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Geodesic completeness and stability

By JOHN K. BEEM and PAUL E. EHRLICH
Mathematics Department, University of Missouri, Columbia, MO 65211, *U.S.A.*

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1. *Introduction*

A (connected) Riemannian manifold M is geodesically complete (i.e. each geodesic may be extended to a geodesic with domain $(-\infty, +\infty)$) iff, as a metric space under the induced Riemannian distance function, M is Cauchy complete ([**12**], p. 138). Furthermore, all compact Riemannian manifolds are complete. On the other hand, compact pseudo-Riemannian manifolds exist which are not geodesically complete. For example Fierz and Jost[**8**] have constructed incomplete metrics of the form $2dx\,dy + h\,dy^2$ on T^2. Furthermore, Williams[**16**] has shown that geodesic completeness may fail to be stable for pseudo-Riemannian manifolds. Here a property is said to be stable if the set of metrics for M with this property is open. This failure of stability may occur for both compact and non-compact manifolds. In particular, Williams has found that $M = S^1 \times S^1 = \mathbb{R}^2/(2\pi Z \times 2\pi Z)$, which may be given the complete Lorentzian metric $g = 2dx\,dy$, also admits Lorentzian metrics $g_n = 2dx\,dy + (\sin(x)/n)\,dy^2$ which are geodesically incomplete yet which for large n are arbitrarily close to g in the Whitney fine C^r topologies. In this example the set $S = \{(0, y) | 0 \leqslant y \leqslant 2\pi\}$ represents the image of a closed null geodesic for (M, g) and also for all (M, g_n). However, for the metrics (M, g_n) this closed null geodesic has affine parametrizations which are incomplete because the velocity vector of this null geodesic returns to a scalar multiple of itself after each complete circuit of S. In general, a closed null geodesic will be complete iff its velocity vector returns exactly to itself after each trip around its image.

In this paper we obtain criteria for the stability of geodesic completeness for pseudo-Riemannian manifolds. It had previously been thought [**10**] that geodesic completeness would be C^2-stable in the Whitney topology for all Lorentzian manifolds. The above example shows that for arbitrary Lorentzian manifolds the property of geodesic completeness fails to be C^r-stable for all $r \geqslant 0$. Thus assumptions on the background metric in addition to geodesic completeness are needed in order to guarantee stability of geodesic completeness.

Let (M, g) be a pseudo-Riemannian manifold of arbitrary signature. We will say that (M, g) has a *pseudoconvex null* (resp. *non-spacelike*, *non-timelike*) *geodesic system* if given any compact set K there exists a compact set K' such that any null (resp. non-spacelike, non-timelike) geodesic segment with both endpoints in K must lie in K'. We will say that (M, g) has *no imprisoned null* (resp. *non-spacelike*, *non-timelike*) *geodesics* if each inextendible null (resp. non-spacelike, non-timelike) geodesic fails to be contained in any compact set. The main purpose of this paper is to establish the following result on the stability of geodesic completeness for pseudo-Riemannian manifolds. Let (M, g) be a pseudo-Riemannian manifold with all null (resp. non-

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spacelike, non-timelike) geodesics complete and suppose (M, g) has a pseudoconvex null (resp. non-spacelike, non-timelike) geodesic system with no imprisoned null (resp. non-spacelike, non-timelike) geodesics. Then there exists a C^1-fine neighbourhood $U(g)$ in the space Pseudo (M) of the pseudo-Riemannian metrics on M such that each $\bar{g} \in U(g)$ is null (resp. non-spacelike, non-timelike) geodesically complete. As a corollary both null and non-spacelike geodesic completeness are C^1-stable for globally hyperbolic space-times. Hence our results yield the C^r-stability $(1 \leqslant r \leqslant \infty)$ of geodesic completeness for Minkowski space-time and the C^r-stability $(1 \leqslant r \leqslant \infty)$ of non-spacelike geodesic completeness for de Sitter space-time and the Einstein static universe. Since these last two space-times are not pseudoconvex for spacelike geodesics, our methods do not yield results for stability of their spacelike completeness. Let (H, h) be a complete (positive definite) Riemannian manifold and consider the Lorentzian product space-time $(\mathbb{R} \times H, \ -dt^2 + h)$. If H has no imprisoned geodesics, then geodesic completeness is C^r-stable $(1 \leqslant r \leqslant \infty)$ at the background metric $-dt^2 + h$. Even if H has imprisoned geodesics, then we can still conclude the C^r-stability $(1 \leqslant r \leqslant \infty)$ of non-spacelike completeness.

The pseudoconvexity and non-imprisonment conditions are well known in the study of pseudodifferential operators [**3**, **14**, **15**].

In [**4**] the stability of pseudoconvexity and non-imprisonment has been studied in the space of symbols of order k in order to find conditions for the stability of solvability of pseudodifferential equations. Taken together these conditions are C^1-stable ([**4**], p. 18) for symbols of order $k \geqslant 0$. On the other hand non-imprisonment by itself need not be C^r-stable for any $r \geqslant 0$, ([**4**], p. 13). Furthermore, pseudoconvexity need not be C^0-stable ([**4**], p. 13).

