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COMPARISONS OF DEPENDENCE FOR STATIONARY MARKOV PROCESSES*

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Results and conditions which quantify the decrease in dependence with lag for a stationary Markov process and enable one to compare the dependence for two stationary Markov processes are obtained. The notions of dependence used in this article are the supermodular ordering and the concordance ordering. Both discrete-time and continuous-time Markov processes are considered. Some applications of the main results are given. In queueing theory, the monotonicity results of the waiting time of the nth customer as well as the stationary waiting time in an $MR/GI/1$ queue and the stationary workload in a Markov-modulated queue are established, thus strengthening previous results while simplifying their derivation. This article is a continuation of those by Fang et al. [7] and Hu and Joe [10].

1. INTRODUCTION

Many dependence orderings have been developed to compare the strength of dependence between two random vectors, among which are "the multivariate concordance ordering" (see Joe [11]), "the supermodular dependence ordering" (see Shaked and Shanthikumar [23]), "the positive function dependence ordering" (also called "the more positive dependence ordering," see Rinott and Pollak [19] and Gleser and Moore [9]), and "the more regression dependence ordering" (see Yanagimoto and Okamoto [27], Schriever [21], and Fang and Joe [8]), and so forth. It may be intuitive that the dependence decreases with lag or time for a *stationary homogeneous*

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Markov process (SHMP). Fang et al. [7] and Hu and Joe [10] came up with conditions under which this is true with respect to the multivariate concordance ordering for discrete-time SHMPs, and the positive function dependence ordering for both discrete-time and continuous-time SHMPs, respectively. Joe [12, Thms. 8.7 and 8.8] considered the same problem with respect to the more regression dependence ordering for one discrete-time SHMP and also gave a result [12, Thm. 8.9] to compare two discrete-time SHMPs in the sense of the bivariate concordance ordering. Bäuerle [1,2] and Bäuerle and Rolski [5] compared two SHMPs, in both discrete and continuous time, with special transition probability matrices and intensity matrices.

The purpose of this article, which is a continuation of Fang et al. [7] and Hu and Joe [10], is to investigate conditions which quantify the decrease for an SHMP and also enable one to compare two SHMPs with respect to the supermodular dependence ordering or the concordance ordering. Such a study is meaningful for a couple of reasons. First, in applications to SHMP models, it is useful to know the conditions needed for the behavior of decrease in dependence with lag and to identify a dependence parameter for the model (i.e., to compare two SHMPs) in the sense of different dependence orderings. Second, the supermodular dependence ordering has recently been found to be of increasing interest in several areas of applied probability. See, for example, Meester and Shanthikumar [16], Szekli et al. [25], Bäuerle [1,2], Bäuerle and Rieder [4], Müller [17], Shaked and Shanthikumar [23], Bäuerle and Müller [3], Bäuerle and Rolski [5], and Müller and Scarsini [18] and references therein.

This article is organized as follows. Section 2 consists of definitions and notations. Results for discrete-time SHMPs and continuous-time SHMPs are given in Sections 3 and 4, respectively. Some applications of the main results are illustrated in Section 5. In queueing theory, the monotonicity results of the waiting time of the nth customer as well as the stationary waiting time in an $MR/GI/1$ queue, and the stationary workload in a Markov-modulated queue are established, thus strengthening previous results while simplifying their derivation.

2. PRELIMINARIES

A real-valued function $g:\Re^m \to \Re$ is said to be supermodular (also called L-superadditive) if $g(\mathbf{x}\vee\mathbf{y})+g(\mathbf{x}\wedge\mathbf{y}) \geq g(\mathbf{x})+g(\mathbf{y})$, $\forall \mathbf{x},\mathbf{y}\in\Re^m$. Here, $\vee$ and $\wedge$ denote respectively the componentwise maximum and the componentwise minimum. A function $g(x_1,\ldots,x_m)$ is supermodular if and only if $g(\ldots,x_i,\ldots,x_j,\ldots)$ is supermodular in (x_i,x_j) for any $i\neq j$, with the other variables held fixed. This follows from Kemperman [14]. In what follows, we denote by $\mathcal{L}_m$ the set of all m-dimensional Borel-measurable supermodular functions.

DEFINITION 2.1: *Let* $\mathbf{X}=(X_1,\ldots,X_m)$ *and* $\mathbf{Y}=(Y_1,\ldots,Y_m)$ *be two random vectors with the same univariate marginals.*

(a) $\mathbf{Y}$ *is said to be more concordant than* $\mathbf{X}$ (*denoted by* $\mathbf{Y}\geq_c \mathbf{X}$) *if*

$$\Pr(Y_1>x_1,\ldots,Y_m>x_m)\geq \Pr(X_1>x_1,\ldots,X_m>x_m)$$

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and

$$\Pr(Y_1 \le x_1, \ldots, Y_m \le x_m) \ge \Pr(X_1 \le x_1, \ldots, X_m \le x_m)$$

hold for all x_i, $i = 1, \ldots, m$ (*see Joe* [11]).

(b) $\mathbf{Y}$ *is said to be more supermodularly dependent than* $\mathbf{X}$ (*denoted by* $\mathbf{Y} \ge_{sm} \mathbf{X}$) *if* $E[\phi(\mathbf{Y})] \ge E[\phi(\mathbf{X})]$ *holds for all* $\phi \in \mathcal{L}_m$ *for which the expectations exist* (*see, for example, Shaked and Shanthikumar* [23]).

(c) *For two stochastic processes* $\{X(t), t \ge 0\}$ *and* $\{Y(t), t \ge 0\}$, *we write*

$$\{Y(t), t \ge 0\} \succ \{X(t), t \ge 0\}$$

if for all $0 \le t_1 \le t_2 \le \cdots \le t_m < \infty$, $m = 1, 2, \ldots$,

$$(Y(t_1), Y(t_2), \ldots, Y(t_m)) \succ (X(t_1), X(t_2), \ldots, X(t_m)),$$

where $\succ$ *is one of the orderings defined above.*

It is well known that the supermodular ordering implies the concordance ordering. Moreover, it is known since Tchen [26] that they are equivalent in the bivariate case. For dimension $m \ge 3$, Müller and Scarsini [18] has given an example, which shows that the supermodular ordering is strictly stronger than the concordance ordering.

The following definitions will be used in this article.

DEFINITION 2.2 (Shaked and Shanthikumar [22]): *Let* $\mathbf{X} = (X_1, \ldots, X_m)$ *and* $\mathbf{Y} = (Y_1, \ldots, Y_m)$ *be two random vectors, and let U and V be two random variables.*

(a) $\mathbf{X}$ *is smaller than* $\mathbf{Y}$ *in the usual stochastic order* (*denoted by* $\mathbf{X} \le_{\text{st}} \mathbf{Y}$) *if* $E[h(\mathbf{X})] \le E[h(\mathbf{Y})]$ *for all increasing functions h for which the expectations exist.*

(b) *U is smaller than V in the increasing convex order* (*denoted by* $U \le_{\text{icx}} V$) *if* $E[h(U)] \le E[h(V)]$ *for all increasing convex functions h for which the expectations exist.*

(c) $\mathbf{X}$ *is stochastically increasing in V* (*denoted by* $\mathbf{X} \uparrow_{\text{st}} V$) *if* $E[h(\mathbf{X})|V = v]$ *is increasing in v for all increasing functions h for which the conditional expectations exist.*

DEFINITION 2.3 (Daley [6], Keilson and Kester [13]):

(a) *A discrete-time homogeneous Markov process* $\{X_n, n \ge 1\}$ *is said to be stochastically monotone if* $X_2 \uparrow_{\text{st}} X_1$.

