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Fully Discrete Galerkin Finite Element Method for the Cubic Nonlinear Schrödinger Equation

Jianyun Wang[1] and Yunqing Huang[2,*]

[1] *School of Science, Hunan University of Technology, Zhuzhou 412007, China*
[2] *Hunan Key Laboratory for Computation and Simulation in Science and Engineering, School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, China*

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Abstract. This paper is concerned with numerical method for a two-dimensional time-dependent cubic nonlinear Schrödinger equation. The approximations are obtained by the Galerkin finite element method in space in conjunction with the backward Euler method and the Crank-Nicolson method in time, respectively. We prove optimal L^2 error estimates for two fully discrete schemes by using elliptic projection operator. Finally, a numerical example is provided to verify our theoretical results.

AMS subject classifications: 65M60, 65M50, 65M12

Key words: Finite element method, nonlinear Schrödinger equations, backward Euler scheme, Crank-Nicolson scheme.

1. Introduction

In this paper, we consider the following initial boundary value problem for the two-dimensional time-dependent cubic nonlinear Schrödinger equation: we seek a complex-valued function $u = u(x, y, t)$ defined on $\Omega \times [0, T]$ satisfying

$$\begin{cases} iu_t = -\Delta u + Vu + |u|^2 u + f, & (x, y, t) \in \Omega \times [0, T], \\ u(x, y, t) = 0, & (x, y, t) \in \partial\Omega \times [0, T], \\ u(x, y, 0) = u_0(x, y), & (x, y) \in \Omega, \end{cases} \tag{1.1}$$

where $i = \sqrt{-1}$ is the complex unit, $\Omega \in R^2$ is a convex polygonal domain, Δ is the usual Laplace operator and $u_0(x, y)$ is a given complex-valued initial data. Function $V(x, y)$ is a given real-valued external potential and non-negative bounded for all $(x, y) \in \Omega$. Function $f(x, y, t)$ is complex-valued.

*Corresponding author. *Email addresses:* wjy8137@163.com (J. Y. Wang), huangyq@xtu.edu.cn (Y. Q. Huang)

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The Schrödinger equation is the fundamental equation of physics for describing quantum mechanical behavior. Especially, the nonlinear Schrödinger equation (NLSE) can be met in many different areas of physics and chemistry. There are numerous works in the literature to solve the Schrödinger equations. At present, the common numerical methods are finite difference method [1, 7, 10], spectral method [6, 8], two-grid method [12, 29], mixed finite method [19, 30], discontinuous Galerkin method [16, 20, 28] and finite element method [2–5,9,11,13–15,17,18,21–27]. In [1], Akrivis et al. analyzed the discretization of an initial-boundary value problem for the cubic Schrödinger equation in one space dimension by a Crank-Nicolson-type finite difference scheme. In [7], Chang et al. presented several finite difference schemes for the generalized nonlinear Schrödinger equation. Their numerical results demonstrate that the linearized Crank-Nicolson scheme is efficient and robust. In [6], Bao et al. studied the performance of time-splitting spectral approximations for general nonlinear Schrödinger equations in the semiclassical regimes. In [29], Jin et al. solved the time-dependent Schrödinger equation by the finite element two-grid method and analyzed the convergence. The semi-discrete schemes are proved to be convergent with an optimal convergence order and the full-discrete schemes are verified by a numerical example. In [30], Zhao et al. established a new mixed finite element approximate formulation with less degree of freedoms for nonlinear Schrödinger equation based on spaces of bilinear finite element and its gradient. They analyzed error estimates under both semi-discrete and fully-discrete schemes. In [16], Karakashian et al. analyzed the convergence of the discontinuous Galerkin method for the nonlinear (cubic) Schrödinger equation. They showed the existence of the resulting approximations and proved optimal order error estimates in $L^\infty(L^2)$. In [20], Huang et al. analyzed a class of mass preserving direct discontinuous Galerkin (DDG) schemes for Schrödinger equations subject to both linear and nonlinear potentials. In an earlier work [21], Sanz-Serna established Optimal L^2 rates of convergence for several fully-discrete schemes for the numerical solution of the nonlinear Schrödinger equation. In a later work [2], Akrivis et al. approximated the solutions of a nonlinear Schrödinger equations by two fully discrete finite element schemes based on the standard Galerkin method in space and two implicit Crank-Nicolson method in time. The authors studied the existence and uniqueness of solutions and proved L^2 error bounds of optimal order of accuracy with time step condition $\tau = o(h^{\frac{d}{4}}), (d = 2, 3)$. In [24], Tourigny obtained the optimal H^1 error estimates of the backward Euler and Crank-Nicolson Galerkin FEMs for the generalized nonlinear Schrödinger equation with the time step conditions $\tau = o(h^{\frac{d}{2}})$ and $\tau = o(h^{\frac{d}{4}}), (d = 1, 2, 3)$, respectively. In a recent work [26], Wang studied linearized Crank-Nicolson Galerkin FEMs for a generalized nonlinear Schrödinger equation. The author used a complex time-discrete system to present the optimal L^2 error estimate without time-step restrictions. More recently, In [27], Huang et al. considered a two-dimensional time-dependent linear Schrödinger equation with the rectangular Lagrange type finite element of order p, and analyzed the superconvergence error estimate in the semi-discrete scheme and the fully discrete scheme, respectively. In this paper, we consider a cubic nonlinear Schrödinger Eq. (1.1) and analyze the optimal L^2 error estimate in the backward Euler and the Crank-Nicolson fully discrete schemes,

