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# LARGE DEVIATIONS ANALYSIS OF A COUPLED-PROCESSORS SYSTEM

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In this article, we derive closed-form expressions of the large deviations local rate function in terms of the arrival and service parameters for the zero path in exponential queueing networks corresponding to a coupled-processors system.

## 1. INTRODUCTION

In [4] and [5], the sample path large deviation problem has been studied for random walks (rw) of low dimension with boundaries. More precisely, the local rate function for a face-homogeneous rw in $\mathbb{R}_+ \times \mathbb{R}$ is expressed as a function of the logarithmic moment-generating functions of the jump distributions on the two faces. This method also gives the local rate function in $\mathbb{R}_+^2$ for all linear paths not equal to the path identically zero (see [4]). It provides a complete solution for the ergodic face-homogeneous rw in $\mathbb{R}_+^2$, since the local rate for the identically zero path is zero for an ergodic rw in $\mathbb{R}_+^2$. The nonergodic case was left as an open problem in [4]. In the companion article [2], we partially solved this problem and we were able to find an expression for this local rate in terms of the logarithmic moment-generating functions of the three faces in which the state is unequal to zero (see Section 2 for the formula). However, we needed, in addition to some mild regularity conditions, an extra condition which seems to be unnecessary. In a recent article [3] the local rate function is related to the convergence parameter of associated local transform matrices. Our expressions make an easy numerical calculation of the local rate possible and also determine this convergence radius for the models we studied. Moreover, for specific queueing networks, closed-form expressions can be derived, as we do in this

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article. Indeed, in this companion article of [2], we focus on the queueing network in $\mathbb{R}_+^2$, which is the mathematical model for two processors that handle jobs of two types. The arrival processes are supposed to be Markovian and the service times are assumed to be exponentially distributed. If there are jobs of both types then both processors proceed independently, but if the buffer of one job type is empty then the processor of the other job type changes its service speed. This kind of coupled-processors model allows for various specifications: "the full capacity given to the nonempty buffer" and "switched-off processors if one of the buffers is empty." It turns out that for these special coupled-processors models, we can give closed-form expressions for the local rate function for the identically zero path in terms of the arrival and service rates. These expressions turn out to be surprisingly simple. As a by-product, we show that the extra condition of [2] (see relation (4) of Theorem 2.1) is indeed not necessary in these specific models. Also, we characterize the condition for the general coupled-processors model.

The outline of this article is as follows. In Section 2 we give the mathematical model (i.e., the class of rw in $\mathbb{R}_+^2$ corresponding to the induced queueing networks). Also in Section 2 we formulate the main result of [2] with the extra condition and we recall some lemmas needed for our analysis. In Section 3 we start with formulating the lemma of [2], which is the key to the analysis in this article. In Section 3.1 we derive the characterization of the extra condition for the coupled-processors model, and we derive the closed-form expression for the local rate in case it is satisfied. In Section 3.2 we treat the "full capacity model with full capacity given to the nonempty buffer." In the case that the queueing network is nonergodic, there is one expression for the local rate for all parameter instances. In Section 3.3 the "system with switched-off processors if one of the buffers is empty" is analyzed; it turns out that in this model we have two expressions for different regions in the space of service and arrival rates. Also, we provide some numerical results for a model that has only twisted processes which are either null recurrent or transient. To conclude, we add in Section 3.4, as a special case, the one-processor model and we show that its local rate is a special case of the expressions derived in Sections 3.2 and 3.3.

## 2. MODEL AND LARGE DEVIATIONS BOUNDS

### 2.1. Model Description

Consider a system of two processors indexed by 1 and 2. By $\lambda_1 > 0$ and $\lambda_2 > 0$ we denote their input rates, respectively. If both queues are not empty, then processors 1 and 2 work with service rate $\mu_1 > 0$ and $\mu_2 > 0$, respectively. If queue 1 becomes empty the service rate of processor 2 switches to $\mu_2^0 \geq 0$, and if queue 2 becomes empty the service rate of processor 1 switches to $\mu_1^0 \geq 0$.

We consider the time-discretized version obtained by uniformization, with transition probabilities $p_{ij}$, $i, j \in \mathbb{Z}_+^2$, depicted in Figure 1, where we assume that relation (1) holds.

$$\lambda_1 + \lambda_2 + \mu_1 + \mu_2 \leq 1 \quad \text{and} \quad \lambda_1 + \lambda_2 + \mu_i^0 \leq 1, \qquad i = 1,2. \tag{1}$$

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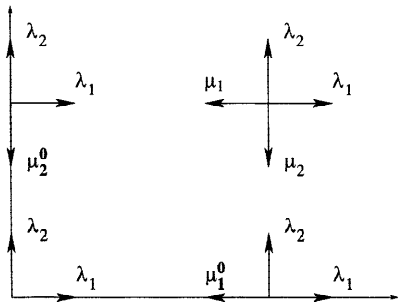



**FIGURE 1.** Transition probabilities on the faces.

This model is a Markov chain (MC), with state space $\mathbb{Z}_+^2$ and with four faces of homogeneity:

$$\Lambda^1 = \{i \in \mathbb{Z}^2 : i_1 > 0, i_2 = 0\},$$

$$\Lambda^2 = \{i \in \mathbb{Z}^2 : i_1 = 0, i_2 > 0\},$$

$$\Lambda^3 = \{i \in \mathbb{Z}^2 : i_1 > 0, i_2 > 0\},$$

and

$$\Lambda^0 = \{0\}.$$

Note that it is a face-homogeneous random walk on $\mathbb{Z}_+^2$.