(b) *A continuous-time homogeneous Markov process* $\{X(t), t \ge 0\}$ *is said to be stochastically monotone if* $X(t) \uparrow_{\text{st}} X(0)$ *for all* $t \ge 0$.

(c) *If the state space of the Markov process in* (a) [(b)] *is denumerable, then the transition probability matrix* [*function*] *is also said to be stochastically monotone.*

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Keilson and Kester [13] studied monotone Markov processes with discrete state spaces, in both discrete and continuous time. They also gave several examples of monotone Markov processes arising in applications. Moreover, they studied the algebraic properties of monotone transition matrices. For a thorough treatment of stochastic monotonicity, one can refer to Stoyan [24].

Some notations needed for subsequent sections are as follows. For an integer $m \geq 1$ and $\mathbf{r} = (r_1, r_2, \ldots, r_m) \in \mathcal{N}^m \equiv \{0,1,2,\ldots\}^m$ or $\Re_+^m \equiv [0,\infty)^m$, let $\mathbf{R} = (R_1, R_2, \ldots, R_m)$ be defined with $R_j = \sum_{i=1}^{j} r_i$, $j = 1, \ldots, m$. Note that $R_j = R_{j-1}$ if $r_j = 0$. For two vectors $\mathbf{r}, \mathbf{s} \in \mathcal{N}^m$ or $\Re_+^m$, $\mathbf{r} \geq \mathbf{s}$ means $r_i \geq s_i$ for $i = 1, \ldots, m$. We denote by $\mathbf{1}$ the identity vector in $\mathcal{N}^m$.

Throughout, the terms "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing," respectively. All expectations are implicitly assumed to exist whenever they appear. Let $[W|B]$ denote any random variable that has as its distribution the conditional distribution of W given event B.

3. DEPENDENCE FOR DISCRETE-TIME SHMPs

A discrete-time SHMP $\{X_n, n \geq 1\}$ can be constructed from a given bivariate cumulative distribution function (cdf) of (X_1, X_2) with a common univariate cdf. One property of each SHMP to be used often is that, for all $n \geq 2$ and $m \geq 1$, $\{X_i, i \leq m\}$ and $\{X_j, j \geq m + n\}$ are independent given any nonempty subset of $\{X_j, j = m + 1, \ldots, n + m - 1\}$.

3.1. The Decrease in Dependence with Lag for an SHMP

In this subsection, we will give conditions which quantify the decrease in the supermodular dependence with lag for an SHMP.

THEOREM 3.1: *Let $\{X_n, n \geq 1\}$ be an SHMP with state space $\mathcal{E}$ being the subset of the real line, and let $m > 1$ be an integer. If $X_1 \uparrow_{\text{st}} X_2$ and $X_2 \uparrow_{\text{st}} X_1$, then*

$$(X_{S_1}, X_{S_2}, \ldots, X_{S_m}) \geq_{\text{sm}} (X_{R_1}, X_{R_2}, \ldots, X_{R_m}) \tag{3.1}$$

for all $\mathbf{r}, \mathbf{s} \in \mathcal{N}^m$ with $\mathbf{r} \geq \mathbf{s} \geq \mathbf{1}$.

PROOF: First, consider $m = 2$. Because the bivariate concordance ordering is equivalent to the bivariate supermodular ordering, the result follows from Theorem 3.1 of Fang et al. [7].

Now, we proceed with a proof by induction. Let $k \geq 3$ and suppose that Eq. (3.1) holds for any $m < k$. By induction, it suffices to show that Eq. (3.1) holds for $m = k$. Because the supermodular ordering is transitive, it suffices to show the conclusion for $\mathbf{r}, \mathbf{s} \in \mathcal{N}^k$ with the first component equal to 1, which differ by 1 in one coordinate only, say the pth coordinate, so that $r_p = s_p + 1$.

Let $\phi \in \mathcal{L}_k$. If $p < k$, then define

$$\varphi_1(x_1, \ldots, x_p) = E[\phi(x_1, \ldots, x_p, X_{S_{p+1}}, \ldots, X_{S_k}) | X_{S_p} = x_p].$$

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It is easy to check that $\varphi_1(x_1,\ldots,x_p)$ is supermodular in (x_i,x_j) for all $1\le i<j<p$. On the other hand, the supermodularity of ϕ implies that $\phi(x_i',x_j;\mathbf{x}^{(i,j)})-\phi(x_i,x_j;\mathbf{x}^{(i,j)})$ is increasing in x_j for fixed $\mathbf{x}^{(i,j)}$, $i\neq j$, and $x_i<x_i'$. Here, we use the notation $\mathbf{x}^{(i,j)}$ is the vector $\mathbf{x}$ with ith and jth components deleted and $\phi(x_i',x_j;\mathbf{x}^{(i,j)})$ is the value of function ϕ at point $\mathbf{x}$ with ith and jth components being x_i' and x_j, respectively. By Theorem 3.2(a) of Fang et al. [7], $(X_{T_1},X_{T_2},\ldots,X_{T_l})\uparrow_{\mathrm{st}} X_1$ for any $\mathbf{t}\in\mathcal{N}^l$, $l\ge 2$ and, hence, we conclude that, for $x_i<x_i'$, $x_p<x_p'$, and $i<p$,

$$\begin{aligned}
&E[\phi(\ldots,x_i',\ldots,x_p,X_{p+1},\ldots,X_k)-\phi(\ldots,x_i,\ldots,x_p,X_{p+1},\ldots,X_k)|X_{S_p}=x_p]\\
&\quad\le E[\phi(\ldots,x_i',\ldots,x_p,X_{p+1},\ldots,X_k)-\phi(\ldots,x_i,\ldots,x_p,X_{p+1},\ldots,X_k)|X_{S_p}=x_p']\\
&\quad\le E[\phi(\ldots,x_i',\ldots,x_p',X_{p+1},\ldots,X_k)-\phi(\ldots,x_i,\ldots,x_p',X_{p+1},\ldots,X_k)|X_{S_p}=x_p'];
\end{aligned}$$

that is, for $x_i<x_i'$, $x_p<x_p'$, and $i<p$,

$$\varphi_1(\ldots,x_i',\ldots,x_p')+\varphi_1(\ldots,x_i,\ldots,x_p)\ge\varphi_1(\ldots,x_i',\ldots,x_p)+\varphi_1(\ldots,x_i,\ldots,x_p').$$