The stability question for Riemannian (i.e. positive definite) manifolds is much simpler. In §2 we give a short proof of the C^0-stability of (geodesic) completeness and the C^0-stability of (geodesic) incompleteness.

Stability of null geodesic completeness under conformal perturbations of the given background metric may be studied by direct methods. In §4 we show that if (M, g) is null geodesically complete (resp. incomplete), then there exists a C^0-fine neighbourhood of g in the space $C(M, g)$ of metrics conformal to g such that each $\bar{g}$ in this neighbourhood is null geodesically complete (resp. incomplete). Most of the usual exact solutions of the Einstein equations are null incomplete and thus our methods yield conformal stability of null incompleteness for these models. For the special case of Robertson–Walker (big-bang) models we have previously [**2**] investigated both conformal and non-conformal stability of incompleteness. A different approach to the study of stability under conformal change has been taken by Canarutto and Dodson [**5**]. They have established the stability of b-incompleteness under conformal change. Furthermore, Canarutto and Michor [**6**] have investigated the stability of b-incompleteness using the Whitney topology on the space of connections.

In §5 we obtain an instability result for pseudo-Riemannian manifolds which contain a complete closed null geodesic. Generalizing a technique used in [**8**] and [**16**], we show that one may always make an arbitrarily small C^∞ perturbation of the metric near a point in the image of any complete closed null geodesic such that the image remains a closed null geodesic for the perturbed metric but becomes incomplete for

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this metric. Together with §4, this implies that whenever (M, g) has a closed null geodesic, then there are incomplete metrics in each C^∞-fine neighbourhood of g.

Most of our notational conventions are standard and may be found in [**1**], [**9**], and [**12**]. Lorentzian manifolds have signatures $(-, +, \ldots, +)$. A tangent vector $\mathbf{v}$ in a pseudo-Riemannian manifold is null (resp. timelike, spacelike) if $g(\mathbf{v}, \mathbf{v}) = 0$ (resp. < 0, > 0). The Whitney fine C^r-topologies are described in ([**1**], p. 28). If $\{B_i\}$ is a locally finite cover of M by charts and $\delta: M \to (0, +\infty)$, then $|g_1 - g_2|_r < \delta$ means that for each $p \in M$ all the corresponding coefficients and derivatives up to order r of the two tensors g_1 and g_2 are $\delta(p)$ close when calculated in the coordinates of B_i for all B_i with $p \in B_i$. The fine C^r-topology on Pseudo (M) has a basis of the form $\{\bar{g} \in \text{Pseudo}\,(M) | \, |g_0 - \bar{g}|_r < \delta\}$ where $g_0 \in \text{Pseudo}\,(M)$ and the continuous function $\delta: M \to (0, +\infty)$ are arbitrary. If $r < s$, then the C^s-topology is finer than the C^r-topology. Thus C^r-stable implies C^s-stable for all $r \leqslant s \leqslant \infty$.

2. *Riemannian manifolds*

Let (M, h) be a connected Riemannian manifold with a sequence of compact sets $\{C_k\}$ such that

$$\varnothing = C_1 \subset C_2 \subset \ldots \subset C_k \subset \ldots \quad \text{and} \quad M = \bigcup C_k.$$

For each fixed $k \geqslant 1$ let $\epsilon_k > 0$ be a constant such that, if r is any Riemannian metric on M and $|h - r|_0 < \epsilon_k$ on $C_k - C_{k-1}$, then for each non-zero tangent vector $\mathbf{v} \in T(C_k - C_{k-1})$ one has

$$\frac{1}{4} < \frac{h(\mathbf{v}, \mathbf{v})}{r(\mathbf{v}, \mathbf{v})} < \frac{9}{4}.$$

Let $\epsilon: M \to (0, \infty)$ be any continuous function with $0 < \epsilon(p) < \epsilon_k$ whenever $p \in C_k - C_{k-1}$. Now suppose that r is any Riemannian metric for M with $|r - h|_0 < \epsilon$. If $\gamma: [a, b] \to M$ is any curve in M, then the length $L_h(\gamma)$ of γ with respect to h is related to the length $L_r(\gamma)$ of γ with respect to r by

$$L_r(\gamma) \leqslant 2L_h(\gamma) \leqslant 3L_r(\gamma).$$

Hence the distance functions of (M, h) and (M, r) are related by

$$d_r(p, q) \leqslant 2d_h(p, q) \leqslant 3d_r(p, q)$$

for all $p, q \in M$. Thus a sequence $\{p_n\}$ is Cauchy for d_r iff it is Cauchy for d_h. In view of the Hopf–Rinow Theorem, this establishes the C^0-stability of both completeness and incompleteness in the space of Riemannian metrics for M.