### 2.2. Large Deviations Bounds

Let us introduce the $H$-functions (i.e., the logarithmic moment-generating functions):

$$H^0(\alpha) = \log\{1 - \lambda_1 - \lambda_2 + \lambda_1 e^{\alpha_1} + \lambda_2 e^{\alpha_2}\},$$

$$H^1(\alpha) = \log\{1 - \lambda_1 - \lambda_2 - \mu_1^0 + \lambda_1 e^{\alpha_1} + \lambda_2 e^{\alpha_2} + \mu_1^0 e^{-\alpha_1}\},$$

$$H^2(\alpha) = \log\{1 - \lambda_1 - \lambda_2 - \mu_2^0 + \lambda_1 e^{\alpha_1} + \lambda_2 e^{\alpha_2} + \mu_2^0 e^{-\alpha_2}\},$$

$$H^3(\alpha) = \log\{1 - \lambda_1 - \lambda_2 - \mu_1 - \mu_2 + \lambda_1 e^{\alpha_1} + \lambda_2 e^{\alpha_2} + \mu_1 e^{-\alpha_1} + \mu_2 e^{-\alpha_2}\}.$$

We use the following points in $\mathbb{R}^2$:

$$\alpha^1 = \arg\min_{\alpha} \max\{H^1(\alpha), H^3(\alpha)\},$$

$$\alpha^2 = \arg\min_{\alpha} \max\{H^2(\alpha), H^3(\alpha)\}. \tag{2}$$

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In [2], a general face-homogeneous rw has been analyzed under certain conditions. It is easily checked that the conditions hold for the model of this article. It has been proved in [2] that $\alpha^1$ and $\alpha^2$ exist and are finite. Let

$$\hat{\alpha} = \arg\max\{H^3(\alpha^1), H^3(\alpha^2)\}. \tag{3}$$

In [2], the following large deviations (LD) theorem has been proved for a face-homogeneous rw on $\mathbb{Z}_+^2$, with the Euclidean norm $\|i\| = \sqrt{i_1^2 + i_2^2}$.

THEOREM 2.1: *If for $\Lambda = 0,1,2$ we have*

$$H^\Lambda(\hat{\alpha}) \le H^3(\hat{\alpha}), \tag{4}$$

*then the following LD bounds are satisfied with local rate function $L^0 = H^3(\hat{\alpha})$:*

***LD upper bound:*** *For any $\delta > 0$ there exists $N(\delta)$ such that*

$$\mathsf{P}\left\{S_0 = 0, \sup_{t=0,\dots,[\tau N]} \|S_t\| < \delta N\right\} \le \exp\{+\delta N + N\tau L^0\} \tag{5}$$

*for all $N > N(\delta)$.*

***LD lower bound:*** *For any $\delta > 0$, $\delta' > 0$, there exists $N(\delta, \delta')$ such that*

$$\mathsf{P}\left\{S_0 = 0, \sup_{t=0,\dots,[\tau N]} \|S_t\| < \delta N\right\} \ge \exp\{-\delta' N + N\tau L^0\} \tag{6}$$

*for all $N > N(\delta, \delta')$.*

### 2.3. Twisted Process and Classification Lemma

For any $\alpha \in \mathbb{R}^2$ we define a MC

$$\mathcal{M}(\alpha) = \{S_t^\alpha, t = 0,1,2,\dots\}$$

on the state space $\mathbb{Z}_+^2$ with transition probabilities

$$p_{ij}(\alpha) \triangleq \frac{p_{ij}\exp\{(\alpha, j) - (\alpha, i)\}}{\sum_j p_{ij}\exp\{(\alpha, j) - (\alpha, i)\}}, \tag{7}$$

where $(\cdot,\cdot)$ is the scalar product in $\mathbb{R}^2$. The MC $\mathcal{M}(\alpha)$ is said to be a *twisted MC*. Note that $\mathcal{M}(0) = \mathcal{M}$.

For any $\alpha$ and any face $\Lambda$, we have that

$$p_{ij}(\alpha) = p_{j-i}^\Lambda(\alpha) \quad \text{for any } i \in \Lambda.$$

Hence, for any $\alpha$ the twisted process $\mathcal{M}(\alpha)$ is a face-homogeneous random walk on $\mathbb{Z}_+^2$ with the same faces of homogeneity.

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Now we define the vector $M^{\Lambda}(\alpha): \mathbb{R}^2 \to \mathbb{R}^2$, $\Lambda = 0,1,2,3$, as the mean drift from a state $i \in \Lambda$ for the twisted process $\mathcal{M}(\alpha)$. We have

$$M^{\Lambda}(\alpha) \triangleq \sum_{k \in \mathbb{Z}_+^2} k p_k^{\Lambda}(\alpha). \tag{8}$$

One can easily verify that the functions $H^{\Lambda}(\alpha)$, $\Lambda = 0,1,2,3$, are continuously differentiable. In [2] the following relation has been proved.