Then, $\varphi_1(x_1,\ldots,x_p)$ is supermodular in (x_i,x_p) for $i<p$ and, hence, φ_1 is in $\mathcal{L}_p$. By the induction hypotheses,

$$\begin{aligned}
E[\phi(X_{S_1},X_{S_2},\ldots,X_{S_k})]&=E\{E[\phi(X_{S_1},X_{S_2},\ldots,X_{S_k})|X_{S_1},\ldots,X_{S_p}]\}\\
&=E[\varphi_1(X_{S_1},X_{S_2},\ldots,X_{S_p})]\\
&\ge E[\varphi_1(X_{R_1},X_{R_2},\ldots,X_{R_p})]\\
&=E[\phi(X_{R_1},X_{R_2},\ldots,X_{R_k})]. \qquad \textbf{(3.2)}
\end{aligned}$$

If $p=k$, a similar argument to the above, which makes use of the assumption $X_1\uparrow_{\mathrm{st}} X_2$, yields that

$$\varphi_2(x_2,\ldots,x_k)=E[\phi(X_1,x_2,\ldots,x_k)|X_{S_2}=x_2]$$

is also in $\mathcal{L}_k$, so by the induction hypotheses,

$$\begin{aligned}
E[\phi(X_{S_1},X_{S_2},\ldots,X_{S_k})]&=E[\varphi_2(X_{S_2},\ldots,X_{S_k})]\\
&\ge E[\varphi_2(X_{R_2},\ldots,X_{R_k})]\\
&=E[\phi(X_{R_1},X_{R_2},\ldots,X_{R_k})]. \qquad \textbf{(3.3)}
\end{aligned}$$

Conclusion (3.1) now follows from Eqs. (3.2) and (3.3). ■

3.2. Comparisons of Dependence for Two SHMPs

This subsection consists of the comparison of dependence of two SHMPs. Because the properties of an SHMP $\{X_n,n\ge 1\}$ are determined by the joint cdf of (X_1,X_2), to compare dependence of two SHMPs $\{X_n,n\ge 1\}$ and $\{Y_n,n\ge 1\}$, we might hope that the conditions are imposed on the cdf's of (X_1,X_2) and (Y_1,Y_2).

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THEOREM 3.2: *Let* $\{X_n, n \geq 1\}$ *and* $\{Y_n, n \geq 1\}$ *be two SHMPs with a common univariate marginal distribution. Assume that* $X_1 \uparrow_{\text{st}} X_2$ *and* $Y_2 \uparrow_{\text{st}} Y_1$ *(or* $X_2 \uparrow_{\text{st}} X_1$ *and* $Y_1 \uparrow_{\text{st}} Y_2$*). If*

$$(Y_1, Y_2) \geq_c (X_1, X_2), \tag{3.4}$$

then

$$(Y_1, Y_2, \ldots, Y_n) \geq_{\text{sm}} (X_1, X_2, \ldots, X_n), \quad \forall n \geq 2 \tag{3.5}$$

and, hence,

$$(Y_1, Y_2, \ldots, Y_n) \geq_c (X_1, X_2, \ldots, X_n), \quad \forall n \geq 2. \tag{3.6}$$

PROOF: Because the supermodular ordering implies the concordance ordering, to prove Eq. (3.6), it suffices to prove Eq. (3.5). First, assume that $X_1 \uparrow_{\text{st}} X_2$ and $Y_2 \uparrow_{\text{st}} Y_1$. For $n = 2$, Eq. (3.5) is trivial. We now proceed with a proof by induction. Let $m \geq 3$ and suppose that Eq. (3.5) holds for any $n < m$. Let $f(x)$ be the probability density function (pdf) of X_1 and $f_{j|i}(\cdot|\cdot)[g_{j|i}(\cdot|\cdot)]$ be the conditional pdf of X_j given X_i [Y_j given Y_i] whenever all densities exist. We now construct m "bridge" random vectors $\mathbf{Z}^{(j)} = (Z_{j,1}, Z_{j,2}, \ldots, Z_{j,m})$, $j = 0,1,\ldots,m-1$, such that $\mathbf{Z}^{(j)}$ has the joint pdf

$$f(z_1) f_{2|1}(z_2|z_1) \cdots f_{2|1}(z_{m-j}|z_{m-j-1}) g_{2|1}(z_{m-j+1}|z_{m-j}) \cdots g_{2|1}(z_m|z_{m-1}). \tag{3.7}$$

If all densities do not exist, we can also construct m "bridge" random vectors by considering the conditional distributions in place of the conditional pdf's. It is clear that

$$\mathbf{Z}^{(0)} \stackrel{\text{st}}{=} (X_1, X_2, \ldots, X_m) \quad \text{and} \quad \mathbf{Z}^{(m-1)} \stackrel{\text{st}}{=} (Y_1, Y_2, \ldots, Y_m),$$

where $\stackrel{\text{st}}{=}$ means "equal in distribution." By induction, to prove Eq. (3.5), it suffices to show that Eq. (3.5) holds for $n = m$ and, hence, it suffices to show that

$$\mathbf{Z}^{(p+1)} \geq_{\text{sm}} \mathbf{Z}^{(p)} \quad \text{for all } p = 0, \ldots, m-2,$$

because the supermodular ordering is transitive.

Let $\phi \in \mathcal{L}_m$. For $p = 0,1,\ldots,m-2$, define

$$\varphi_1(x_1, \ldots, x_{m-p}) = E[\phi(x_1, \ldots, x_{m-p}, Z_{p,m-p+1}, \ldots, Z_{p,m}) | Z_{p,m-p} = x_{m-p}].$$

Then,

$$\begin{aligned} E[\phi(\mathbf{Z}^{(p)})] &= E\{E[\phi(\mathbf{Z}^{(p)}) | Z_{p,1}, \ldots, Z_{p,m-p}]\} \\ &= E[\varphi_1(Z_{p,1}, \ldots, Z_{p,m-p})] \\ &= E\{E[\varphi_1(Z_{p,1}, \ldots, Z_{p,m-p}) | Z_{p,m-p-1}, Z_{p,m-p}]\} \\ &= E[\varphi_2(Z_{p,m-p-1}, Z_{p,m-p})], \end{aligned} \tag{3.8}$$

where

$$\varphi_2(x_{m-p-1}, x_{m-p}) = E[\varphi_1(Z_{p,1}, \ldots, Z_{p,m-p-2}, x_{m-p-1}, x_{m-p}) | Z_{p,m-p-1} = x_{m-p-1}].$$

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Similarly,

$$E[\phi(\mathbf{Z}^{(p+1)})] = E[\varphi_2(Z_{p+1,m-p-1}, Z_{p+1,m-p})]. \tag{3.9}$$

A similar argument to that in the proof of Theorem 3.1, which makes use of the assumption $Y_2 \uparrow_{\text{st}} Y_1$, yields $\varphi_1 \in \mathcal{L}_{m-p}$ and, hence, $\varphi_2 \in \mathcal{L}_2$ because $X_1 \uparrow_{\text{st}} X_2$. It follows from Eq. (3.7) that

$$(Z_{p,m-p-1}, Z_{p,m-p}) \overset{\text{st}}{=} (X_1, X_2) \quad \text{and} \quad (Z_{p+1,m-p-1}, Z_{p+1,m-p}) \overset{\text{st}}{=} (Y_1, Y_2).$$

From Eqs. (3.4), (3.8), and (3.9), we thus conclude that

$$\begin{aligned} E[\phi(\mathbf{Z}^{(p)})] &= E[\varphi_2(Z_{p,m-p-1}, Z_{p,m-p})] \\ &\le E[\varphi_2(Z_{p+1,m-p-1}, Z_{p+1,m-p})] \\ &= E[\phi(\mathbf{Z}^{(p+1)})]; \end{aligned}$$

that is, $\mathbf{Z}^{(p+1)} \ge_{\text{sm}} \mathbf{Z}^{(p)}$.