3. *Pseudo-Riemannian manifolds*

Let (M, g) be a pseudo-Riemannian manifold. If (M, g) has a pseudo-convex null geodesic system and no imprisoned null geodesics, then theorem 3·3 of [**4**] yields the existence of a C^1-fine neighbourhood $U(g)$ of g in Pseudo (M) such that each $\bar{g} \in U(g)$ has a pseudoconvex null geodesic system and no imprisoned null geodesics. As stated, the result of [**4**] only yields the stability of these two conditions for null geodesics. However, the very same arguments used in [**4**] may be applied to the non-spacelike geodesic system of (M, g) or to the non-timelike geodesic system of (M, g). Conse-

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quently, one also has the fine C^1-stability of pseudoconvexity and non-imprisonment for non-spacelike geodesic systems and non-timelike geodesic systems. A key consideration here is that at each $p \in M$ the null directions, the non-spacelike directions and the non-timelike directions each form a closed set of directions at p. Since the timelike directions and the spacelike directions at p do not form closed sets of directions, the proof method of [4] does not apply to either the system of timelike geodesics or the system of spacelike geodesics. The goal of this section is to obtain in Theorem 3·6 the C^1-stability of null, non-spacelike and non-timelike geodesic completeness (assuming pseudoconvexity and non-imprisonment). Since the proof is the same for the null, non-spacelike and non-timelike cases, we only give the proof for the null geodesic system.

Assume then that (M, g) is a pseudo-Riemannian manifold which is null geodesically complete, has a pseudoconvex null geodesic system and has no imprisoned null geodesics. Let h be a complete (positive definite) Riemannian metric on M. The existence of h is guaranteed by a result of Nomizu and Ozeki [11]. Furthermore, the Hopf–Rinow Theorem yields the compactness of all subsets of M which are closed and bounded with respect to the associated distance function d_h. The metric h is an auxiliary device and is not unique. Thus, the particular constants which will be obtained in Lemmas 3·2–3·5 depend on the choice of h, but the results of Theorem 3·6 and Corollaries 3·7 and 3·8 will be independent of h.

We now construct an increasing sequence $\{A(n)\}$ of compact sets which exhausts M using the procedure given in ([4], p. 17). Fixing $x_0 \in M$, set $A(0) = \{x_0\}$, and $A(1) = \{x \in M \mid d_h(x_0, x) \leqslant 2\}$. If $A(n)$ has been defined, then let $A(n+1)$ be a compact set containing $A(n)$ such that, if $\gamma : [a, b] \to M$ is a null geodesic segment with endpoints in $A(n)$, then the image of γ lies in $A(n+1)$. Assume also that $d_h(x, M - A(n+1)) > 2$ for all $x \in A(n)$. Lemma 3·2 and theorem 3·3 of [4] yield

LEMMA 3·1. *Let (M, g) be a pseudo-Riemannian manifold with no imprisoned null geodesics and a pseudoconvex null geodesic system. There exists a continuous function $\delta : M \to (0, +\infty)$ defining a C^1-fine neighbourhood $U(g) = \{\bar{g} \in Pseudo\ (M) \mid |g - \bar{g}|_1 < \delta\}$ such that, if $\bar{g} \in U(g)$ has a null geodesic $\bar{\gamma} : [a, b] \to M$ with $\bar{\gamma}(a)$, $\bar{\gamma}(b) \in A(n)$, then $\bar{\gamma}(t) \notin A(n+4) - A(n+3)$ for all $a \leqslant t \leqslant b$. Furthermore, each $\bar{g} \in U(g)$ has no imprisoned null geodesics and has a pseudoconvex null geodesic system.*

In the rest of this section, the sequence $\{A(n)\}$, continuous function $\delta : M \to (0, \infty)$, and neighbourhood $U(g)$ will always be fixed as above. Furthermore, h will be a complete (positive definite) Riemannian metric on M.

If $c : (a, b) \to M$ is a smooth curve of M, then the h-speed of c at t_0 is $[h(c'(t_0), c'(t_0))]^{\frac{1}{2}}$. If c is a geodesic of (M, g), then the h-speed of c never vanishes. Also if c_1 and c_2 are different affine parametrizations of the same geodesic, there exists a constant $\lambda \geqslant 0$ such that at each fixed point q of the geodesic, the ratio of the h-speed of c_1 to the h-speed of c_2 is equal to λ.

We now show that if the h-speed of a null geodesic γ is bounded on an annular ring $A(n) - A(n-1)$, there exists an annular ring $A(k) - A(k-1)$ with $k > n$ such that γ must increase in affine parameter by at least two units in going from the first ring to the second.

LEMMA 3·2. *Let (M, g) be a null geodesically complete pseudo-Riemannian manifold with a pseudoconvex null geodesic system and no imprisoned null geodesics. Let n be given*

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and let $L > 0$ be fixed. There exists an integer $k = k(n, L, h) > n+5$ such that if $\gamma: \mathbb{R} \to M$ is any null geodesic of (M, g) with $\gamma(0) \in \mathrm{cl}\,(A(n) - A(n-1))$, $h(\gamma'(0), \gamma'(0)) \leqslant L^2$, and $\gamma(t) \in \mathrm{cl}\,(A(k) - A(k-1))$, then $|t| > 2$.

