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PROBLEMS AND SOLUTIONS

PROBLEM

04.3.1 An $I(2)$ Model for VAR(1) Processes

Paolo Paruolo
University of Insubria, Varese, Italy

This problem discusses an $I(2)$ model in the VAR(1) case. The $I(2)$ representation theorem of Johansen (1992) (JRT) holds also for VAR(1) processes. The $I(2)$ model for VAR(k) processes has been discussed for $k \geq 2$ in Johansen (1996, Ch. 9; 1997). We here discuss a parametrization for the $I(2)$ case of VAR(1) that differs from the VAR(k) model.

Consider the VAR(1) process with no deterministics,

$$\Delta X_t = \Pi X_{t-1} + \epsilon_t,$$

with ϵ_t *i.i.d.* $N(0, \Omega)$ of dimension p, Ω positive definite. It is assumed that $|I - (I + \Pi)z| = 0$ has solutions either at $z = 1$ or $|z| > 1$ (Assumption A). Under Assumption A, the JRT states that X_t is $I(2)$ iff the following three conditions hold.

(a) $\Pi = \alpha\beta'$, with α, β, full rank $p \times r$ matrices, $r < p$,

(b) $P_{\alpha_\perp} P_{\beta_\perp} = \alpha_1 \beta_1'$

for α_1, β_1 $p \times s$ full column rank matrices, $s < p - r$, and

(c) $\alpha_2' \theta \beta_2$ is square and invertible, where $\theta := I + \bar{\beta}\bar{\alpha}'$, $\alpha_2 := (\alpha : \alpha_1)_\perp$, $\beta_2 := (\beta : \beta_1)_\perp$, $\bar{\alpha} := a(a'a)^{-1}$, $P_a := \bar{\alpha}a'$. In the following $p_2 := p - r - s$.

1. Let a VAR(1) process be $I(2)$, i.e., let Assumption A and conditions (a)–(c) hold. Prove that $p_2 \leq r$.
2. Define the $I(1)$ model as the collection of VAR(1) processes indexed by the parameters $\tau := (\alpha, \beta, \Omega)$. Find the restrictions on τ that ensure that (b) holds. Also find an explicit parametrization. This model is called the $I(2)$ model.
3. Let $Y_t := \beta' X_t + \delta(\beta_2' \beta_2)^{-1} \beta_2' \Delta X_t$ describe the (possibly multi-) cointegrating relations. Show that δ is of full column rank p_2 and find δ in terms of the parameters in part 2. Indicate also within Y_t the $r - p_2$ polynomially cointegrating relations and the p_2 $CI(2,2)$ relations.
4. X_t has representation $\Delta^2 X_t = C_2(L)\epsilon_t$. Find the expression for $C_2 := C_2(1)$ in terms of the parameters in part 2.
5. Discuss the identification of τ.

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REFERENCES

Johansen, S. (1992) A representation of vector autoregressive processes integrated of order 2. *Econometric Theory* 8, 188–202.

Johansen, S. (1996) *Likelihood-Based Inference in Cointegrated Vector Auto-regressive Models*. Oxford University Press.

Johansen, S. (1997) A likelihood analysis of the $I(2)$ model. *Scandinavian Journal of Statistics* 24, 433–462.

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