Next, assume that $X_2 \uparrow_{\text{st}} X_1$ and $Y_1 \uparrow_{\text{st}} Y_2$. Equation (3.5) can be dealt with in a manner similar to that in the preceding paragraph by showing that

$$\mathbf{W}^{(p)} \ge_{\text{sm}} \mathbf{W}^{(p+1)} \quad \text{for all } p = 0, \ldots, m-2,$$

where the joint pdf of $\mathbf{W}^{(j)}$ is given by

$$f(w_1) g_{2|1}(w_2|w_1) \cdots g_{2|1}(w_{m-j}|w_{m-j-1}) f_{2|1}(w_{m-j+1}|w_{m-j}) \cdots f_{2|1}(w_m|w_{m-1}).$$

This completes the proof. ■

Remark 3.1: For $n = 2$, Conclusion (3.6) was obtained by Joe [12, Thm. 8.9], but our proofs are quite different. Because the class of supermodular functions is broad, Eq. (3.5) for the supermodular ordering is sometimes more useful than Eq. (3.6) for the concordance ordering and can be used to get many interesting and important inequalities.

Lemma 1 of Bäuerle and Rolski [5] is a direct corollary of Theorem 3.2 (see Corollary 3.1 below). Consider two SHMPs $\{X_n, n \ge 1\}$ and $\{Y_n, n \ge 1\}$, both with stationary distribution $\pi = (\pi_0, \pi_1, \ldots, \pi_\tau)$ on a denumerable state space $\mathcal{E} \equiv \{0, 1, \ldots, \tau\}$ and one-step transition probability matrices $\mathbf{P}$ and $\tilde{\mathbf{P}}$, where τ is a positive integer or infinity. In what follows, we denote by $\mathbf{\Pi}$ the diagonal matrix $\text{diag}(\pi_0, \pi_1, \ldots, \pi_\tau)$, and by $\mathbf{I}$ the identity matrix.

COROLLARY 3.1: *If $X_1 \uparrow_{\text{st}} X_2$ and $X_2 \uparrow_{\text{st}} X_1$, and if $\tilde{\mathbf{P}} = (1-c)\mathbf{I} + c\mathbf{P}$, $0 \le c \le 1$, then*

$$(Y_1, Y_2, \ldots, Y_n) \ge_{\text{sm}} (X_1, X_2, \ldots, X_n) \quad \textit{for all } n \ge 2.$$

PROOF: First note that Eq. (3.4) is equivalent to

$$\sum_{k=i}^{\tau} \sum_{l=j}^{\tau} (\mathbf{\Pi}\tilde{\mathbf{P}})_{kl} \ge \sum_{k=i}^{\tau} \sum_{l=j}^{\tau} (\mathbf{\Pi}\mathbf{P})_{kj} \quad \text{for all } i, j \in \mathcal{E}, \tag{3.10}$$

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where $(\mathbf{\Pi P})_{kj}$ is the (k,j)th entry of matrix $\mathbf{\Pi P}$. It is easy to check that, for $i < j$,

$$\begin{aligned}\sum_{k=i}^{\tau}\sum_{l=j}^{\tau}(\mathbf{\Pi}\tilde{\mathbf{P}})_{kl} &= c\sum_{k=i}^{\tau}\sum_{l=j}^{\tau}\pi_k P_{kl} + (1-c)\sum_{l=j}^{\tau}\pi_l \\ &= c\sum_{l=j}^{\tau}\sum_{k=i}^{\tau}\pi_k P_{kl} - c\sum_{l=j}^{\tau}\sum_{k=0}^{\tau}\pi_k P_{kl} + \sum_{l=j}^{\tau}\pi_l \\ &= -c\sum_{l=j}^{\tau}\sum_{k=0}^{i-1}\pi_k P_{kl} + \sum_{l=j}^{\tau}\pi_l,\end{aligned}$$

and, for $i > j$,

$$\begin{aligned}\sum_{k=i}^{\tau}\sum_{l=j}^{\tau}(\mathbf{\Pi}\tilde{\mathbf{P}})_{kl} &= 1 - \sum_{k=0}^{i-1}\pi_k - \sum_{l=0}^{j-1}\pi_l + \sum_{k=0}^{i-1}\sum_{l=0}^{j-1}(\mathbf{\Pi}\tilde{\mathbf{P}})_{kl} \\ &= 1 - \sum_{k=0}^{i-1}\pi_k + c\sum_{k=0}^{i-1}\sum_{l=0}^{j-1}\pi_k P_{kl} - c\sum_{l=0}^{j-1}\pi_l \\ &= \sum_{k=i}^{\tau}\pi_k - c\sum_{l=0}^{j-1}\sum_{k=i}^{\tau}\pi_k P_{kl}.\end{aligned}$$

Hence, Eq. (3.10) is true. Moreover, the assumptions $X_1 \uparrow_{\mathrm{st}} X_2$ and $X_2 \uparrow_{\mathrm{st}} X_1$ imply that $Y_1 \uparrow_{\mathrm{st}} Y_2$ and $Y_2 \uparrow_{\mathrm{st}} Y_1$. Therefore, Theorem 3.2 can be applied. ■

4. DEPENDENCE FOR CONTINUOUS-TIME SHMPs

Consider now a continuous-time SHMP $\{X(t), t \geq 0\}$ on a denumerable state space $\mathcal{E} \subset \{0,1,2,\ldots\}$ with transition probability matrix function $\mathbf{P}(t) = (P_{ij}(t))$ (where $P_{ij}(t) = \Pr(X(t) = j \,|\, X(0) = i)$) and intensity matrix $\mathbf{Q} = \lim_{t\to 0}((\mathbf{P}(t) - \mathbf{I})/t)$, where we make use of the usual assumption that $\mathbf{P}(t) \to \mathbf{I}$ as $t \to 0$. The entries of $\mathbf{Q}$ will be denoted by q_{ij}. All SHMPs $\{X(t), t \geq 0\}$ appearing in this article are assumed to be *uniformizable* (i.e.,

$$\nu^* := \sup_i |q_{ii}| < \infty) \tag{4.1}$$

and to have infinite state space $\mathcal{E} = \{0,1,2,\ldots\}$. The results below are also valid for finite SHMPs.