Proof. It is sufficient to consider the case $h(\gamma'(0), \gamma'(0)) = L^2$. Assume the lemma is false. Then there exists a sequence $\gamma_m: \mathbb{R} \to M$ of null geodesics with $\gamma_m(0) \in \mathrm{cl}\,(A(n) - A(n-1))$ and $h(\gamma'(0), \gamma'(0)) = L^2$ together with a sequence $\{t_m\}$ with $|t_m| \leqslant 2$ but $\gamma_m(t_m) \in \mathrm{cl}\,(A(m) - A(m-1))$. Since the sphere bundle with h radius L over the compact set $\mathrm{cl}\,(A(n) - A(n-1))$ is compact, there is a convergent subsequence $\{\gamma_j'(0)\}$ of $\{\gamma_m'(0)\}$. Thus $\gamma_j'(0) \to v_0$, where v_0 is null and $h(v_0, v_0) = L^2$. Let $\beta: \mathbb{R} \to M$ be the inextendible null geodesic of (M, g) with $\beta'(0) = v_0$. Since $\beta([-2, 2])$ is compact, there is some fixed N such that $\beta([-2, 2]) \subset A(N)$. Since $\gamma_j'(0) \to \beta'(0)$, it follows that $\gamma_j([-2, 2]) \subset A(N+1)$ for all large j. This contradicts $\gamma_j(t_j) \in \mathrm{cl}\,(A(j) - A(j-1))$ and $|t_j| \leqslant 2$ for all $j > N+2$. ∎

We now show that for a Lorentzian metric $\bar{g} \in U(g)$ which is close to g on the set $A(k) - A(n-4)$, any null geodesic $\bar{\gamma}$ of $\bar{g}$ which has $\bar{\gamma}(0) \in \mathrm{cl}\,(A(n) - A(n-1))$, $\bar{\gamma}(a, 0] \cap A(n-5) \neq \varnothing$ and satisfies $h(\bar{\gamma}'(0), \bar{\gamma}'(0)) \leqslant L^2$ must increase in affine length by at least one unit in going from $\bar{\gamma}(0) \in A(n) - A(n-1)$ to some point of $A(k) - A(k-1)$. Recall from Lemma 3·1 that if $\bar{g} \in U(g)$, then $\bar{\gamma}$ cannot pass through $A(n-5)$, reach $\mathrm{cl}\,(A(n) - A(n-1))$, and return to $A(n-5)$. Thus in the next lemma it is not necessary to control the separation of $\bar{g} \in U(g)$ from g on $A(n-5)$.

LEMMA 3·3. *Let (M, g) be a null geodesically complete pseudo-Riemannian manifold with a pseudoconvex null geodesic system and no imprisoned null geodesics. Let $L > 0$ and $n > 5$ be given. There exist positive constants $k > n+5$ and $\epsilon > 0$ such that if $\bar{g} \in U(g)$ satisfies $|g - \bar{g}|_1 < \epsilon$ on $A(k) - A(n-5)$, then for any inextendible null geodesic $\bar{\gamma}: (a, b) \to M$ of $(M, \bar{g})$ with $\bar{\gamma}(a, 0] \cap A(n-5) \neq \varnothing$, $\bar{\gamma}(0) \in \mathrm{cl}\,(A(n) - A(n-1))$, $h(\bar{\gamma}'(0), \bar{\gamma}'(0)) \leqslant L^2$, $\bar{\gamma}(t_0) \in A(k) - A(k-1)$ and $t_0 > 0$, it must be true that $t_0 > 1$.*

Proof. Assume the lemma is false and let k be as in Lemma 3·2. There must then exist a sequence of pseudo-Riemannian metrics $\{\bar{g}_m\} \subset U(g)$ with $|g - \bar{g}_m|_1 < 1/m$ on $A(k) - A(n-5)$ such that each metric $\bar{g}_m$ has an inextendible null geodesic $\bar{\gamma}_m$ and a parameter value t_m with $0 < t_m \leqslant 1$ failing to satisfy the conclusion of the lemma. Thus, $\bar{\gamma}_m$ passes through $A(n-5)$ at some negative parameter values, and satisfies $\bar{\gamma}_m(0) \in \mathrm{cl}\,(A(n) - A(n-1))$, $h(\bar{\gamma}_m'(0), \bar{\gamma}_m'(0)) = L^2$ and $\bar{\gamma}_m(t_m) \in A(k) - A(k-1)$. Using the compactness of the h sphere bundle of radius L over $\mathrm{cl}\,(A(n) - A(n-1))$, we obtain a convergent subsequence $\bar{\gamma}_j'(0) \to v_1$ of $\{\bar{\gamma}_m'(0)\}$. We may assume without loss of generality that $t_j \to \tau \in (0, 1]$. Let β be the unique inextendible null geodesic of (M, g) with $\beta'(0) = v_1$. Standard ordinary differential equation theory implies that $\bar{\gamma}_j(t_j) \to \beta(\tau)$, whence $\beta(\tau) \in \mathrm{cl}\,(A(k) - A(k-1))$. But Lemma 3·2 implies $\beta(t) \notin \mathrm{cl}\,(A(k) - A(k-1))$ for all $t \in [0, 2)$, in contradiction. ∎