LEMMA 2.2: *For any $\alpha$ we have that*

$$M^{\Lambda}(\alpha) = (M_1^{\Lambda}(\alpha), M_2^{\Lambda}(\alpha)) \triangleq \left( \frac{\partial}{\partial \alpha_1} H^{\Lambda}(\alpha), \frac{\partial}{\partial \alpha_2} H^{\Lambda}(\alpha) \right).$$

In the following, we give the classification for the twisted process. The following lemma is a consequence of Theorems 3.3.1 and 3.3.2 of [1].

LEMMA 2.3: *With the mean drift vectors (8), we define two functions:*

$$V^1(\alpha) \triangleq M_1^3(\alpha) M_2^1(\alpha) - M_2^3(\alpha) M_1^1(\alpha),$$

$$V^2(\alpha) \triangleq M_2^3(\alpha) M_1^2(\alpha) - M_1^3(\alpha) M_2^2(\alpha).$$

*Then for any given $\alpha$ we have the following:*

*(a) If $M_1^3(\alpha) < 0, M_2^3(\alpha) < 0$, then $\mathcal{M}(\alpha)$ is*
  *(i) ergodic iff $V^1(\alpha) < 0$ and $V^2(\alpha) < 0$*
  *(ii) null recurrent iff*

$$\text{either} \quad \begin{matrix} V^2(\alpha) = 0 \\ V^1(\alpha) \le 0 \end{matrix} \quad \text{or} \quad \begin{matrix} V^1(\alpha) = 0 \\ V^2(\alpha) \le 0 \end{matrix} \tag{9}$$

  *(iii) transient iff $V^1(\alpha) > 0$ or $V^2(\alpha) > 0$.*

*(b) If $M_1^3(\alpha) \ge 0, M_2^3(\alpha) < 0$, then $\mathcal{M}(\alpha)$ is*
  *(i) ergodic iff $V^1(\alpha) < 0$*
  *(ii) null recurrent iff $V^1(\alpha) = 0$*
  *(iii) transient iff $V^1(\alpha) > 0$.*

*(c) If $M_1^3(\alpha) < 0, M_2^3(\alpha) \ge 0$, then $\mathcal{M}(\alpha)$ is*
  *(i) ergodic iff $V^2(\alpha) < 0$*
  *(ii) null recurrent iff $V^2(\alpha) = 0$*
  *(iii) transient iff $V^2(\alpha) > 0$.*

*(d) If $M_1^3(\alpha) \ge 0, M_2^3(\alpha) \ge 0$ and $M_1^3(\alpha) + M_2^3(\alpha) > 0$ then $\mathcal{M}(\alpha)$ is transient.*

*Remark 2.1:* If $M_1^3(\alpha) = M_2^3(\alpha) = 0$ then $\mathcal{M}(\alpha)$ can be ergodic, null recurrent, or transient (see [1]).

For simplicity, we will write $M^{\Lambda}$, $V^1$, and $V^2$ instead of $M^{\Lambda}(0)$, $V^1(0)$, and $V^2(0)$, respectively.

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## 3. THE LOCAL RATE FUNCTION FOR THE ZERO PATH

In the following subsections we derive explicit expressions for the local rate $L^0$ in terms of the input and service rates. The analysis is based on the following lemma, which has been proved in [2].

LEMMA 3.1: *The points $\alpha^1$ and $\alpha^2$ are finite. Either*

$$H^1(\alpha^1) \le H^3(\alpha^1), \qquad M^3(\alpha^1) = 0$$

*or $\alpha^1$ is the unique solution of the system*

$$H^1(\alpha) = H^3(\alpha), \qquad V^1(\alpha) = 0, \qquad M_2^3(\alpha) < 0.$$

*Either*

$$H^2(\alpha^2) \le H^3(\alpha^2), \qquad M^3(\alpha^2) = 0$$

*or $\alpha^2$ is the unique solution of the system*

$$H^2(\alpha) = H^3(\alpha), \qquad V^2(\alpha) = 0, \qquad M_1^3(\alpha) < 0.$$

### 3.1. The Case That the Local Rate Function Is Determined by $H^3$

In this subsection we give criteria in terms of $\mu_i^0$, $\mu_i$, and $\lambda_i$, $i = 1,2$, for $\hat{\alpha}$ to be equal to $\alpha^3$, where

$$\alpha^3 = \arg\min_{\alpha} H^3(\alpha).$$

It has been proved in [2] that $\alpha^3$ is finite.

For $\lambda_i \neq \mu_i, i = 1,2$, we define the constants

$$a_i = \frac{\sqrt{\mu_i}}{\sqrt{\lambda_i} - \sqrt{\mu_i}} \left(\sqrt{\lambda_1 \mu_1} + \sqrt{\lambda_2 \mu_2} - \mu_1 - \mu_1\right).$$

LEMMA 3.2: *The condition*

$$H^\Lambda(\alpha^3) \le H^3(\alpha^3) \quad \text{for } \Lambda = 0,1,2, \tag{10}$$

*holds if and only if one of the following conditions is satisfied:*

*(a)* $\mu_1 < \lambda_1$, $\mu_2 < \lambda_2$, *and* $\mu_1^0 < a_1$, $\mu_2^0 < a_2$
*(b)* $\mu_1 = \lambda_1$, $\mu_2^0 \le \mu_2 < \lambda_2$, *or* $\mu_2 = \lambda_2$, $\mu_1^0 \le \mu_1 < \lambda_1$
*(c)* $\mu_1 > \lambda_1$, $\mu_2 < \lambda_2$, *and* $\mu_2^0 \le a_2$
*(d)* $\mu_1 < \lambda_1$, $\mu_2 > \lambda_2$, *and* $\mu_1^0 \le a_1$
*(e)* $\mu_1 = \lambda_1$, $\mu_2 = \lambda_2$.