Uniformizable SHMPs have the advantage that their behavior can be studied in the following discrete-time setting (see Stoyan [24]). For any $\nu > 2\nu^*$, there exists a discrete-time SHMP $\{N_k, k \geq 0\}$ with one-step transition probability matrix

$$\mathbf{P}_\nu = \mathbf{I} + \frac{1}{\nu}\mathbf{Q} \tag{4.2}$$

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and initial distribution π, the same as that of $\{X(t), t \geq 0\}$, such that

$$X(t) = N_{\kappa_\nu(t)} \quad \text{for all } t \geq 0, \tag{4.3}$$

where $\{\kappa_\nu(t)\}$ is a homogeneous Poisson process with parameter ν, independent of $\{N_k, k \geq 0\}$. Then the transition probability matrix of $\{X(t), t \geq 0\}$ is given by

$$\mathbf{P}(t) = \sum_{k=0}^{\infty} e^{-\nu t} \frac{(\nu t)^k}{k!} \mathbf{P}_\nu^k = \exp\{-\nu t[\mathbf{I} - \mathbf{P}_\nu]\} \quad \text{for all } t \geq 0. \tag{4.4}$$

In order to prove the results below, we need the following lemmas which give necessary and sufficient conditions for transition probability matrices and intensity matrices to be stochastically monotone. First, define the infinite matrix

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot \\ 1 & 1 & 0 & 0 & \cdot \\ 1 & 1 & 1 & 0 & \cdot \\ 1 & 1 & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

It is easy to see that

$$\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot \\ -1 & 1 & 0 & 0 & \cdot \\ 0 & -1 & 1 & 0 & \cdot \\ 0 & 0 & -1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

LEMMA 4.1 (Keilson and Kester [13], Lindqvist [15]):

(a) $\mathbf{P}_\nu$ *is stochastically monotone if and only if* $(\mathbf{T}^{-1}\mathbf{P}_\nu\mathbf{T})_{ij} \geq 0$ *for all* i *and* j.

(b) $\mathbf{Q}$ *is stochastically monotone if and only if* $(\mathbf{T}^{-1}\mathbf{Q}\mathbf{T})_{ij} \geq 0$ *for all* $i \neq j$, *or, equivalently*,

$$\sum_{k=l}^{\infty} q_{ik} \leq \sum_{k=l}^{\infty} q_{jk} \quad \textit{for all } i < j < l$$

and

$$\sum_{k=0}^{l} q_{ik} \geq \sum_{k=0}^{l} q_{jk} \quad \textit{for all } l < i < j.$$

Let $\mathbf{P}_\nu^R$ denote the one-step transition probability matrix of the time-reverse SHMP of $\{N_k, k \geq 0\}$, and let $\mathbf{Q}^R$ denote the intensity matrix of the time-reverse SHMP of $\{X(t), t \geq 0\}$.

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LEMMA 4.2:

(a) *If* $\mathbf{Q}$ *is stochastically monotone, then* $\mathbf{P}_\nu$ *is also stochastically monotone.*
(b) *If* $\mathbf{Q}^R$ *is stochastically monotone, then* $\mathbf{P}_\nu^R$ *is also stochastically monotone.*

PROOF: It is easy to see that

$$\mathbf{Q}^R = \mathbf{\Pi}^{-1}\mathbf{Q}^\tau\mathbf{\Pi} \quad \text{and} \quad \mathbf{P}_\nu^R = \mathbf{\Pi}^{-1}\mathbf{P}_\nu^\tau\mathbf{\Pi} = \mathbf{I} + \frac{1}{\nu}\mathbf{Q}^R,$$

where $\mathbf{Q}^\tau$ is the transpose of $\mathbf{Q}$. By Lemma 4.1, to prove (a), it suffices to verify that $(\mathbf{T}^{-1}\mathbf{P}_\nu\mathbf{T})_{ii} \geq 0$ for all i. In fact,

$$\begin{aligned}(\mathbf{T}^{-1}\mathbf{P}_\nu\mathbf{T})_{ii} &= 1 + \frac{1}{\nu}(\mathbf{T}^{-1}\mathbf{Q}\mathbf{T})_{ii} \\ &= 1 + \frac{1}{\nu}[(\mathbf{Q}\mathbf{T})_{ii} - (\mathbf{Q}\mathbf{T})_{i-1,i}] \\ &= 1 + \frac{1}{\nu}\sum_{j=i}^{\infty}(q_{ij} - q_{i-1,j}) \geq 1 + \frac{1}{\nu}[q_{i-1,i-1} + q_{ii}] \geq 1 - \frac{2\nu^*}{\nu} > 0.\end{aligned}$$

The proof for (b) is similar. ■

4.1. The Decrease in Dependence with Lag for an SHMP

THEOREM 4.1: *Let* $\{X(t), t \geq 0\}$ *be an SHMP with intensity matrix* $\mathbf{Q}$. *If* $\mathbf{Q}$ *is stochastically monotone, then*

$$(X(S_1), X(S_2), \ldots, X(S_m)) \geq_c (X(R_1), X(R_2), \ldots, X(R_m)) \tag{4.5}$$

for all $\mathbf{r}, \mathbf{s} \in \Re_+^m$ *with* $\mathbf{r} \geq \mathbf{s}$ *and* $m \geq 2$.

PROOF: For given $\mathbf{r}, \mathbf{s} \in \Re_+^m$ with $\mathbf{r} \geq \mathbf{s}$, define independent random variables $\{W_i, U_i, i = 1, \ldots, m\}$ with Poisson distributions as follows:

$$W_i \sim \text{Poisson}(s_i\nu), \qquad U_i \sim \text{Poisson}((r_i - s_i)\nu), \quad i = 1, \ldots, m,$$

which are also independent of $\{N_k, k \geq 0\}$. It is easy to see that

$$(\kappa_\nu(S_1), \ldots, \kappa_\nu(S_m)) \stackrel{\text{st}}{=} \left(W_1, W_1 + W_2, \ldots, \sum_{k=1}^{m} W_k\right) \stackrel{\text{def}}{=} (W_1^*, \ldots, W_m^*)$$

and

$$(\kappa_\nu(R_1), \ldots, \kappa_\nu(R_m)) \stackrel{\text{st}}{=} \left(W_1 + U_1, \ldots, \sum_{k=1}^{m}(W_k + U_k)\right) \stackrel{\text{def}}{=} (W_1^{**}, \ldots, W_m^{**}).$$

