Linearly periodic continued fractions

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1. *Introduction*

An infinite simple continued fraction representation of a real number α is in the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}},$$

where a_0 is an integer, and a_i are positive integers for $i \geqslant 1$. This is often written more compactly in one of the following ways:

$$a_0 + \frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{a_3 +} \ldots \text{ or } [a_0;\, a_1,\, a_2,\, a_3,\, \ldots\,].$$

A finite simple continued fraction $[a_0;\, a_1,\, a_2,\, \ldots\,,\, a_n]$ is a rational number. Continued fractions are important in many branches of mathematics, and particularly in the theory of approximation of real numbers by rationals.

The continued fraction $[a_0;\, a_1,\, a_2,\, \ldots\,,\, a_n,\, \ldots\,]$ is called (eventually) periodic if $[a_0;\, a_1,\, a_2,\, \ldots\,,\, a_n,\, \ldots\,] = [a_0;\, a_1,\, \ldots\,,\, a_k,\, \overline{a_{k+1},\, \ldots\,,\, a_l}]$ starts with a preperiod $a_0, a_1, a_2, \ldots, a_k$ and then a period $a_{k+1}, \ldots, a_l$ is repeated an infinite number of times.

The continued fraction expansion of a real number α is periodic from a point onward if, and only if, $\alpha = x + y\sqrt{d}$ is the root of some quadratic equation $az^2 + bz + c = 0$ with integer coefficients, that is,

$$\alpha = [a_0;\, a_1,\, \ldots\,,\, a_n,\, \overline{a_{n+1},\, a_{n+2},\, \ldots\,,\, a_{n+m}}]$$

and we call that α is a *quadratically periodic continued fraction*.

Definition 1: A linearly periodic continued fraction of period m and introductory sequence length $n + 1$ is of the form

$$[a_0;\, a_1,\, \ldots\,,\, a_n,\, \overline{f_1(i),\, \ldots\,,\, f_m(i)}]_{i=0}^{\infty}$$

where a_0 is an integer, $a_1, \ldots, a_n$ are positive integers and $f_1, \ldots, f_m$ are linear functions of i, that is $f_j(i) = b_j + c_j i$ for some integers b_j, c_j and $j = 1, 2, \ldots, m$.

Well-known examples (see e.g. [1, 2, 3]) are

$$e = [2;\, 1,\, 2,\, 1,\, 1,\, 4,\, 1,\, 1,\, 6,\, 1,\, 1, \ldots] = [2; \overline{1,\, 2i+2,\, 1}]_{i=0}^{\infty},$$

$$\tan 1 = [1;\, 1,\, 1,\, 3,\, 1,\, 5,\, 1,\, 7,\, 1,\, 9,\, 1, \ldots] = [1; \overline{2i+1,\, 1}]_{i=0}^{\infty},$$

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$$e^{1/k} = [1; k-1,\ 1,\ 1,\ 3k-1,\ 1,\ 1,\ 5k-1,\ 1,\ 1,\ 7k-1,\ 1, \dots]$$
$$= \left[1; \overline{(2i+1)k-1,\ 1,\ 1}\right]_{i=0}^{\infty}.$$

Note that any quadratically periodic continued fraction is also a linearly periodic continued fraction with $c_j = 0$ for all j.

The purpose of this Article is to give the m different series for the linearly periodic continued fractions of period m.

2. *Results*

Let $p_n/q_n = [a_0; a_1, a_2, \dots, a_n]$ be the nth convergent of the simple continued fraction expansion for some real number α. It is well-known (see e.g. Chapter X of [4]), that p_i and q_i satisfy the recurrences

$$p_{-2} = 0,\ p_{-1} = 1 \qquad \text{and} \qquad p_n = a_n p_{n-1} + p_{n-2},$$
$$q_{-2} = 1,\ q_{-1} = 0 \qquad \text{and} \qquad q_n = a_n q_{n-1} + q_{n-2}.$$

Using the recurrences of p_n and q_n, we get $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$, so

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_{n-1}q_n}.$$

Using telescoping summation, we have (see e.g. [4, 5])

$$\frac{p_n}{q_n} = a_0 + \frac{1}{q_0 q_1} - \frac{1}{q_1 q_2} + \dots + \frac{(-1)^{n-1}}{q_{n-1}q_n}.$$