*We have that $\mu_1 < a_1$, $\mu_2 < \lambda_2$ in case (a), $a_2 < \mu_2$ in case (c), and $a_1 < \mu_1$ in case (d).*

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PROOF: We have that $M^3(\alpha^3) = 0$. It is easily checked that

$$M_1^3(\alpha) = 0 \quad \text{iff } \lambda_1 e^{\alpha_1} - \mu_1 e^{-\alpha_1} = 0 \text{ iff } \alpha_1 = \log\sqrt{\mu_1/\lambda_1},$$
$$M_2^3(\alpha) = 0 \quad \text{iff } \lambda_2 e^{\alpha_2} - \mu_2 e^{-\alpha_2} = 0 \text{ iff } \alpha_2 = \log\sqrt{\mu_2/\lambda_2};$$

that is,

$$\alpha^3 = (\log\sqrt{\mu_1/\lambda_1}, \log\sqrt{\mu_2/\lambda_2}).$$

Then

$$H^0(\alpha^3) = \log\{1 - \lambda_1 - \lambda_2 + \sqrt{\lambda_1\mu_1} + \sqrt{\lambda_2\mu_2}\},$$
$$H^1(\alpha^3) = \log\{1 - \lambda_1 - \lambda_2 - \mu_1^0 + \sqrt{\lambda_1\mu_1} + \sqrt{\lambda_2\mu_2} + \mu_1^0\sqrt{\lambda_1/\mu_1}\},$$
$$H^2(\alpha^3) = \log\{1 - \lambda_1 - \lambda_2 - \mu_2^0 + \sqrt{\lambda_1\mu_1} + \sqrt{\lambda_2\mu_2} + \mu_2^0\sqrt{\lambda_2/\mu_2}\},$$
$$H^3(\alpha^3) = \log\{1 - \lambda_1 - \lambda_2 - \mu_1 - \mu_2 + 2\sqrt{\lambda_1\mu_1} + 2\sqrt{\lambda_2\mu_2}\}. \tag{11}$$

Now we obtain that

$$H^0(\alpha^3) \le H^3(\alpha^3) \quad \text{iff } 0 \le \sqrt{\lambda_1\mu_1} + \sqrt{\lambda_2\mu_2} - \mu_1 - \mu_2, \tag{12}$$

$$H^1(\alpha^3) \le H^3(\alpha^3) \quad \text{iff } \mu_1^0 \frac{\sqrt{\lambda_1} - \sqrt{\mu_1}}{\sqrt{\mu_1}} \le \sqrt{\lambda_1\mu_1} + \sqrt{\lambda_2\mu_2} - \mu_1 - \mu_2, \tag{13}$$

$$H^2(\alpha^3) \le H^3(\alpha^3) \quad \text{iff } \mu_2^0 \frac{\sqrt{\lambda_2} - \sqrt{\mu_2}}{\sqrt{\mu_2}} \le \sqrt{\lambda_1\mu_1} + \sqrt{\lambda_2\mu_2} - \mu_1 - \mu_2. \tag{14}$$

(a) Let $\mu_1 < \lambda_1$, $\mu_2 < \lambda_2$. Then (12) follows immediately. Conditions (13) and (14) hold iff

$$\mu_1^0 \le a_1 \quad \text{and} \quad \mu_2^0 \le a_2.$$

(b) Let $\mu_1 = \lambda_1$. Then, (12) and (13) hold iff $\mu_2 \le \lambda_2$. If $\mu_2 < \lambda_2$, then (14) holds iff $\mu_2^0 \le \mu_2$. Therefore, (10) follows from the condition $\mu_1 = \lambda_1, \mu_2^0 \le \mu_2 < \lambda_2$. Similarly, we prove the case $\mu_2 = \lambda_2$ and $\mu_1^0 \le \mu_1 < \lambda_1$.

(c) Let $\mu_1 > \lambda_1$ and $\mu_2 < \lambda_2$. Since $\mu_2 < \lambda_2$, then (12) holds iff $a_2 \ge 0$. Clearly, if (12) is true, then (13) holds since $\mu_1 > \lambda_1$. Now, one can easily see that (14) is satisfied iff $\mu_2^0 \le a_2$.

(d) This can be proved similar to part (c).

(e) The proof is obvious.