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It follows from Eq. (4.3) that, for every $x_1, \ldots, x_m$,

$$\begin{aligned}
&\Pr(X(R_i) > x_i, i = 1, \ldots, m) \\
&\quad = \Pr(N_{\kappa_\nu(R_i)} > x_i, i = 1, \ldots, m) \\
&\quad = \Pr(N_{W_i^{**}} > x_i, i = 1, \ldots, m) \\
&\quad = E[\Pr(N_{W_i^{**}} > x_i, i = 1, \ldots, m | W_1^{**}, \ldots, W_m^{**})] \\
&\quad = E\{E[\Pr(N_{W_i^{**}} > x_i, i = 1, \ldots, m | W_1^{**}, \ldots, W_m^{**}; W_1^{*}, \ldots, W_m^{*}) | W_1^{**}, \ldots, W_m^{**}]\} \\
&\quad \le E\{E[\Pr(N_{W_i^{*}} > x_i, i = 1, \ldots, m | W_1^{**}, \ldots, W_m^{**}; W_1^{*}, \ldots, W_m^{*}) | W_1^{**}, \ldots, W_m^{**}]\} \\
&\quad = E[\Pr(N_{W_i^{*}} > x_i, i = 1, \ldots, m | W_1^{**}, \ldots, W_m^{**})] \\
&\quad = \Pr(N_{W_i^{*}} > x_i, i = 1, \ldots, m) \\
&\quad = \Pr(X(S_i) > x_i, i = 1, \ldots, m),
\end{aligned}$$

where the inequality follows from Theorem 3.2(b) of Fang et al. [7] and Lemma 4.2(a). Similarly,

$$\Pr(X(R_i) \le x_i, i = 1, \ldots, m) \le \Pr(X(S_i) \le x_i, i = 1, \ldots, m).$$

This completes the proof. ■

In Theorem 4.1, if the time-reverse SHMP of $\{X(t), t \ge 0\}$ is also stochastically monotone, we can get a stronger result; this is shown in Theorem 4.2.

THEOREM 4.2: *Let $\{X(t), t \ge 0\}$ be an SHMP with intensity matrix $\mathbf{Q}$. If $\mathbf{Q}$ and $\mathbf{Q}^R$ are both stochastically monotone, then*

$$(X(S_1), X(S_2), \ldots, X(S_m)) \ge_{\text{sm}} (X(R_1), X(R_2), \ldots, X(R_m)) \tag{4.6}$$

for all $\mathbf{r}, \mathbf{s} \in \Re_+^m$ with $\mathbf{r} \ge \mathbf{s}$ and $m \ge 2$.

PROOF: By Lemma 4.2, the monotonicity of $\mathbf{Q}$ *and* $\mathbf{Q}^R$ implies that $\mathbf{P}_\nu$ and $\mathbf{P}_\nu^R$ are also stochastically monotone for all $\nu > 2\nu^*$. Let $\phi \in \mathcal{L}_m$ and $\mathbf{r}, \mathbf{s} \in \Re_+^m$ with $\mathbf{r} \ge \mathbf{s}$. Using the same notations as those in the proof of Theorem 4.1, we have

$$\begin{aligned}
&E[\phi(X(R_1), \ldots, X(R_m))] \\
&\quad = E\{E[E(\phi(N_{W_1^{**}}, \ldots, N_{W_m^{**}}) | W_1^{*}, \ldots, W_m^{*}; W_1^{**}, \ldots, W_m^{**}) | W_1^{**}, \ldots, W_m^{**}]\} \\
&\quad \le E\{E[E(\phi(N_{W_1^{*}}, \ldots, N_{W_m^{*}}) | W_1^{*}, \ldots, W_m^{*}; W_1^{**}, \ldots, W_m^{**}) | W_1^{**}, \ldots, W_m^{**}]\} \\
&\quad = E\{E[\phi(N_{W_1^{*}}, \ldots, N_{W_m^{*}}) | W_1^{**}, \ldots, W_m^{**}]\} \\
&\quad = E[\phi(X(S_1), \ldots, X(S_m))],
\end{aligned}$$

where the inequality follows from Theorem 3.1. This completes the proof. ■

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4.2. Comparisons of Dependence for Two SHMPs

THEOREM 4.3: *Let* $\{X(t), t \geq 0\}$ *and* $\{Y(t), t \geq 0\}$ *be two SHMPs with initial distribution* π *and intensity matrices* $\mathbf{Q} = (q_{ij})$ *and* $\tilde{\mathbf{Q}} = (\tilde{q}_{ij})$, *respectively. Assume that* $\mathbf{Q}$ *and* $\tilde{\mathbf{Q}}^R$ *are stochastically monotone, or* $\mathbf{Q}^R$ *and* $\tilde{\mathbf{Q}}$ *are stochastically monotone. Then,*

$$\sum_{k=i}^{\infty}\sum_{l=j}^{\infty}\pi_k q_{kl} \leq \sum_{k=i}^{\infty}\sum_{l=j}^{\infty}\pi_k \tilde{q}_{kl} \quad \text{for all } i \text{ and } j, \tag{4.7}$$

if and only if

$$\{Y(t), t \geq 0\} \geq_{\text{sm}} \{X(t), t \geq 0\}. \tag{4.8}$$

PROOF: First, note that $\{Y(t), t \geq 0\}$ have the following stochastic representation:

$$Y(t) = \tilde{N}_{\kappa_\nu(t)} \quad \text{for all } t \geq 0, \tag{4.9}$$

where $\{\tilde{N}_k, k \geq 0\}$ is a discrete-time SHMP with initial distribution π and one-step transition probability matrix

$$\tilde{\mathbf{P}}_\nu = \mathbf{I} + \frac{1}{\nu}\tilde{\mathbf{Q}}, \tag{4.10}$$

independent of $\{\kappa_\nu(t), t \geq 0\}$ as defined in Eq. (4.3). Choose

$$\nu > 2\max\left\{\sup_i |q_{ii}|, \sup_i |\tilde{q}_{ii}|\right\}.$$

By Lemma 4.2, the monotonicity of $\mathbf{Q}$ and $\tilde{\mathbf{Q}}^R$ [resp. $\mathbf{Q}^R$ and $\tilde{\mathbf{Q}}$] implies that $\mathbf{P}_\nu$ and $\tilde{\mathbf{P}}_\nu^R$ [resp. $\mathbf{P}_\nu^R$ and $\tilde{\mathbf{P}}_\nu$] are both stochastically monotone.

Necessity. Assume that Eq. (4.7) is true. Note that Eq. (4.7) is equivalent to

$$(\tilde{N}_0, \tilde{N}_1) \geq_c (N_0, N_1).$$

Applying Theorem 3.2 to SHMPs $\{N_k, k \geq 0\}$ and $\{\tilde{N}_k, k \geq 0\}$ yields that

$$(\tilde{N}_0, \tilde{N}_1, \ldots, \tilde{N}_n) \geq_{\text{sm}} (N_0, N_1, \ldots, N_n) \quad \text{for all } n \geq 1$$

and, hence,

$$(\tilde{N}_{k_1}, \tilde{N}_{k_2}, \ldots, \tilde{N}_{k_m}) \geq_{\text{sm}} (N_{k_1}, N_{k_2}, \ldots, N_{k_m}) \quad \text{for all } 0 \leq k_1 \leq k_2 \leq \cdots \leq k_m < \infty.$$

It then follows from Eqs. (4.3) and (4.9) that, for each $\phi \in \mathcal{L}_m$ and all $0 \leq t_1 \leq t_2 \leq \cdots \leq t_m < \infty$, $m \geq 2$,

$$\begin{aligned}
E[\phi(X(t_1), \ldots, X(t_m))] &= E[\phi(N_{\kappa_\nu(t_1)}, \ldots, N_{\kappa_\nu(t_m)})] \\
&= E\{E[\phi(N_{\kappa_\nu(t_1)}, \ldots, N_{\kappa_\nu(t_m)}) | \kappa_\nu(t_1), \ldots, \kappa_\nu(t_m)]\} \\
&\leq E\{E[\phi(\tilde{N}_{\kappa_\nu(t_1)}, \ldots, \tilde{N}_{\kappa_\nu(t_m)}) | \kappa_\nu(t_1), \ldots, \kappa_\nu(t_m)]\} \\
&= E[\phi(Y(t_1), \ldots, Y(t_m))],
\end{aligned}$$

which implies that Eq. (4.8) is true.

