99.18 Justifying the $\sqrt{2\pi}$ factor in Stirling's formula

Proofs of Stirling's formula usually split into two parts. The first part establishes the existence of a convergent sequence (K_n) such that $n! = K_n n^{n+\frac{1}{2}} e^{-n}$. This is often done by a graphical argument; here, we establish this by observing that (K_n) is a decreasing sequence because $\dfrac{K_{n+1}}{K_n} = \dfrac{e}{(1+\frac{1}{n})^{n+\frac{1}{2}}} < 1$ since the sequence $\left(1+\dfrac{1}{n}\right)^{n+\frac{1}{2}} \downarrow e$. (The latter may be found in [1], but there was also a spate of recent proofs in *The American Mathematical Monthly*: see [2], which generated several follow-up pieces.) Moreover, the limit $K = \lim_{n\to\infty} K_n$ is greater than 0 because (K_n) is bounded away from zero since $\ln\left(\dfrac{K_{n+1}}{K_n}\right) = O\left(\dfrac{1}{n^2}\right)$. Again, there are many references for this such as [3, 4]. The second part identifies K as $\sqrt{2\pi}$. This is often accomplished by invoking Wallis's product

$$\prod_{k=1}^{\infty} \frac{(2k-1)(2k+1)}{2k \cdot 2k} = \frac{2}{\pi};$$

here we present an alternative argument based on two elementary properties of the gamma function, $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$.

- $\Gamma(x)^2 \leqslant \Gamma(x-\frac{1}{2})\Gamma(x+\frac{1}{2})$.

  This comes from applying the Cauchy-Schwarz inequality for integrals, $\left(\int_a^b f(t)g(t)dt\right)^2 \leqslant \left(\int_a^b f(t)^2dt\right)\left(\int_a^b g(t)^2dt\right)$ to the functions $f(t) = t^{\frac{1}{2}x-\frac{1}{4}}e^{-\frac{1}{2}t}$ and $g(t) = t^{\frac{1}{2}x-\frac{1}{4}}e^{-\frac{1}{2}t}$:

  $$\Gamma(x)^2 = \left(\int_0^\infty \left(t^{\frac{1}{2}x-\frac{3}{4}}e^{-\frac{1}{2}t}\right)\left(t^{\frac{1}{2}x-\frac{1}{4}}e^{-\frac{1}{2}t}\right)dt\right)^2 \leqslant \left(\int_0^\infty t^{x-\frac{3}{2}}e^{-t}dt\right)\left(\int_0^\infty t^{x-\frac{1}{2}}e^{-t}dt\right) =$$
  $$\Gamma(x-\tfrac{1}{2})\Gamma(x+\tfrac{1}{2}).$$

- $\Gamma\left(n+\frac{1}{2}\right) = \dfrac{(2n)!}{2^{2n}n!}\sqrt{\pi}$ when n is a non-negative integer.

  For $\Gamma(x) = (x-1)\Gamma(x-1)$ on integrating by parts, so $\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) = \ldots = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\ldots\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \dfrac{(2n)!}{2^{2n}n!}\Gamma\left(\frac{1}{2}\right)$ where $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}}e^{-t}dt = 2\int_0^\infty e^{-x^2}dx = \sqrt{\pi}$ on substituting $t = x^2$.

For positive integers n, from the first property we then have

$$\frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(n)\Gamma(n+1)} \leqslant 1 \leqslant \frac{\Gamma(n-\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(n)^2}$$

which, using the second property together with $\Gamma(n) = (n-1)!$, simplifies to give

$$\frac{(2n)!^2\, n\pi}{2^{4n}(n!)^4} \leqslant 1 \leqslant \frac{(2n)!^2\, n\pi}{2^{4n}(n!)^4} \cdot \frac{4n^2}{2n(2n-1)}. \qquad (*)$$

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But $\dfrac{(2n)!^2\, n\pi}{2^{4n}(n!)^4} = \dfrac{K_{2n}^2 (2n)^{4n+1} e^{-4n} n\pi}{2^{4n} K_n^4 n^{4n+2} e^{-4n}} = \dfrac{K_{2n}^2}{K_n^4} \cdot 2\pi$ so, letting $n \to \infty$ in (*), we thus see that $\dfrac{2\pi}{K^2} = 1$ and $K = \sqrt{2\pi}$.

Notice that (*) on its own provides a proof of Wallis's product:

$$\prod_{k=1}^{\infty} \frac{(2k-1)(2k+1)}{2k \,.\, 2k} = \lim_{n\to\infty} \frac{(2n)!\,(2n+1)}{2^{4n}(n!)^4} = \frac{2}{\pi}.$$

Alternatively, we can leave $\Gamma\left(\frac{1}{2}\right)$ unevaluated. The line (*) and its sequel then show that Wallis's product is equal to $\dfrac{2}{\Gamma\left(\frac{1}{2}\right)^2}$ and that $\dfrac{2\Gamma\left(\frac{1}{2}\right)^2}{K^2} = 1$ or $K = \sqrt{2}\Gamma\left(\frac{1}{2}\right)$. Thus knowing that Wallis's product is $\dfrac{2}{\pi}$ or that $K = \sqrt{2\pi}$ in Stirling's formula suffices to show that

$$\sqrt{\pi} = \Gamma\left(\tfrac{1}{2}\right) = 2\int_0^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-x^2} dx$$

which in turn establishes the normalising factor for the standard normal distribution, $\dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = 1$.

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99.19 When do the inradius and exradii form progressions?

Let ABC be a non-degenerate triangle with area Δ and side-lengths a, b, c where $a \leqslant b \leqslant c$ and $s = \frac{1}{2}(a + b + c)$. Then the inradius, r, and the exradii r_a, r_b, r_c are given by the standard formulae $r = \dfrac{\Delta}{s}$ and $r_a = \dfrac{\Delta}{s-a}$, etc. so that $r \leqslant r_a \leqslant r_b \leqslant r_c$. In this short note we

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