Definition 2: For a simple continued fraction $[a_0; a_1, a_2, a_3, \dots]$ and fixed non-negative integer k, define the two sequences $\{A_{k,n}\}$ and $\{B_{k,n}\}$ by the second order linear recurrences

$$A_{k,-2} = 0,\ A_{k,-1} = 1,\ A_{k,n} = a_{k+n} A_{k,n-1} + A_{k,n-2} \qquad (n \geqslant 0),$$
$$B_{k,-2} = 1,\ B_{k,-1} = 0,\ B_{k,n} = a_{k+n} B_{k,n-1} + B_{k,n-2} \qquad (n \geqslant 0).$$

If $k = 0$, we have $A_{0,n} = p_n$ and $B_{0,n} = q_n$. From the recurrence relations of $A_{k,n}$ and $B_{k,n}$, we see that, by induction on n,

$$\begin{bmatrix} A_{k,n} & A_{k,n-1} \\ B_{k,n} & B_{k,n-1} \end{bmatrix} = \begin{bmatrix} A_{k,n-1} & A_{k,n-2} \\ B_{k,n-1} & B_{k,n-2} \end{bmatrix} \begin{bmatrix} a_{k+n} & 1 \\ 1 & 0 \end{bmatrix}$$
$$\vdots$$
$$= \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{k+1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{k+n} & 1 \\ 1 & 0 \end{bmatrix}.$$

By taking the determinant of both sides, we immediately obtain

$$A_{k,n} B_{k,n-1} - B_{k,n} A_{k,n-1} = (-1)^{n-1}.$$

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Also, it can be seen that

$$\frac{A_{k,n}}{B_{k,n}} = [a_k; a_{k+1}; a_{k+2}, \ldots, a_{k+n}].$$

The proof of above identity is analogous to the proof, when $k = 0$, of a standard formula for the convergents of $[a_0; a_1, a_2, \ldots, a_n]$.

We first prove some auxiliary results about numerators and denominators of the convergents.

Lemma: Let $\alpha = [a_0; a_1, a_2, a_3, \ldots]$ and p_n/q_n be its corresponding nth convergents. Let $A_{k,n}, B_{k,n}$ be defined above. Then

(i) $p_{k+n}q_k - p_kq_{k+n} = (-1)^k A_{k+2,n-2}$, (1)

(ii) $q_{k+1}A_{k+2,n-2} + q_kB_{k+2,n-2} = q_{k+n}$. (2)

Proof: Since $p_{k+1}q_k - p_kq_{k+1} = (-1)^k$, we have

$$\begin{aligned} p_{k+2}q_k - p_kq_{k+2} &= (a_{k+2}p_{k+1} + p_k)q_k - p_k(a_{k+2}q_{k+1} + q_k) \\ &= a_{k+2}(p_{k+1}q_k - p_kq_{k+1}) \\ &= (-1)^k a_{k+2}. \end{aligned}$$

Observe that $A_{k+2,-1} = 1$ and $A_{k+2,0} = a_{k+2}$, the identity (i) holds for $n = 1, 2$. Assume it holds for $n\,(> 1)$. By the recurrence relations of $A_{k,n}$, $B_{k,n}$ and the inductive hypothesis, we get

$$\begin{aligned} p_{k+n+1}q_k - p_kq_{k+n+1} &= (a_{k+n+1}p_{k+n})q_k - p_k(a_{k+n+1}q_{k+n} + q_{k+n-1}) \\ &= a_{k+n+1}(p_{k+n}q_k - p_kq_{k+n}) + (p_{k+n-1}q_k - p_kq_{k+n-1}) \\ &= (-1)^k a_{k+n+1}A_{k+2,n-2} + (-1)^n A_{k+2,n-3} \\ &= (-1)^n A_{k+2,n-1}. \end{aligned}$$

The identity (i) follows by induction.

In the same way, the proof of the identity (ii) proceeds by induction on n. The $n = 0, 1$ cases are immediate. Assume (ii) holds for $n\,(> 1)$. Using the recurrence relations of $A_{k,n}$, $B_{k,n}$, q_n and the inductive hypothesis, we get

$$\begin{aligned} & q_{k+1}A_{k+2,n-1} + q_kB_{k+2,n-1} \\ &= q_{k+1}(a_{k+n+1}A_{k+2,n-2} + A_{k+2,n-3}) + q_k(a_{k+n+1}B_{k+2,n-2} + B_{k+2,n-3}) \\ &= a_{k+n+1}q_{k+n} + q_{k+n-1} \\ &= q_{k+n+1}, \end{aligned}$$

so the identity (2) holds for $n + 1$.