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respectively.

The paper is organized as follows. In Section 2, the notation and preliminaries are introduced. Moreover, we present the finite element semi-discrete scheme and some results which are relevant to our study. In Section 3, we present the backward Euler finite element fully discrete scheme and obtain the error estimate in L^2 norm with order $\mathcal{O}(h^2+\tau)$. In Section 4, we construct the Crank-Nicolson finite element fully discrete scheme and prove the error estimate in L^2 norm with order $\mathcal{O}(h^2+\tau^2)$. In Section 5, a numerical example is presented to demonstrate our theoretical analysis. Conclusions are given in the last section. Throughout this paper, the symbol C is used for a positive constant which may at times depend on the solution u and the data of the problem (1.1) but is independent of the mesh size h.

2. Preliminaries

For $1 \le p \le \infty$, $L^p = L^p(\Omega)$ will denote the Banach space of complex-valued measurable functions defined on Ω equipped with the norm $\|\cdot\|_{L^p}$. In the sequel, we use $\|\cdot\|$ instead of $\|\cdot\|_{L^2}$.

For an integer $s \ge 0$ and $1 \le p \le \infty$, $W^{s,p}$ will denote the Banach space of complex-valued measurable functions, together with their distributional derivatives of order up to s, are in L^p. To simplify the notation, we shall use the symbols H^s to denote the usual complex-valued Sobolev space $W^{s,2}$, and $\|\cdot\|_s$ instead of $\|\cdot\|_{s,2}$. Further, $\|\cdot\|_{s,\infty}$ represents the norm on $L^\infty([0,T],W^{s,2})$.For any complex-valued function $u(x,y), v(x,y) \in L^2(\Omega)$, let (u,v) denote the inner product

$$(u,v) = \int_\Omega u(x,y)v(x,y)dx$$

and $\|u\|$ denotes the corresponding L^2 norm $\|u\| = \sqrt{(u,u)}$,where $\bar{v}$ denotes the complex conjugate of v.

Then the weak solution $u(x,y,t)$ of problem (1.1) is defined as follows: for all $t \in [0,T]$, find $u(x,y,t) \in H_0^1(\Omega)$ such that

$$\begin{cases} i(u_t,v) = a(u,v) + (|u|^2u,v) + (f,v), & \forall v \in H_0^1(\Omega),\ t>0, \\ u(x,y,0) = u_0(x,y), & (x,y) \in \Omega, \end{cases} \tag{2.1}$$

where

$$a(u,v) = (\nabla u, \nabla v) + (Vu,v), \quad \text{and} \quad \nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right).$$

Let T_h be a quasi-uniform rectangular partition of Ω with mesh size $0 < h < 1$, and S^h be the corresponding piecewise bilinear finite element subspace in $H_0^1(\Omega)$. In general, for any fixed $t \in [0,T]$ and given $w(x,y,t) \in H_0^1(\Omega)$, we can define its elliptic projection $P_h w(x,y,t) \in S^h$ such that

$$a(P_h w, v_h) = a(w, v_h), \quad \forall v_h \in S^h. \tag{2.2}$$

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Then, we can define the semi-discrete finite element solution $u_h(x,y,t)$ of problem (1.1) as follows: for all $t \in [0,T]$, find $u_h(x,y,t) \in S^h$ such that

$$\begin{cases} i(u_{ht}, v_h) = a(u_h, v_h) + (|u_h|^2 u_h, v_h) + (f, v_h), & \forall v_h \in S^h,\ t > 0, \\ u_h(x,y,0) = P_h u_0(x,y), & (x,y) \in \Omega, \end{cases} \tag{2.3}$$

Lemma 2.1. ([29]) *If for any $t \in [0,T]$, $w(x,y,t) \in H^2(\Omega)$, then $P_h w(x,y,t)$ has the following estimate*

$$\|w - P_h w\| \le Ch^2 \|w\|_2, \tag{2.4}$$

where $P_h w(x,y,t)$ is the elliptic projection of $w(x,y,t)$.