Fix the numbers n, k and L with $L \geqslant 0$ and $k > n+5$. Let S denote the compact set of all null vectors attached at points of $\mathrm{cl}\,(A(n) - A(n-1))$ with h length L. Our non-imprisonment assumptions on (M, g) together with the compactness of S guarantee that there is some number N_1 such that if γ is a null geodesic of (M, g) with $\gamma'(0) \in S$, then $\gamma(t) \notin A(k+5)$ for all $|t| > N_1$. The map $\mathbb{R} \times S \to \mathbb{R}$ defined by $(t, v) \to h(\gamma'(t), \gamma'(t))$ where $\gamma'(0) = v$ is continuous and thus attains a maximum N^2 on the compact set $[-N_1, N_1] \times S$. Consequently, we obtain the following result.

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LEMMA 3·4. *Let (M, g) be a null geodesically complete pseudo-Riemannian manifold with a pseudoconvex null geodesic system and no imprisoned null geodesics. Let n, k, and L be given with $L > 0$, $k > n+5$. There exists an $N > 0$ such that if γ is a null geodesic of (M, g) with $\gamma(0) \in A(n)$, and $h(\gamma'(0), \gamma'(0)) \leqslant L^2$, then $h(\gamma'(t), \gamma'(t)) \leqslant N^2$ for all t with $\gamma(t) \in \mathrm{cl}\,(A(k+5)) = A(k+5)$.*

Using standard results from ordinary differential equations and Lemmas 3·3 and 3·4, one may establish the following result.

LEMMA 3·5. *Let (M, g) be a null geodesically complete pseudo-Riemannian manifold which has a pseudoconvex null geodesic system and no imprisoned null geodesics. Let n, k, and L be given with $L > 0$ and $k > n+5$. There exist constants N and $\epsilon > 0$ such that if $\bar{g} \in U(g)$ with $|g - \bar{g}|_1 < \epsilon$ on $A(k+5) - A(n-5)$ and if $\bar{\gamma}: (a, b) \to M$ is a null geodesic of $\bar{g}$ with $\bar{\gamma}(a, 0] \cap A(n-5) \neq \varnothing$, $\bar{\gamma}(0) \in A(n) - A(n-1)$, $h(\bar{\gamma}'(0), \bar{\gamma}'(0)) \leqslant L^2$, $\bar{\gamma}(t_0) \in \mathrm{cl}\,(A(k) - A(k-1))$ and $t_0 > 0$, then $h(\bar{\gamma}'(t_0), \bar{\gamma}'(t_0) \leqslant N^2 + 1$.*

We are now ready to establish our main result on the stability of geodesic completeness.

THEOREM 3·6. *Let (M, g) be a null (resp. non-spacelike, non-timelike) geodesically complete pseudo-Riemannian manifold with a pseudoconvex null (resp. non-spacelike, non-timelike) geodesic system and no imprisoned null (resp. non-spacelike, non-timelike) geodesics. Then there exists a C^1-fine neighbourhood $U_1(g)$ of g in Pseudo (M) such that each $\bar{g} \in U_1(g)$ is null (resp. non-spacelike, non-timelike) geodesically complete, has a pseudoconvex null (resp. non-spacelike, non-timelike) geodesic system and no imprisoned null (resp. non-spacelike, non-timelike) geodesics.*

Proof. As mentioned above, we will only give the proof for the null case. First, let $\delta: M \to (0, +\infty)$ be chosen according to Lemma 3·1. Second, inductively construct four sequences $\{n_j\}$, $\{k_j\}$, $\{L_j\}$ and $\{\epsilon_j\}$ for $j = 1, 2, \ldots$ as follows. Set $n_1 = 6$ and $L_1 = 1$. Given n_j and L_j choose $k_j = k(n_j, L_j, h)$ according to Lemma 3·2. Now given n_j, k_j and L_j, choose ϵ and N according to Lemma 3·5 and set $\epsilon_j = \epsilon$ and $L_{j+1} = [N^2 + 1]^{\frac{1}{2}}$. Defining $n_{j+1} = k_j$ and continuing generates all four required sequences. Also let $\epsilon: M \to (0, +\infty)$ be any continuous function such that $\epsilon(x) < \delta(x)$ for all $x \in M$ and $0 < \epsilon(x) < \epsilon_j$ whenever $x \in A(k_j+5) - A(n_j-5)$. Since each point x of M lies in only a finite number of sets of the form $A(k_j+5) - A(n_j-5)$, such a function ϵ exists. Set $U_1(g) = \{\bar{g} \in U(g) \mid |g - \bar{g}|_1 < \epsilon\}$. Since $\epsilon(x) < \delta(x)$ for all x, each $\bar{g} \in U_1(g)$ has a pseudoconvex null geodesic system and has no imprisoned null geodesics. Fix $\bar{g} \in U_1(g)$ and let $\bar{\gamma}$ be a null geodesic of $(M, \bar{g})$. It remains to prove $\bar{\gamma}$ is complete.