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Sufficiency. Assume that Eq. (4.8) is true. Then, $(Y(0), Y(t)) \geq_c (X(0), X(t))$ for all $t > 0$; that is,

$$\sum_{k=i}^{\infty}\sum_{l=j}^{\infty} \pi_k \tilde{P}_{kl}(t) \geq \sum_{k=i}^{\infty}\sum_{l=j}^{\infty} \pi_k P_{kl}(t) \quad \text{for all } i \text{ and } j$$

and, hence,

$$\sum_{k=i}^{\infty}\sum_{l=j}^{\infty} \pi_k \frac{\tilde{P}_{kl}(t) - \mathbf{I}_{kl}}{t} \geq \sum_{k=i}^{\infty}\sum_{l=j}^{\infty} \pi_k \frac{P_{kl}(t) - \mathbf{I}_{kl}}{t} \quad \text{for all } i \text{ and } j.$$

Letting $t \to 0$ yields Eq. (4.7). This completes the proof. ■

Remark 4.1: From stochastic representations (4.3) and (4.9), we know that the properties of $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ are determined by the underlying SHMPs $\{N_k, k \geq 0\}$ and $\{\tilde{N}_k, k \geq 0\}$. In order to compare the supermodular dependence structure between $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$, it is natural to impose the condition (4.7) on (N_0, N_1) and $(\tilde{N}_0, \tilde{N}_1)$. Equation (4.7) is also equivalent to

$$\sum_{k=0}^{i}\sum_{l=0}^{j} \pi_k q_{kl} \leq \sum_{k=0}^{i}\sum_{l=0}^{j} \pi_k \tilde{q}_{kl} \quad \text{for all } i \text{ and } j. \tag{4.11}$$

An immediate consequence of Theorem 4.3 is the following corollary, due to Bäuerle and Rolski [5].

COROLLARY 4.1: *Let* $\{X(t), t \geq 0\}$ *and* $\{Y(t), t \geq 0\}$ *be two SHMPs, both with initial distribution* π *and intensity matrices* $\mathbf{Q}$ *and* $\tilde{\mathbf{Q}} = c\mathbf{Q}$, *respectively, where* $0 < c \leq 1$. *If* $\mathbf{Q}$ *and* $\mathbf{Q}^R$ *are stochastically monotone, then*

$$\{Y(t), t \geq 0\} \geq_{\text{sm}} \{X(t), t \geq 0\}.$$

PROOF: It is easy to check that, for each $\nu > 2 \sup_i |q_{ii}|$,

$$\mathbf{P}_\nu = 1 + \frac{1}{\nu}\mathbf{Q} \quad \text{and} \quad \tilde{\mathbf{P}}_\nu = 1 + \frac{c}{\nu}\mathbf{Q} = (1-c)\mathbf{I} + c\mathbf{P}_\nu$$

are both stochastic matrices with stationary distribution π. By Lemma 4.2, the monotonicity of $\mathbf{Q}$ and $\mathbf{Q}^R$ implies that $\mathbf{P}$ and $\mathbf{P}^R$ are both stochastically monotone. By Corollary 3.1, we have Eq. (3.10) and, hence,

$$\sum_{k=i}^{\infty}\sum_{l=j}^{\infty} (\mathbf{\Pi Q})_{kl} \leq \sum_{k=i}^{\infty}\sum_{l=j}^{\infty} (\mathbf{\Pi}\tilde{\mathbf{Q}})_{kl} \quad \text{for all } i \text{ and } j.$$

Therefore, Theorem 4.3 can be applied. ■

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5. APPLICATIONS

5.1. Birth–Death Processes

Let $\{X(t), t \geq 0\}$ be a homogeneous birth–death process with a denumerable state space $\mathcal{E} = \{n_1, n_1 + 1, \ldots, n_2\}$ and intensity matrix $\mathbf{Q} = (q_{ij})$, where $0 \leq n_1 < n_2 \leq \infty$ and $q_{i,i-1} = \mu_i$, $q_{i-1,i} = \lambda_{i-1}$, $i = n_1 + 1, \ldots, n_2$, and all other $q_{ij} = 0$. Suppose that $\lambda_{i-1} > 0$ and $\mu_i > 0$ for each $i = n_1 + 1, \ldots, n_2 - 1$. Define

$$\delta_i = \frac{\lambda_{i-1}}{\mu_i}, \quad i = n_1 + 1, \ldots, n_2. \tag{5.1}$$

A stationary distribution $\pi = (\pi_{n_1}, \ldots, \pi_{n_2})$ should satisfy

$$\pi_\nu = \pi_{n_1} \prod_{i=n_1+1}^{\nu} \delta_i, \quad \nu = n_1 + 1, \ldots, n_2.$$

This solution is well defined if $\{\lambda_{i-1}\}$ and $\{\mu_i\}$ are such that

$$\sum_{\nu=n_1+1}^{n_2} \sum_{i=n_1+1}^{\nu} \delta_i < \infty$$

(see Ross [20]). For a given distribution π, there can be many choices of $\{\lambda_{i-1}, \mu_i\}$ with π as a univariate stationary distribution of $\{X(t), t \geq 0\}$.