One can easily see that

$$a_i = \mu_i + \sqrt{\mu_1\mu_2}\,\frac{\sqrt{\lambda_j} - \sqrt{\mu_j}}{\sqrt{\lambda_i} - \sqrt{\mu_i}}, \qquad i, j \in \{1,2\}, i \ne j.$$

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Now, if $\mu_1 < \lambda_1$ and $\mu_2 < \lambda_2$, then $\mu_1 < a_1$ and $\mu_2 < a_2$. If $\mu_1 < \lambda_1$ and $\mu_2 > \lambda_2$, then $a_1 < \mu_1$. If $\mu_2 < \lambda_2$ and $\mu_1 > \lambda_1$, then $a_2 < \mu_2$. ■

THEOREM 3.3: *Let one of the conditions of Lemma 3.2 be satisfied. Then the LD bounds hold with the local rate function*

$$L^0 = H^3(\alpha^3) = \log\{1 - (\sqrt{\lambda_1} - \sqrt{\mu_1})^2 - (\sqrt{\lambda_2} - \sqrt{\mu_2})^2\}. \tag{15}$$

PROOF: If condition (10) holds, then $\alpha^1 = \alpha^3 = \alpha^2$ and so $\hat{\alpha} = \alpha^3$. Hence, (4) of Theorem 2.1 is satisfied and the local rate function $L^0 = H^3(\alpha^3)$. Note that relation (11) is a slightly different expression of (15). ■

### 3.2. The Coupled-Processors System with Full Capacity Given to the Nonempty Buffer

In this subsection, we consider the system with $\mu_1^0 = \mu_2^0 = \mu_1 + \mu_2$.

LEMMA 3.4: *Let $\mu_1 \neq \lambda_1$ and $\mu_2 \neq \lambda_2$. Then the system is transient iff $\mu_1 + \mu_2 < \lambda_1 + \lambda_2$. It is null recurrent iff $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$ and it is ergodic iff $\mu_1 + \mu_2 > \lambda_1 + \lambda_2$.*

PROOF: We have that

$$V^1 = (\lambda_1 - \mu_1)\lambda_2 - (\lambda_2 - \mu_2)(\lambda_1 - \mu_1 - \mu_2) = \mu_2(\lambda_1 + \lambda_2 - \mu_1 - \mu_2),$$

$$V^2 = (\lambda_2 - \mu_2)\lambda_1 - (\lambda_1 - \mu_1)(\lambda_2 - \mu_1 - \mu_2) = \mu_1(\lambda_1 + \lambda_2 - \mu_1 - \mu_2).$$

Now the criteria for the system to be transient, null recurrent, or ergodic follow immediately from Lemma 2.3, and within parentheses, we indicate the part that is applicable from Lemma 2.3.

(a) If $\mu_1 > \lambda_1, \mu_2 > \lambda_2$ (part (a)), then $V^1 < 0, V^2 < 0$ and so the system is ergodic.

(b, c) If $\lambda_1 \geq \mu_1, \lambda_2 < \mu_2$ (part (b)) or $\lambda_1 < \mu_1, \lambda_2 \geq \mu_2$ (part (c)), then the system is ergodic if $\lambda_1 + \lambda_2 < \mu_1 + \mu_2$, it is null recurrent if $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$, and it is transient if $\lambda_1 + \lambda_2 > \mu_1 + \mu_2$.

(d) If $\lambda_1 \geq \mu_1$, $\lambda_2 \geq \mu_2$, and $\lambda_1 + \lambda_2 > \mu_1 + \mu_2$ (part (d)), then the system is transient. ■

In the next theorem we determine the local rate function $L^0$ for the nonergodic random walk. Since $L^0 = 0$ for the ergodic case, we have obtained a complete solution for this model.

THEOREM 3.5: *Let the system be nonergodic. Then the LD bounds hold with the local rate function*

$$L^0 = \log\{1 - (\sqrt{\lambda_1 + \lambda_2} - \sqrt{\mu_1 + \mu_2})^2\}. \tag{16}$$

PROOF: Let us calculate $\hat{\alpha}$. Recall that either $\hat{\alpha} = \alpha^1$ or $\hat{\alpha} = \alpha^2$ by (3).

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By Lemma 3.1 either $\alpha^1 = \alpha^3 = \alpha^2$ or $\alpha^1$ satisfies the system

$$H^1(\alpha) = H^3(\alpha), \qquad V^1(\alpha) = 0, \qquad M_2^3(\alpha) < 0,$$

and $\alpha^2$ satisfies the system

$$H^2(\alpha) = H^3(\alpha), \qquad V^2(\alpha) = 0, \qquad M_1^3(\alpha) < 0.$$

From the equation $H^1(\alpha) = H^3(\alpha)$, it follows that

$$(\mu_1 + \mu_2)e^{-\alpha_1} = \mu_1 e^{-\alpha_1} + \mu_2 e^{-\alpha_2},$$

and so $\alpha_1 = \alpha_2$. The equation $V^1(\alpha) = 0$ implies that

$$(\lambda_1 e^{\alpha_1} - \mu_1 e^{-\alpha_1})\lambda_2 e^{\alpha_2} - (\lambda_2 e^{\alpha_2} - \mu_2 e^{-\alpha_2})(\lambda_1 e^{\alpha_1} - (\mu_1 + \mu_2)e^{-\alpha_1}) = 0.$$

Now, taking in account that $\alpha_1 = \alpha_2$, we get that

$$\mu_2\lambda_1 + \mu_2\lambda_2 - \mu_2(\mu_1 + \mu_2)e^{-2\alpha_1} = 0$$

and, therefore,

$$\alpha_1 = \log\sqrt{\frac{\mu_1 + \mu_2}{\lambda_1 + \lambda_2}}.$$

Thus, we found the solution of $H^1(\alpha) = H^3(\alpha)$ and $V^1(\alpha) = 0$. Let us show that this solution is exactly $\hat{\alpha}$; that is, we will prove that

$$\hat{\alpha} = \left(\log\sqrt{\frac{\mu_1 + \mu_2}{\lambda_1 + \lambda_2}}, \log\sqrt{\frac{\mu_1 + \mu_2}{\lambda_1 + \lambda_2}}\right). \tag{17}$$