There exists a smallest $j \geqslant 1$ such that the image of $\bar{\gamma}$ intersects the interior of $A(n_j) - A(n_j - 1)$. Let J denote this value of j. By reparametrizing γ if necessary, we may assume that $\bar{\gamma}(0) \in A(n_{J+1}) - A(n_{J+1} - 1)$, $h(\bar{\gamma}(0), \bar{\gamma}'(0) \leqslant L_{J+1}^2$ and that $\bar{\gamma}$ intersects $A(n_J) - A(n_J - 1)$ for some negative values of its affine parameter. Applying Lemmas 3·3 and 3·5 to each successive pair (n_j, k_j) and recalling that $n_{j+1} = k_j$ we find $\bar{\gamma}$ increases in affine parameter by at least one unit starting from $t = 0$ as $\bar{\gamma}$ goes from $A(n_j) - A(n_j - 1)$ to $A(k_j) - A(k_j - 1)$ for all $j \geqslant J+1$. Thus $\bar{\gamma}$ is defined on an interval of the form (a, ∞). Reversing the direction of $\bar{\gamma}$ we find that a must be $-\infty$. ∎

Applying Theorem 3·6 to n-dimensional Minkowski space-time $(\mathbb{R}^n, \eta)$ we obtain the following corollary.

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COROLLARY 3·7. *There exists a C^1-fine neighbourhood $U(\eta)$ of the Minkowski metric η such that each $\bar{g} \in U(\eta)$ is geodesically complete. Furthermore, $U(\eta)$ may be chosen such that each inextendible geodesic of each $\bar{g} \in U(\eta)$ has non-compact closure and such that the entire geodesic system of $(\mathbb{R}^n, \bar{g})$ is pseudoconvex.*

Since the geodesics of a pseudo-Riemannian manifold are solutions to differential equations involving the first derivatives of the metric tensor, the fine C^0-topology gives little control over geodesic systems. In particular, every C^0-fine neighbourhood $U(\eta)$ of Lorentzian metrics about the Minkowski metric η contains a geodesically incomplete metric. Thus Corollary 3·7 is the best possible result using the Whitney fine topologies.

A globally hyperbolic space-time has a pseudoconvex null (resp. non-spacelike) geodesic system and no imprisoned null (resp. non-spacelike) geodesics. Consequently, we obtain the following corollary to Theorem 3·6.

COROLLARY 3·8. *Let (M, g) be a globally hyperbolic space-time which is null (resp. non-spacelike) geodesically complete. Then there exists a C^1-fine neighbourhood $U(g)$ in the space* Lor (M) *of all Lorentzian metrics on M such that each $\bar{g} \in U(g)$ is null (resp. non-spacelike) geodesically complete.*

4. *Conformal stability of incompleteness*

Let (M, g) be a pseudo-Riemannian manifold and let $f: M \to \mathbb{R}$ be a smooth function. The metric $\bar{g} = e^{2f} g$ is (globally) conformal to g. The relation $g_1 \sim g_2$ if g_1 and g_2 are conformal on M is an equivalence relation and the equivalence class of g will be denoted by $C(M, g)$. Since $C(M, g) \subset$ Pseudo (M), the Whitney C^r-fine topologies on Pseudo (M) induce corresponding C^r-fine topologies on $C(M, g)$. Intuitively, $\bar{g}$ is close to g in the C^0-fine topology on $C(M, g)$ if the above f is close to zero.

Let ∇ and $\bar{\nabla}$ denote the Levi–Civita connections for g and $\bar{g} = e^{2f} g$, respectively. Using a well known formula ([7], p. 2), the connections ∇ and $\bar{\nabla}$ may be related by

$$\bar{\nabla}_X Y = \nabla_X Y + X(f)\, Y + Y(f) X - g(X, Y)\, \mathrm{grad}\,(f),$$

where grad (f) denotes the gradient of f with respect to the metric g. In particular, if σ is a null geodesic of g we find

$$\bar{\nabla}_{\sigma'} \sigma' = 2\sigma'(f)\, \sigma', \tag{4·1}$$

which shows that σ is a (null) pregeodesic for $(M, \bar{g})$. Standard formulas (cf. [13], pp. 6–35 ff) may be used to reparametrize σ to a curve $\bar{\sigma}$ which is a null *geodesic* of $(M, \bar{g})$. To obtain this reparametrization we assume zero is in the domain of σ and set

$$\left.\begin{aligned} S(u) &= \exp\left(\int_0^u 2\sigma'(f)\, ds\right), \\ &= \exp\left(\int_0^u 2(f \circ \sigma)'\, ds\right), \\ &= \exp\left(2f(\sigma(u)) - 2f(\sigma(0))\right). \end{aligned}\right\} \tag{4·2}$$

Then, defining $p(t)$ and $\bar{\sigma}$ by

$$p(t) = \int_0^t S(u)\, du \tag{4·3}$$

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and $$\bar{\sigma}(t) = \sigma(p^{-1}(t)), \tag{4·4}$$

one obtains an (affine) parametrization of $\bar{\sigma}$ as a null geodesic of $\bar{g} = e^{2f}g$. If σ is defined on the interval (a, b) and $p:(a, b)\to(A, B)$ then $\bar{\sigma}$ is defined on (A, B).