Lemma 2.2. ([21]) *For any $v(\mathbf{x}) \in S^h$, there is the following inverse inequality*

$$\|v\|_{0,\infty} \le Ch^{-\frac{d}{2}} \|v\|, \tag{2.5}$$

where $d = 2, 3$ is the dimension of variable $\mathbf{x}$.

3. The backward Euler method

In this section, we focus on the time discretization with the backward Euler scheme. Let $\tau = T/N$ be the time step of the interval $[0,T]$, the time nodes $t_j = j\tau, \quad j = 0, 1, \cdots, N$ and time elements

$$I_j = (t_j,\ t_{j+1}), \quad j = 0, 1, \cdots, N-1.$$

To simplify the notation, we denote $u(x,y,t_n)$ by u^n. Then we discretize (2.3) by backward Euler scheme and the fully discrete finite element solution

$$U^n(x,y) \in S^h, \quad 0 \le n \le N$$

of problem (1.1) can be defined by

$$\begin{cases} i(\partial_t U^n, v_h) = a(U^n, v_h) + (|U^n|^2 U^n, v_h) + (f^n, v_h), & \forall v_h \in S^h, \\ U^0(x,y) = P_h u_0(x,y), \end{cases} \tag{3.1}$$

where

$$\partial_t U^n = \frac{1}{\tau}(U^n - U^{n-1}).$$

Lemma 3.1 Let function series $U^n(x,y), 0 \le n \le N$, be the fully discrete finite element solution defined in (3.1), if $\tau < 0.5$, then we have

$$\|U^n\| \le C. \tag{3.2}$$

Proof. Taking $v_h = U^n$ in (3.1), we get

$$\frac{i}{\tau}(U^n - U^{n-1}, U^n) = a(U^n, U^n) + (|U^n|^2 U^n, U^n) + (f^n, U^n). \tag{3.3}$$

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By comparing the imaginary parts of (3.3), we have

$$\frac{1}{\tau}\big(\|U^n\|^2 - Re(U^n - U^{n-1})\big) = Im\{(f^n, U^n)\},$$

which implies that

$$\|U^n\|^2 \le \|U^n\|\|U^{n-1}\| + \tau\|f^n\|\|U^n\|,$$

that is $\|U^n\| \le \|U^{n-1}\| + \tau\|f^n\|$. Thus

$$\begin{aligned}\|U^n\| &\le \|U^0\| + \tau\sum_{j=1}^{n}\|f^j\| \\ &= \|P_h u_0\| + \tau\sum_{j=1}^{n}\|f^j\|, \le C\|u_0\| + T\max_{1\le j\le N}\|f^j\|, \end{aligned} \tag{3.4}$$

therefore, (3.2) follows from (3.4). □

Assumption 3.1. *Assume that there exists* $0 < h_1 < 1$ *such that for* $0 < h < h_1$, *when* $\tau = o(h)$, *there holds*

$$\|U^n - u^n\|_{0,\infty} < K_1, \quad 0 \le n \le N, \tag{3.5}$$

where $U^n(x, y)$ *be the fully discrete finite element solution defined in* (3.1), *and* K_1 *is a positive constant.*

Theorem 3.1. *Let* $u(x, y, t)$ *be the solution defined in* (2.1), *function series* $U^n(x, y), 0 \le n \le N$, *be the fully discrete finite element solution defined in* (3.1), *if* $\tau = o(h)$ *and* $\tau < (2M_1 + 8)^{-1}$, *then we have error estimates*

$$\|U^n - P_h u^n\| \le Ch^2 + C\tau, \tag{3.6a}$$

$$\|u^n - U^n\| \le Ch^2 + C\tau, \tag{3.6b}$$

where

$$M_1 = \max_{1\le j\le N}\|u^j\|_{0,\infty}^2\big(\|u^j\|_{0,\infty} + \|U^j\|_{0,\infty}\big)^2.$$

Proof. From (2.1) and (3.1), we can get

$$i(u_t^n - \partial_t U^n, v_h) = a(u^n - U^n, v_h) + (|u^n|^2 u^n - |U^n|^2 U^n, v_h). \tag{3.7}$$