The identity (1) can be found in [6, Lemma 1].

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Theorem 1: Let α be a positive irrational number whose linearly periodic continued fractions of period m is

$$\alpha = \left[a_0;\, a_1,\, \ldots,\, a_n, \overline{f_1(i),\, \ldots,\, f_m(i)}\right]_{i=0}^{\infty}.$$

Then α, when approximated by its nth convergent p_n/q_n, can be explicitly represented by

$$\alpha = \frac{p_n}{q_n} + \sum_{1=0}^{\infty} \frac{(-1)^{mi+n} C_i}{q_{mi+n}\left(q_{mi+n+1} C_i + q_{mi+n} D_i\right)},$$

where $\dfrac{C_i}{D_i} = \left[f_2(i),\, \ldots,\, f_m(i)\right]$ $(i \geqslant 0)$.

Proof: Rewrite (1) as

$$\frac{p_{k+n}}{q_{k+n}} = \frac{p_k}{q_k} + \frac{(-1)^k A_{k+2,n-2}}{q_k q_{k+n}}.$$

Replacing q_{k+n} in the right-hand side of the above equation by using (2), we obtain

$$\frac{p_{k+n}}{q_{k+n}} = \frac{p_k}{q_k} + \frac{(-1)^k A_{k+2,n-2}}{q_k\left(q_{k+1} A_{k+2,n-2} + q_k B_{k+2,n-2}\right)}.$$

Replacing k by $mi + n$ and n by m in above equation, we get

$$\frac{p_{m(i+1)+n}}{q_{m(i+1)+n}} - \frac{p_{mi+n}}{q_{mi+n}} = \frac{(-1)^{mi+n} A_{mi+n+2,m-2}}{q_{mi+n}\left(q_{mi+n+1} A_{mi+n+2,m-2} + q_{mi+n} B_{mi+n+2,m-2}\right)}.$$

Summing over i from 0 to ∞, we get

$$\alpha = \frac{p_n}{q_n} + \sum_{i=0}^{\infty} \frac{(-1)^{mi+n} A_{mi+n+2,m-2}}{q_{mi+n}\left(q_{mi+n+1} A_{mi+n+2,m-2} + q_{mi+n} B_{mi+n+2,m-2}\right)}$$

For non-negative integer i, observe that

$$\frac{A_{mi+n+2,m-2}}{B_{mi+n+2,m-2}} = \left[a_{mi+n+2},\, a_{mi+n+3},\, a_{m(i+1)+n}\right] = \left[f_2(i),\, \ldots,\, f_m(i)\right],$$

completing the proof.

We give some examples of Theorem 1 for e, $\tan 1$ and $\sqrt{e}$.

Example 1: We obtain three distinct infinite series for

$$e = \left[2;\ 1,\ 2,\ 1,\ 1,\ 4,\ 1,\ 1,\ 6,\ 1,\ 1,\ \ldots\right].$$

(i) $e = \left[2; \overline{1,\ 2i+2,\ 1}\right]_{i=0}^{\infty}$; we have $\dfrac{C_i}{D_i} = \left[2i+2;\, 1\right] = 2i+3$ for $i \geqslant 0$, so

$$e = 2 + \sum_{i=0}^{\infty} \frac{(-1)^i (2i+3)}{q_{3i}\left((2i+3) q_{3i+1} + q_{3i}\right)}$$

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$$= 2 + \frac{3}{1(3+1)} - \frac{5}{4(7\times 5+4)} + \frac{7}{39(71\times 7+39)} - \cdots$$

$$= 2 + \frac{3}{1\times 4} - \frac{5}{4\times 39} + \frac{7}{39\times 536} - \cdots .$$

(ii) $e = \left[2;\ 1, \overline{2i+2,\ 1,\ 1}\right]_{i=0}^{\infty}$; we have $\frac{C_i}{D_i} = [1;\ 1] = 2$ for $i \geqslant 0$, so

$$e = 3 + \sum_{i=0}^{\infty} \frac{(-1)^{i+1}\times 2}{q_{3i+1}(2q_{3i+2} + q_{3i+1})}$$

$$= 3 - \frac{2}{1(2\times 3+1)} + \frac{2}{7(2\times 32+7)} - \frac{2}{71(2\times 465+71)} + \cdots$$

$$= 3 - \frac{2}{1\times 7} + \frac{2}{7\times 71} - \frac{2}{71\times 1001} + \cdots .$$