From equation (4·2) it follows that if $|f|$ is bounded by L on M, then $S(u)$ satisfies

$$e^{-4L} \leqslant S(u) \leqslant e^{4L}.$$

Using equation (4·3) and recalling that $a < 0 < b$, one finds

$$ae^{4L} \leqslant p(a) \leqslant ae^{-4L}$$

and

$$be^{-4L} \leqslant p(b) \leqslant be^{4L}.$$

Thus, $p(a)$ is finite iff a is finite and $p(b)$ is finite iff b is finite. It follows that whenever $|f|$ is bounded on M, a null geodesic σ of (M, g) is complete iff the corresponding null geodesic $\bar{\sigma} = \sigma \circ p^{-1}$ is complete for $(M, \bar{g})$. Since $U(g) = \{\bar{g}\in C(M, g) \mid \bar{g} = e^{2f}g$ and $|f| < L$ on $M\}$ is an open C^0-fine neighbourhood of (M, g) in $C(M, g)$, we obtain the following conformal stability theorem for null completeness and incompleteness.

THEOREM 4·1. *If (M, g) is a null complete (resp. incomplete) pseudo-Riemannian manifold, then there exists a C^0-fine neighbourhood $U(g)$ of g in the conformal class $C(M, g)$ such that all $\bar{g}\in C(M, g)$ are null complete (resp. incomplete).*

When M is compact, every f is bounded and it follows that the metrics in $C(M, g)$ are either all null complete or else they are all null incomplete.

5. *Closed null geodesics*

In this section we indicate how the techniques of [**8**] and [**16**] may be generalized. We find that if (M, g) is a pseudo-Riemannian manifold with a closed and complete null geodesic σ, then any C^r-fine neighbourhood $U(g)$ of g in Pseudo (M) contains a metric $\bar{g}$ such that the pointset of σ is a null geodesic of $\bar{g}$ which is incomplete for $\bar{g}$. Thus null completeness is never stable in the presence of a closed null geodesic. Of course, Theorem 4·1 implies that for a sufficiently small neighbourhood $U(g)$ the metric $\bar{g}$ cannot be conformal to g.

We first show that a closed null geodesic is complete iff its velocity vector returns exactly to itself after each circuit of its image.

LEMMA 5·1. *Let (M, g) have an inextendible null geodesic $\sigma:(a, b)\to M$ with $\sigma(0) = \sigma(1)$ and $\sigma'(1) = \lambda\sigma'(0)$ for $\lambda > 0$. If $\lambda = 1$, then σ is complete. If $\lambda \neq 1$, then σ is incomplete.*

Proof. If $\lambda = 1$, then $\sigma([0, 1])$ is a closed geodesic pointset which is traversed over and over at the same rate. Consequently, the domain of σ is $(-\infty, +\infty)$.

If $\lambda > 1$, then the speed of σ increases by a factor λ each time $\sigma([0, 1])$ is traversed. In particular, if $\gamma(s) = \sigma(\lambda s)$, then $\gamma(\lambda^{-1}) = \sigma(1) = \sigma(0) = \gamma(0)$ and $\gamma'(0) = \lambda\sigma'(0) = \sigma'(1)$. Thus $\gamma(t) = \sigma(1+t)$ and the first circuit of γ (i.e. $0 \leqslant t \leqslant \lambda^{-1}$) corresponds to the second circuit of σ (i.e. $1 \leqslant t \leqslant 1+\lambda^{-1}$). Especially, $\sigma(1+\lambda^{-1}) = \sigma(0)$. Similarly, if we let $\eta(s) = \sigma(\lambda^2 s)$, it may be checked that $\eta(t) = \sigma(1+\lambda^{-1}+t)$ and the first circuit of η (i.e. $0 \leqslant t \leqslant \lambda^{-2}$) corresponds to the third

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circuit of σ and $\sigma(1+\lambda^{-1}+\lambda^{-2}) = \sigma(0)$. Inductively, $\sigma(1+\lambda^{-1}+\lambda^{-2}+\ldots+\lambda^{-n}) = \sigma(0)$ for all positive integers n. Since $\lambda > 1$ and

$$\sum_{n=0}^{\infty} \lambda^{-n} = [1-\lambda^{-1}]^{-1},$$

the domain of σ has finite least upper bound equal to $[1-\lambda^{-1}]^{-1}$. Thus σ is incomplete. If $\lambda < 1$, a similar argument shows the domain of σ is bounded below. ∎