First, note that from similar calculations, it follows that (17) is the solution of $H^2(\alpha) = H^3(\alpha)$ and $V^2(\alpha) = 0$ as well. Now, we need to check whether (17) satisfies one of the inequalities

$$M_1^3(\hat{\alpha}) < 0 \quad \text{or} \quad M_2^3(\hat{\alpha}) < 0.$$

We have that

$$\operatorname{sign}(M_1^3(\hat{\alpha})) = \operatorname{sign}(\lambda_1 e^{\hat{\alpha}_1} - \mu_1 e^{-\hat{\alpha}_1}) = \operatorname{sign}(\lambda_1\mu_2 - \lambda_2\mu_1),$$

$$\operatorname{sign}(M_2^3(\hat{\alpha})) = \operatorname{sign}(\lambda_2 e^{\hat{\alpha}_2} - \mu_2 e^{-\hat{\alpha}_2}) = \operatorname{sign}(\lambda_2\mu_1 - \lambda_1\mu_2).$$

Therefore, one of the following cases holds:

1. If $\lambda_1\mu_2 > \lambda_2\mu_1$, then $M_2^3(\hat{\alpha}) < 0$ and, so, $\hat{\alpha} = \alpha^1$.
2. If $\lambda_1\mu_2 < \lambda_2\mu_1$, then $M_1^3(\hat{\alpha}) < 0$ and, so, $\hat{\alpha} = \alpha^2$.
3. If $\lambda_1\mu_2 = \lambda_2\mu_1$, then $M_1^3(\hat{\alpha}) = 0 = M_2^3(\hat{\alpha})$. It means that $\hat{\alpha} = \alpha^3$ since $M^3(\alpha^3) = 0$.

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Finally, we check whether $H^0(\hat{\alpha}) \leq H^3(\hat{\alpha})$. We have that

$$H^3(\hat{\alpha}) = \log\{1 - \lambda_1 - \lambda_2 - \mu_1 - \mu_2 + 2\sqrt{(\lambda_1 + \lambda_2)(\mu_1 + \mu_2)}\} \tag{18}$$

and

$$H^0(\hat{\alpha}) = \log\{1 - \lambda_1 - \lambda_2 + \sqrt{(\lambda_1 + \lambda_2)(\mu_1 + \mu_2)}\}.$$

Since we consider a nonergodic MC we have by Lemma 3.4 that $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$. Taking this inequality into account, one can easily see that $H^0(\hat{\alpha}) \leq H^3(\hat{\alpha})$ (i.e., the conditions of Theorem 2.1 are satisfied with the local rate function $L^0 = H^3(\hat{\alpha})$). To conclude, it is easily seen that (16) follows from (18). ■

*Remark 3.1:* Close reading of the proofs of Lemma 3.2 and Theorem 3.5 gives that only if $\lambda_1\mu_2 = \lambda_2\mu_1$ do we have that the system of this section is one of the cases of Lemma 3.2 (the case $\mu_1 = \lambda_1$, $\mu_2 = \lambda_2$ or the case $\mu_1 < \lambda_1$, $\mu_2 < \lambda_2$). One can easily check that (15) and (16) coincide if $\lambda_1\mu_2 = \lambda_2\mu_1$. Moreover, if $\lambda_1\mu_2 \neq \lambda_2\mu_1$ then from Lemma 2.3 it follows that the twisted process $\mathcal{M}(\hat{\alpha})$ is null recurrent.

### 3.3. The Coupled-Processors System with Switched-Off Processor If One of the Queues Is Empty

In this subsection, we consider the system with $\mu_1^0 = \mu_2^0 = 0$. In the next lemma we show that the MC of this model is always nonergodic.

LEMMA 3.6: *The system is always nonergodic. If $\mu_1 \neq \lambda_1$ or $\mu_2 \neq \lambda_2$ then the system is null recurrent iff*

$$\lambda_1\mu_2 = \lambda_2\mu_1 \quad \textit{and} \quad \mu_1 > \lambda_1, \mu_2 > \lambda_2,$$

*and it is transient otherwise.*

PROOF: We have that

$$V^1 = (\lambda_1 - \mu_1)\lambda_2 - (\lambda_2 - \mu_2)\lambda_1 = \mu_2\lambda_1 - \mu_1\lambda_2 = -V^2.$$

From Lemma 2.3 it follows that one of the following cases holds:

(a) if $\mu_1 > \lambda_1, \mu_2 > \lambda_2$ then the system is null recurrent iff $\lambda_1\mu_2 = \lambda_2\mu_1$, and it is transient otherwise;
(b) if $\mu_1 \leq \lambda_1, \mu_2 > \lambda_2$ then $0 < V^1$ and so the system is transient;
(c) if $\mu_1 > \lambda_1, \mu_2 \leq \lambda_2$ then $0 < V^2$ and so the system is transient;
(d) if $\mu_1 \leq \lambda_1, \mu_2 \leq \lambda_2$ and $\lambda_1 + \lambda_2 > \mu_1 + \mu_2$ then the system is transient, and if $\mu_1 = \lambda_1, \mu_2 = \lambda_2$ then from the analysis in [1] it follows that the system is nonergodic. ■

