}^-}1/d_i}{|\mathcal{S}^-|} - \frac{D^-}{|\mathcal{S}^-|}\frac{\sum_{i\in\mathcal{S}^+}1/d_i}{|\mathcal{S}^+|}\right)\Bigg). \qquad (17)
\end{aligned}$$

The result follows upon applying Lemma 1 with $s = |\mathcal{S}^+|$, $r = |\mathcal{S}^-|$, $(\beta_1, \beta_2, \ldots, \beta_s)$ an ordering of $(d(i) : i \in \mathcal{S}^+)$, and $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ an ordering of $(d(i) : i \in \mathcal{S}^-)$. Note that the quantity in (17) is strictly larger than 0, since if $d(v) = 2$ for all $v \in \mathcal{S}^+$ and $n = 2$, $\mathcal{S}^- = \{1\}$ and $\mathcal{S}^+ = \{2\}$, which contradicts the requirement that M is even. □

Remark. Note that it would be of interest to obtain comparison results akin to Theorem 3 for fixed graphs (similar to Theorem 1). Consideration of the degree-wise effect of further random walk steps remains an open problem.

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