THEOREM 5.1: *Let δ_i, $i = n_1 + 1, \ldots, n_2$, be fixed, and regard $\{\lambda_{i-1}\}$ or $\{\mu_i\}$ as parameters. Then, for all $0 < t_1 < t_2 \leq \cdots \leq t_n < \infty$, $(X(t_1), X(t_2), \ldots, X(t_n))$ is decreasing in $\{\lambda_{i-1}\}$ or $\{\mu_i\}$ with respect to the supermodular dependence ordering.*

PROOF: It is well known that birth–death processes are time reversible and stochastically monotone (see, for example, Keilson and Kester [13]). Note that

$$\sum_{k=i}^{n_2} \sum_{l=j}^{n_2} \pi_k q_{kl} = \begin{cases} -\pi_i \mu_i, & i = j \\ 0, & i \neq j, \end{cases} \quad i, j = n_1, \ldots, n_2,$$

are decreasing in $\{\mu_i\}$ or $\{\lambda_{i-1}\}$. The conclusion now follows from Theorem 4.3. ■

5.2. Discrete-time Markov-Modulated Recursive Models (cf. Bäuerle and Rieder [4])

The system state X_{n+1} at time $n + 1$ evolves from the system state X_n at time n and random inputs Z_n and S_n according to a given transition function $\eta_n : \Re^3 \to \Re$, $n = 0, 1, \ldots$; that is,

$$X_{n+1} = \eta_n(X_n, Z_n, S_n), \quad n = 0, 1, \ldots. \tag{5.2}$$

The random variables S_n, $n = 0, 1, \ldots$, are independent and independent of everything else. Let $\{J_n, n \geq 0\}$ be the environmental process, a discrete-time SHMP with

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state space $\Lambda \subset \Re$. Let $\{F_\lambda, \lambda \in \Lambda\}$ be a family of cdf's on $\mathcal{B}(\Re)$. Given $\{J_n, n \geq 0\}$, the random variables Z_n, $n = 0,1,\ldots$, are assumed to be independent, with the conditional cdf of $[Z_n | J_0 = \lambda_0, \ldots, J_n = \lambda_n]$ being F_{λ_n}. For the system state process $\{X_n, n \geq 0\}$ itself, let the initial state be given by a real-valued random variable X_0, independent of everything else.

Consider another system state process $\{Y_n, n \geq 0\}$ with the corresponding environmental process $\{\tilde{J}_n, n \geq 0\}$ on the state space Λ, input process $\{(\tilde{Z}_n, S_n), n \geq 0\}$, and initial state random variable Y_0. Assume that $X_0 \stackrel{\text{st}}{=} Y_0$, and that the conditional distribution of $\tilde{Z}_m$ given $\{\tilde{J}_n, n \geq 0\}$ is the same as that of Z_m given $\{J_n, n \geq 0\}$ for each m.

The next theorem shows us how the dependence structure of an environment process influences the system state process.

THEOREM 5.2: *Let $Z_0 \uparrow_{\text{st}} J_0$ (or $Z_0 \downarrow_{\text{st}} J_0$), and let $\{J_n, n \geq 0\}$ and $\{\tilde{J}_n, n \geq 0\}$ have a common marginal distribution. Assume that $J_0 \uparrow_{\text{st}} J_1$ and $\tilde{J}_1 \uparrow_{\text{st}} \tilde{J}_0$ (or $J_1 \uparrow_{\text{st}} J_0$ and $\tilde{J}_0 \uparrow_{\text{st}} \tilde{J}_1$). If $(\tilde{J}_0, \tilde{J}_1) \geq_c (J_0, J_1)$ and $\eta_n(x,z,s)$ is convex in x, increasing and supermodular in (x,z) for each n and s, then*

$$X_n \leq_{\text{icx}} Y_n \quad \textit{for all } n \geq 1.$$

PROOF: The proof is similar to that of Theorem 3.3 of Bäuerle and Rieder [4] by using Theorem 3.2. ■

Remark 5.1:

1. Theorem 3.3 of Bäuerle and Rieder [4] is an immediate consequence of Theorem 5.2. They considered environment processes $\{J_n\}$ and $\{\tilde{J}_n\}$ on finite state space $\Lambda = \{1,2,\ldots,m\}$ with one-step transition probability matrices $\mathbf{P} = (P_{ij})$ and $\tilde{\mathbf{P}} = (1-c)\mathbf{I} + c\mathbf{P}$, respectively, where $c \in [0,1]$,

$$P_{ij} = \begin{cases} p_j, & i \neq j \\ 1 - \sum_{\nu \neq j} p_j, & i = j, \end{cases} \quad p_\nu > 0 \text{ and } 0 < \sum_{j=1}^{m} p_j < 1.$$

   It is easy to check that $\mathbf{P}^R = \mathbf{P}$ and that $\mathbf{P}$ and $\mathbf{P}^R$ are both stochastically monotone. Therefore, Corollary 3.1 and Theorem 5.1 can be applied.
2. Theorem 5.2 can now be applied to several interesting stochastic models with environmental processes. For example, for an $MR/GI/1$ queue, the arrival process is assumed to be a specific Markov renewal process. The underlying discrete-time SHMP $\{J_n, n \geq 0\}$ can then be interpreted as an environment which influences the interarrival times $\{Z_n, n \geq 0\}$ of customers. We denote the service time of the nth customer by S_n. The random variables S_n, $n = 1,2,\ldots$, are assumed to be independent and identically distributed, independent of everything else. It is well known that the waiting time X_n of the nth customer is given by $X_1 = 0$ and

$$X_{n+1} = \max\{0, X_n - Z_n + S_n\} \quad \text{for all } n = 1,2,\ldots. \tag{5.3}$$

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This model was considered by Bäuerle [2]. Because Eq. (5.3) is a special form of Eq. (5.2), Theorem 5.2 can now be applied. Therefore, the main results of Bäuerle [2] are generalized.

5.3. Workload in Markov-Modulated Queues (cf. Bäuerle and Rolski [5])

We consider a single-server queue with first come, first served queueing discipline. The arrival process $\{N(t), t \geq 0\}$ is a Markov-modulated Poisson process whose intensity process $\{\lambda(t), t \geq 0\}$ is a continuous-time SHMP with a denumerable state space Λ and intensity matrix $\mathbf{Q}$. The serves times S_i, $i = 1, 2, \ldots$, are supposed to be independent and identically distributed, independent of arrivals. The stationary workload is given by

$$W = \sup_{t \geq 0} (M(t) - t),$$

where $M(t) = \sum_{i=1}^{N(t)} S_i$, $t \geq 0$.

In order to investigate the effect that the dependence structure of intensity process has on the stationary workload, we consider another Markov-modulated queue with the same service process $\{S_n, n \geq 1\}$. Let the second intensity process by $\{\tilde{\lambda}(t), t \geq 0\}$ on state space Λ with intensity matrix $\tilde{\mathbf{Q}}$. We denote the stationary workload of the second queue by $\widetilde{W}$.

The following result generalizes Theorem 1 of Bäuerle and Rolski [5]. Their proofs are similar. Bäuerle and Rolski [5] considered the special case that Λ is finite and $\tilde{\mathbf{Q}} = c\mathbf{Q}$, $0 < c \leq 1$.

THEOREM 5.3: *Let intensity processes* $\{\lambda(t), t \geq 0\}$ *and* $\{\tilde{\lambda}(t), t \geq 0\}$ *have the same initial distribution* π. *Assume that* $\mathbf{Q}$ *and* $\tilde{\mathbf{Q}}^R$ *are stochastically monotone, or* $\mathbf{Q}^R$ *and* $\tilde{\mathbf{Q}}$ *are stochastically monotone. If Eq.* (4.7) *holds and the traffic intensity* $\rho = E[S_1] \cdot E[\lambda(0)] < 1$, *then*

$$W \leq_{\text{icx}} \widetilde{W}.$$

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