Now suppose (M, g) contains a closed and complete null geodesic $\sigma: (-\infty, \infty) \to M$ with $\sigma'(0) = \sigma'(1)$. Let $E_1(t), \ldots, E_n(t)$ be parallel fields along σ which satisfy $E_1(t) = \sigma'(t)$ and are pseudo-orthonormal, i.e. $g(E_1, E_1) = g(E_2, E_2) = 0$, $g(E_1, E_2) = 1$, $g(E_1, E_j) = g(E_2, E_j) = 0$ if $j \geqslant 3$, and $g(E_i, E_j) = \delta_{ij}$ if $i \geqslant 3$, $j \geqslant 3$. Then the map $F: \mathbb{R}^n \to M$ defined by

$$F(x_1, \ldots, x_n) = \exp_{\sigma(x_1)} (x_2 E_2(x_1) + x_3 E_3(x_1) + \ldots + x_n E_n(x_1))$$

defines local coordinates $(x_1, \ldots, x_n)$ on M which are valid near $\sigma(0)$. Furthermore, along σ the metric tensor g_{ij} has the pseudo-orthonormal form

$$[g_{ij}] = \left[\begin{array}{cc|c} 0 & 1 & 0\ldots0 \\ 1 & 0 & 0\ldots0 \\ \hline 0 & 0 & \\ 0 & 0 & I \end{array}\right]$$

In these coordinates we have $\sigma(t) = (t, 0, \ldots, 0)$ and since σ is a geodesic of g it follows that $\Gamma^i_{11} \equiv 0$ along σ for all i. We now construct a C^∞ function N on M which vanishes off an open set U having the form $U = \{(x_1, \ldots, x_n) \mid 0 < x_1 < \delta$ and $|x_i| \leqslant \delta$ for $2 \leqslant i \leqslant n\}$. We will require that on some open set V about $\{(x_1, 0, \ldots, 0) \mid 0 \leqslant x_1 \leqslant \delta\}$ the function $N(x_1, \ldots, x_n)$ only depends on x_1. Furthermore, we may construct N such that it is strictly positive on $\{(x_1, 0, \ldots, 0) \mid 0 < x_1 < \delta\}$. We will write $N = N(x_1)$ and restrict our attention to the neighbourhood V. For any $\epsilon > 0$, set $\bar{g} = g + \epsilon x_2 N(x_1)\, dx_1^2$. It is easily checked that along the x_1-axis in V the Christoffel symbols $\bar{\Gamma}^i_{11}$ of $\bar{g}$ satisfy $\bar{\Gamma}^1_{11} = -\epsilon N(x_1)/2$ and $\bar{\Gamma}^i_{11} = 0$ for $i \geqslant 2$. It follows that if $\phi(t)$ is a solution of the differential equation

$$\phi''(t) - \tfrac{1}{2}\epsilon N(t)\,(\phi'(t))^2 = 0,$$

satisfying $\phi(0) = 0$ and $\phi'(0) = 1$, then

$$\bar{\sigma}(t) = (\phi(t), 0, \ldots, 0)$$

parametrizes the null geodesic of $\bar{g}$ with $\bar{\sigma}'(0) = \sigma'(0)$ for the range of t corresponding to $0 \leqslant \phi(t) \leqslant \delta$. If $\phi(t_1) = \delta$, then $\phi''(t) > 0$ for $0 < t < t_1$ implies $\phi'(t_1) > 1$. Thus $\bar{\sigma}'(t_1) = \phi'(t_1)\,\sigma'(\delta)$ yields that the null geodesic $\bar{\sigma}$ of $\bar{g}$ has the same image as that of σ but must return to $\sigma(0)$ with its velocity vector larger than $\sigma'(0)$. Consequently, by Lemma 5·1 the geodesic $\bar{\sigma}$ must be incomplete. Since ϵ may be chosen arbitrarily small, we obtain metrics $\bar{g}$ arbitrarily close to g in the Whitney C^∞-fine topology such that the image of σ corresponds to an incomplete closed null geodesic of $\bar{g}$.

In contrast, using the techniques of Section 4 the following result on the stability of completeness for closed null geodesics in $C(M, g)$ may be obtained. Let $\beta: [0, 1] \to (M, g)$ be a closed null geodesic in (M, g), i.e. $\beta(0) = \beta(1)$ and $\beta'(1) = \lambda\beta'(0)$ for $\lambda > 0$. If $\lambda = 1$, then for every $\bar{g} \in C(M, g)$, the null pregeodesic β in $(M, \bar{g})$ may be reparametrized to a *complete* null closed geodesic. If $\lambda \neq 1$, then for every

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$\bar{g} \in C(M, g)$, the null pregeodesic β in $(M, \bar{g})$ remains *incomplete* when reparametrized to a null geodesic $\bar{\beta}$ of $(M, \bar{g})$.

The alteration of metric technique discussed in this section together with the results of Section 4 yield our last proposition.

PROPOSITION 5·2. *If (M, g) is a pseudo-Riemannian manifold with a closed null geodesic, then each C^∞-fine neighbourhood $U(g)$ of g in Pseudo(M) contains a null incomplete metric $\bar{g}$.*

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