(iii) $e = \left[2;\ 1,\ 2, \overline{1,\ 1,\ 2i+4}\right]_{i=0}^{\infty}$; we have $\frac{C_i}{D_i} = [1; 2i+4] = \frac{2i+5}{2i+4}$ for $i \geqslant 0$, so

$$e = \frac{8}{3} + \sum_{i=0}^{\infty} \frac{(-1)^i(2i+5)}{q_{3i+2}((2i+5)q_{3i+3} + (2i+4)q_{3i+2})}$$

$$= \frac{8}{3} + \frac{5}{3(5\times 4+4\times 3)} - \frac{7}{32(7\times 39+6\times 32)} + \frac{9}{465(9\times 536+8\times 465)} - \cdots$$

$$= \frac{8}{3} + \frac{5}{3\times 32} - \frac{7}{32\times 465} + \frac{9}{465\times 8544} + \cdots .$$

Example 2: We obtain two distinct infinite series for tan 1.

(i) $\tan 1 = \left[1; \overline{2i+1,\ 1}\right]_{i=0}^{\infty}$; we have $\frac{C_i}{D_i} = [1] = 1$, so

$$\tan 1 = 1 + \sum_{i=1}^{\infty} \frac{1}{q_{2i}(q_{2i+1} + q_{2i})}$$

$$= 1 + \frac{1}{1(1+1)} + \frac{1}{2(7+2)} + \frac{1}{9(52+9)} + \cdots$$

$$= 1 + \frac{1}{1\times 2} + \frac{1}{2\times 9} + \frac{1}{9\times 61} + \cdots .$$

(ii) $\tan 1 = \left[1;\ 1, \overline{1,\ 2i+3}\right]_{i=0}^{\infty}$; we have $\frac{C_i}{D_i} = [2i+3] = 2i+3$, so

$$\tan 1 = 2 - \sum_{i=0}^{\infty} \frac{2i+3}{q_{2i+1}((2i+1)q_{2i+2} + q_{2i+1})}$$

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$$= 2 - \frac{3}{1(3\times 2+1)} - \frac{5}{7(5\times 9+7)} - \frac{7}{52(7\times 61+52)} - \cdots$$

$$= 2 - \frac{3}{1\times 7} - \frac{5}{7\times 52} - \frac{7}{52\times 479} - \cdots .$$

Example 3: We obtain three distinct infinite series for $\sqrt{e}$.

(i) $\sqrt{e} = \left[1; \overline{4i+1,\ 1,\ 1}\right]_{i=0}^{\infty}$; we have $\frac{C_i}{D_i} = [1,\ 1] = 2$, so

$$\sqrt{e} = 1 + \sum_{i=0}^{\infty} \frac{2(-1)^{3i}}{q_{3i}(2q_{3i+1} + q_{3i})}$$

$$= 1 + \frac{2}{1\times 3} - \frac{2}{3\times 37} + \frac{2}{37\times 743} + \cdots .$$

(ii) $\sqrt{e} = \left[1; 1, \overline{1,\ 1,\ 4i+5}\right]_{i=0}^{\infty}$; we have $\frac{C_i}{D_i} = [1,\ 4i+5] = \frac{4i+6}{4i+5}$, for $i \geqslant 0$, so

$$\sqrt{e} = 2 + \sum_{i=0}^{\infty} \frac{(-1)^{3i+1}(4i+6)}{q_{3i+1}((4i+6)q_{3i+2} + (4i+5)q_{3i+1})}$$

$$= 2 - \frac{6}{1\times 17} - \frac{10}{17\times 353} + \frac{14}{353\times 10049} + \cdots .$$

(iii) $\sqrt{e} = \left[1; 1,1, \overline{1,\ 4i+5,\ 1}\right]_{i=0}^{\infty}$; we have $\frac{C_i}{D_i} = [4i+5,\ 1] = 4i+6$, for $i \geqslant 0$, so

$$\sqrt{e} = \frac{3}{2} + \sum_{i=0}^{\infty} \frac{(-1)^{3i+2}(4i+6)}{q_{3i+2}((4i+6)q_{3i+3} + q_{3i+2})}$$

$$= \frac{3}{2} + \frac{6}{2\times 20} - \frac{10}{20\times 390} + \frac{14}{390\times 10792} + \cdots .$$

We give the following corollary for quadratically periodic continued fractions.