THEOREM 3.7: *The LD bounds hold with the local rate function*

$$L^0 = \log\{1 - (\sqrt{\lambda_1} - \sqrt{\mu_1})^2 - (\sqrt{\lambda_2} - \sqrt{\mu_2})^2\} \tag{19}$$

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*if*

$$\mu_1 + \mu_2 \leq \sqrt{\lambda_1 \mu_1} + \sqrt{\lambda_2 \mu_2}. \tag{20}$$

*If (20) does not hold, then the local rate function is*

$$L^0 = \log\left\{1 - \frac{(\sqrt{\lambda_1 \mu_2} - \sqrt{\lambda_2 \mu_1})^2}{\mu_1 + \mu_2}\right\}. \tag{21}$$

PROOF: If $\mu_1^0 = \mu_2^0 = 0$, then

$$H^0(\alpha) = H^1(\alpha) = H^2(\alpha) = \log(1 - \lambda_1 - \lambda_2 + \lambda_1 e^{\alpha_1} + \lambda_2 e^{\alpha_2}). \tag{22}$$

Note that the functions $H^0(\alpha)$ and $H^3(\alpha)$ coincide with those of Lemma 3.5. Then, from (12), it follows that $H^0(\alpha^3) \leq H^3(\alpha^3)$ iff (20) holds. Hence, (4) is satisfied and it follows from Theorem 2.1 that the LD bounds hold with the local rate function $L^0$ as in (15).

Suppose that (20) does not hold; then we have that $\hat{\alpha} \neq \alpha^3$. Therefore, by Lemma 3.1, the point $\alpha^1$ is the solution of the system

$$H^1(\alpha) = H^3(\alpha), \qquad V^1(\alpha) = 0, \qquad M_2^3(\alpha) < 0,$$

and $\alpha^2$ is the solution of the system

$$H^2(\alpha) = H^3(\alpha), \qquad V^2(\alpha) = 0, \qquad M_1^3(\alpha) < 0.$$

The equations $H^1(\alpha) = H^3(\alpha)$ and $V^1(\alpha) = 0$ give the following system:

$$\mu_1 e^{-\alpha_1} + \mu_2 e^{-\alpha_2} = \mu_1 + \mu_2,$$
$$\mu_1 \lambda_2 e^{\alpha_2 - \alpha_1} = \mu_2 \lambda_1 e^{\alpha_1 - \alpha_2}.$$

The solution of this system is

$$\alpha_1 = \log\left\{\frac{\mu_1 + \mu_2\sqrt{\dfrac{\mu_1 \lambda_2}{\mu_2 \lambda_1}}}{\mu_1 + \mu_2}\right\} \quad \text{and} \quad \alpha_2 = \log\left\{\frac{\mu_2 + \mu_1\sqrt{\dfrac{\mu_2 \lambda_1}{\mu_1 \lambda_2}}}{\mu_1 + \mu_2}\right\}. \tag{23}$$

Since $H^1(\alpha) = H^2(\alpha)$, $V^1(\alpha) = -V^2(\alpha)$. Hence, (23) is the solution of $H^2(\alpha) = H^3(\alpha)$ and $V^2(\alpha) = 0$ as well, and so $\hat{\alpha}$ is equal to (23). Substituting (23) in (22), we find with the same calculation that

$$H^3(\hat{\alpha}) = \log\left\{1 - \lambda_1 - \lambda_2 + \frac{(\sqrt{\mu_1 \lambda_1} + \sqrt{\mu_2 \lambda_2})^2}{\mu_1 + \mu_2}\right\}$$
$$= \log\left\{1 - \frac{(\sqrt{\lambda_1 \mu_2} - \sqrt{\lambda_2 \mu_1})^2}{\mu_1 + \mu_2}\right\}.$$

Since $H^0(\hat{\alpha}) = H^1(\hat{\alpha}) = H^2(\hat{\alpha}) = H^3(\hat{\alpha})$, (4) of Theorem 2.1 is satisfied and the local rate function $L^0$ is as in (21).

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Note that (19) and (21) coincide if $\mu_1 + \mu_2 = \sqrt{\lambda_1 \mu_1} + \sqrt{\lambda_2 \mu_2}$. ■

*Remark 3.2:* Since $V^1(\hat{\alpha}) = 0 = V^2(\hat{\alpha})$, it follows from Lemma 2.3 that the twisted MC $\mathcal{M}(\hat{\alpha})$ is null recurrent in all cases of this subsection except the case (20), where the MC $\mathcal{M}(\hat{\alpha})$ can be transient as well. Note that if (20) holds, then $\hat{\alpha} = \alpha^3$ and this is one of the cases of Lemma 3.2.