Let $u - U = \rho - \theta$ with

$$\rho = u - P_h u, \quad \theta = U - P_h u. \tag{3.8}$$

Combining (3.7) with (3.8), we obtain

$$\begin{aligned} &i(u_t^n - \partial_t u^n, v_h) + i(\partial_t \rho^n - \partial_t \theta^n, v_h) \\ =&a(\rho^n - \theta^n, v_h) + \Big(|U^n|^2(\rho^n - \theta^n), v_h + ((|u^n|^2 - |U^n|^2)u^n, v_h\Big). \end{aligned} \tag{3.9}$$

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It follows from (2.2) that

$$a(\rho, v_h) = 0. \tag{3.10}$$

From (3.9) and (3.10), we have

$$\begin{aligned} &i(\partial_t \theta^n, v_h) - i(\partial_t \rho^n, v_h) - i(u_t^n - \partial_t u^n, v_h) \\ =&a(\theta^n, v_h) + (|U^n|^2\theta^n, v_h) - (|U^n|^2\rho^n, v_h) - \left((|u^n|^2 - |U^n|^2)u^n, v_h\right), \end{aligned} \tag{3.11}$$

Taking $v_h = \theta^n$ in (3.11), we can get

$$\begin{aligned} &i(\partial_t \theta^n, \theta^n) - i(\partial_t \rho^n, \theta^n) - i(u_t^n - \partial_t u^n, \theta^n) \\ =&a(\theta^n, \theta^n) + (|U^n|^2\theta^n, \theta^n) - (|U^n|^2\rho^n, \theta^n) - \left((|u^n|^2 - |U^n|^2)u^n, \theta^n\right). \end{aligned} \tag{3.12}$$

Noticing that

$$Re\{(\partial_t \theta^n, \theta^n)\} = \frac{1}{2\tau}\left(\|\theta^n\|^2 - \left\|\theta^{n-1}\right\|^2 + \left\|\theta^n - \theta^{n-1}\right\|^2\right),$$

and comparing the imaginary parts of (3.12), we have

$$\begin{aligned} &\frac{1}{2\tau}(\|\theta^n\|^2 - \left\|\theta^{n-1}\right\|^2 + \left\|\theta^n - \theta^{n-1}\right\|^2) \\ =&Re\left\{(\partial_t \rho^n, \theta^n) + (u_t^n - \partial_t u^n, \theta^n)\right\} - Im\left\{(|U^n|^2\rho^n, \theta^n) + ((|u^n|^2 - |U^n|^2)u^n, \theta^n)\right\}. \end{aligned} \tag{3.13}$$

It follows (3.13) that

$$\begin{aligned} &\|\theta^n\|^2 - \|\theta^{n-1}\|^2 + \|\theta^n - \theta^{n-1}\|^2 \\ \le&2\tau\left(|\,(\partial_t \rho^n, \theta^n)\,| + |\,(u_t^n - \partial_t u^n, \theta^n)\,| + \left|(|U^n|^2\rho^n, \theta^n)\right| + \left|((|u^n|^2 - |U^n|^2)u^n, \theta^n)\right|\right). \end{aligned}$$

Consequently,

$$\begin{aligned} &\|\theta^n\|^2 - \|\theta^{n-1}\|^2 \\ \le&2\tau\left(|\,(\partial_t \rho^n, \theta^n)\,| + |\,(u_t^n - \partial_t u^n, \theta^n)\,| + |\,(|U^n|^2\rho^n, \theta^n)\,| + |\,((|u^n|^2 - |U^n|^2)u^n, \theta^n)\,|\right) \\ \le&2\tau\left(\|\partial_t \rho^n\|\|\theta^n\| + \|u_t^n - \partial_t u^n\|\|\theta^n\| + \||U^n|^2\rho^n\|\|\theta^n\| + \|(|u^n|^2 - |U^n|^2)u^n\|\|\theta^n\|\right), \end{aligned}$$

and by Young inequality, we can obtain

$$\begin{aligned} &\|\theta^n\|^2 - \|\theta^{n-1}\|^2 \\ \le&2\tau\left(\|\partial_t \rho^n\|^2 + \|u_t^n - \partial_t u^n\|^2 + \||U^n|^2\rho^n\|^2 + \|(|u^n|^2 - |U^n|^2)u^n\|^2 + 4\|\theta^n\|^2\right). \end{aligned} \tag{3.14}$$

Combining (3.8) with (2.4) gives

$$\begin{aligned} &\left\|\partial_t \rho^n\right\|^2 = \left\|\partial_t u^n - \partial_t P_h u^n\right\|^2 = \tau^{-2}\left\|(u^n - u^{n-1}) - P_h(u^n - u^{n-1})\right\|^2 \\ \le&C\tau^{-2}h^4\left\|u^n - u^{n-1}\right\|_2^2 \le C\tau^{-2}h^4\left(\int_{t_{n-1}}^{t_n}\|u_t(\cdot, t)\|_2 dt\right)^2 \\ \le&C\tau^{-1}h^4\int_{t_{n-1}}^{t_n}\|u_t(\cdot, t)\|_2^2 dt, \end{aligned} \tag{3.15}$$