Corollary: Let $x + y\sqrt{d}$ be a quadratic number in which d is not perfect square. Suppose that the continued fraction

$$x + y\sqrt{d} = \left[a_0; a_1, \ldots, a_n, \overline{a_{n+1}, a_{n+2}, \ldots, a_{n+m}}\right].$$

Then

$$x + y\sqrt{d} = [a_0; a_1, \ldots, a_n] + \sum_{i=0}^{\infty} \frac{(-1)^{mi+n}C}{q_{mi+n}(q_{mi+n+1}C + q_{mi+n}D)},$$

where $\frac{C}{D} = [a_{n+2}, a_{n+3}, \ldots, a_{n+m}]$.

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Note that $C_i = C$ and $D_i = D$ since $f_j(i) = a_{n+j}$ for all i and any $j = 1, 2, \ldots, m$.

Example 4: Consider $\frac{1}{3}(10+\sqrt{10}) = [2; 3, 1, 1, 2, 1, 1, 2, 1, 1, 2, \ldots]$; we have

$$\frac{10+\sqrt{10}}{3} = [2;\ 3,\ \overline{1,\ 1,\ 2}] = \frac{7}{3} - \frac{3}{3\times 18} + \frac{3}{18\times 111} - \frac{3}{111\times 684} + \ \ldots$$

$$= \left[2;\ 3,\ 1,\ \overline{1,\ 2,\ 1}\right] = \frac{9}{4} + \frac{3}{4\times 25} - \frac{3}{25\times 154} + \frac{3}{154\times 949} - \ \ldots$$

$$= \left[2;\ 3,\ 1,\ 1,\ \overline{2,\ 1,\ 1}\right] = \frac{16}{7} - \frac{2}{7\times 43} + \frac{2}{43\times 265} - \frac{2}{265\times 1633} - \ \ldots.$$

Example 5: We give some infinite series of $\sqrt{p}$ for prime $p \leqslant 11$ in the following table.

$\sqrt{p}$	continued fraction	series
$\sqrt{2}$	$[1; \overline{2,\ 2}]$	$= 1 + \frac{2}{1\times 5} + \frac{2}{5\times 29} + \frac{2}{29\times 169} + \ldots$
	$[1; 2, \overline{2,\ 2}]$	$= \frac{3}{2} - \frac{2}{2\times 12} - \frac{2}{12\times 70} - \frac{2}{70\times 408} - \ldots$
$\sqrt{3}$	$[1; \overline{1,\ 2}]$	$= 1 + \frac{2}{1\times 3} + \frac{2}{3\times 11} + \frac{2}{11\times 41} + \ldots$
	$[1; 1, \overline{2,\ 1}]$	$= 2 - \frac{1}{1\times 4} - \frac{1}{4\times 15} - \frac{1}{15\times 56} - \ldots$
$\sqrt{5}$	$[2; \overline{4,\ 4}]$	$= 2 + \frac{4}{1\times 17} + \frac{4}{17\times 305} + \frac{4}{305\times 5473} + \ldots$
	$[2; 4, \overline{4,\ 4}]$	$= \frac{9}{4} - \frac{4}{4\times 72} - \frac{4}{72\times 1292} - \frac{4}{1292\times 23184} - \ldots$
$\sqrt{7}$	$[2; \overline{1,\ 1,\ 1,\ 4}]$	$= 2 + \frac{9}{1\times 14} + \frac{9}{14\times 223} + \frac{9}{223\times 3554} + \ldots$
	$[2; 1, \overline{1,\ 1,\ 4,\ 1}]$	$= 3 - \frac{6}{1\times 17} - \frac{6}{17\times 271} - \frac{6}{271\times 4319} - \ldots$
	$[2; 1,\ 1, \overline{1,\ 4,\ 1,\ 1}]$	$= \frac{5}{2} + \frac{9}{2\times 31} + \frac{9}{31\times 494} + \frac{9}{494\times 7873} + \ldots$
	$[2; 1,\ 1,\ 1, \overline{4,\ 1,\ 1,\ 1}]$	$= \frac{8}{3} - \frac{3}{3\times 48} - \frac{3}{48\times 765} - \frac{3}{765\times 12192} - \ldots$
$\sqrt{11}$	$[3; \overline{3,\ 6}]$	$= 3 + \frac{6}{1\times 19} + \frac{6}{19\times 379} + \frac{6}{379\times 7561} + \ldots$
	$[3; 3, \overline{6,\ 3}]$	$= \frac{10}{3} - \frac{3}{3\times 60} - \frac{3}{60\times 1197} - \frac{3}{1197\times 23880} - \ldots$

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