Now we give some numerical computations for the model with

$$\lambda_1 = 0.2, \qquad \lambda_2 = 0.5, \qquad \mu_1 = 0.5, \qquad \mu_2 = 0.2.$$

In this case, (20) is not satisfied. Hence, the local rate function $L^0$ is equal to (21). By simple calculations, we find from (21) and (23) that

$$L^0 = \log\left(\frac{61}{70}\right) \approx -0.1376 \quad \text{and} \quad \hat{\alpha} = \left(\log\frac{10}{7}, \log\frac{4}{7}\right) \approx (0.356, -0.56).$$

Figure 2 shows the level lines $H^0(\alpha) = 0$ and $H^3(\alpha) = 0$. Figure 3 shows the level lines of the $H$-functions at the level equal to $L^0$ and the level line $V^1(\alpha) = 0$ (the straight line). Starting from the point $\alpha^3 \approx (0.458, -0.458)$ and forward in a south-west direction, the line $V^1(\alpha) = 0$ contains the points $\alpha$ for which the MC $\mathcal{M}(\alpha)$ is null recurrent. All other points $\alpha$ correspond to the transient MC $\mathcal{M}(\alpha)$. Note that in Figure 2, all of the lines intersect at $\hat{\alpha}$, and the MC $\mathcal{M}(\hat{\alpha})$ is null recurrent.

### 3.4. One Processor System

For completeness we also derive the local rate function for the model with one processor and one buffer. Clearly, (24) is the special case of (15) with $\lambda_2 = \mu_2 = 0$.

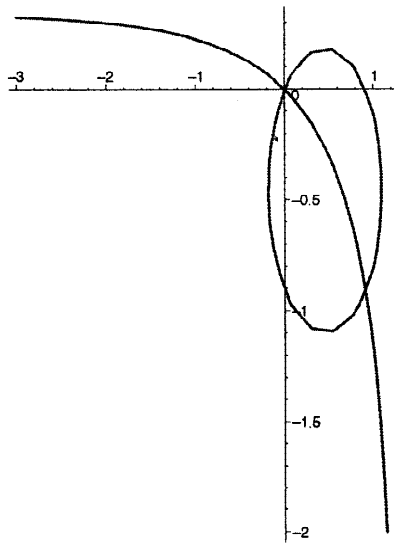



**FIGURE 2.** The level lines $H^0(\alpha) = 0 = H^3(\alpha)$.

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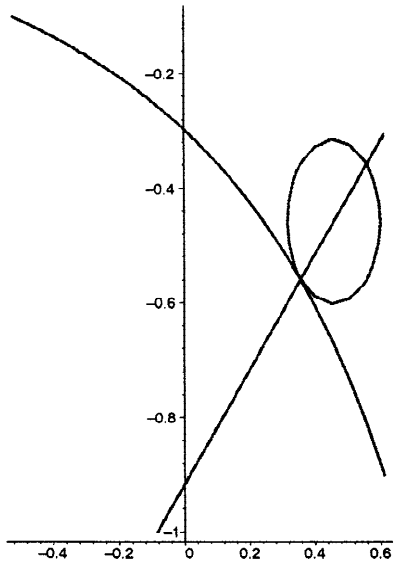



**FIGURE 3.** The level lines $V^1 = 0$, $H^0 = L^0 = H^3$.

Also, it can be argued that by considering the total number of customers at both processors, (16) is equal to (24), with $\lambda = \lambda_1 + \lambda_2$ and $\mu = \mu_1 + \mu_2$.

Let $\lambda > 0$, $\mu > 0$, and $\lambda + \mu \leq 1$. The system is nonergodic iff $\mu \leq \lambda$. We suppose this, and here we have the following $H$-functions:

$$H^0(\alpha) = \log(1 - \lambda + \lambda e^{\alpha}) \quad \text{and} \quad H^1(\alpha) = \log(1 - \lambda - \mu + \lambda e^{\alpha} + \mu e^{-\alpha}).$$

Then $H^0(\alpha) = H^1(\alpha)$ iff $\alpha = 0$. It means that $\hat{\alpha} = \arg\min_{\alpha} H^1(\alpha)$. We have $M^1(\alpha) = 0$ iff $\lambda e^{\alpha} - \mu e^{-\alpha} = 0$ (i.e., $\hat{\alpha} = \log\{\sqrt{\mu/\lambda}\}$) and, hence,

$$L^0 = H^1(\hat{\alpha}) = \log\{1 - \lambda - \mu + 2\sqrt{\mu\lambda}\} = \log\{1 - (\sqrt{\lambda} - \sqrt{\mu})^2\}. \qquad \textbf{(24)}$$

*References*

1. Fayolle, G., Menshikov, M.V., & Malyshev, V.A. (1995). *Topics in the constructive theory of countable Markov Chains.* Cambridge: Cambridge University Press.
2. Hordijk, A. & Popov, N.V. (2003). Large deviations bounds for face-homogeneous random walks in the quarter plane. *Probability in the Engineering and Informational Sciences* 17(3): 369–396.
3. Ignatyuk, I.A. (2002). Sample path large deviations and convergence parameters. *Annals of Applied Probability* 11: 1292–1329.
4. Ignatyuk, I.A., Malyshev, V.A., & Scherbakov, V.V. (1994). Boundary effects in large deviations problems. *Russian Mathematical Surveys* 49(2): 43–102.
5. Shwartz, A. & Weiss, A. (1995). *Large deviations for performance analysis: Queues, communications, and computing*. London: Chapman & Hall.

https://doi.org/10.1017/S0269964803173068 Published online by Cambridge University Press