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and using Schwarz inequality gives

$$\begin{aligned}\left\|u_t^n-\partial_t u^n\right\|^2&=\left\|\tau^{-1}\int_{t_{n-1}}^{t_n}(t-t_{n-1})u_{tt}(\cdot,t)dt\right\|^2\\&=\tau^{-2}\int_\Omega\left(\int_{t_{n-1}}^{t_n}(t-t_{n-1})u_{tt}(\cdot,t)dt\right)^2d\Omega\\&\le\tau^{-2}\int_\Omega\left(\int_{t_{n-1}}^{t_n}(t-t_{n-1})^2dt\right)\left(\int_{t_{n-1}}^{t_n}u_{tt}^2(\cdot,t)dt\right)d\Omega\\&\le C\tau\int_{t_{n-1}}^{t_n}\|u_{tt}^2(\cdot,t)\|dt.\end{aligned}\tag{3.16}$$

Combining (3.5) with (3.8) and (2.4), we can get

$$\||U^n|^2\rho^n\|^2\le\|U^n\|_{0,\infty}^2\|\rho^n\|^2\le C\|u^n-P_hu^n\|^2\le Ch^4\|u^n\|_2^2.\tag{3.17}$$

Furthermore, we have

$$\begin{aligned}&\left\|(|u^n|^2-|U^n|^2)u^n\right\|^2\le\|u^n\|_{0,\infty}^2\left\||u^n|^2-|U^n|^2\right\|^2\\&=\|u^n\|_{0,\infty}^2\|(|u^n|+|U^n|)(|u^n|-|U^n|)\|^2\\&\le\|u^n\|_{0,\infty}^2\left(\left\|u^n\right\|_{0,\infty}+\left\|U^n\right\|_{0,\infty}\right)^2\|u^n-U^n\|^2\\&\le M_1\left(\|\theta^n\|^2+\|\rho^n\|^2\right)\le M_1\|\theta^n\|^2+Ch^4\|u^n\|_2^2.\end{aligned}\tag{3.18}$$

It follows from (3.14)–(3.18) that

$$\begin{aligned}\|\theta^n\|^2-\left\|\theta^{n-1}\right\|^2\le&Ch^4\int_{t_{n-1}}^{t_n}\left\|u_t(\cdot,t)\right\|_2^2dt+C\tau^2\int_{t_{n-1}}^{t_n}\left\|u_{tt}^2(\cdot,t)\right\|dt\\&+C\tau h^4\|u^n\|_2^2+2(M_1+4)\tau\|\theta^n\|^2.\end{aligned}\tag{3.19}$$

Summing up (3.19) for n, we have

$$\begin{aligned}\|\theta^n\|^2-\|\theta^0\|^2&\le Ch^4\int_0^{t_n}\|u_t(\cdot,t)\|_2^2dt+C\tau^2\int_0^{t_n}\left\|u_{tt}^2(\cdot,t)\right\|dt\\&\quad+C\tau h^4\sum_{j=1}^n\|u^j\|_2^2+2(M_1+4)\tau\sum_{j=1}^n\|\theta^j\|^2\\&\le Ch^4\int_0^{t_n}\|u_t(\cdot,t)\|_2^2dt+C\tau^2\int_0^{t_n}\left\|u_{tt}^2(\cdot,t)\right\|dt\\&\quad+CTh^4\max_{1\le j\le N}\|u^j\|_2^2+2(M_1+4)\tau\sum_{j=1}^n\|\theta^j\|^2.\end{aligned}\tag{3.20}$$

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From (2.1), we can get

$$\theta^0 = 0, \tag{3.21}$$

which together with (3.21), gives

$$\|\theta^n\|^2 \le Ch^4 + C\tau^2 + 2(M_1+4)\tau\sum_{j=1}^{n}\|\theta^j\|^2.$$

Consequently,

$$(1-2(M_1+4)\tau)\|\theta^n\|^2 \le Ch^4 + C\tau^2 + 2(M_1+4)\tau\sum_{j=1}^{n-1}\|\theta^j\|^2. \tag{3.22}$$

Assuming that

$$\tau < (2M_1+8)^{-1},$$

and using discreted Gronwall inequality in (3.22), we get

$$\|\theta^n\|^2 \le Ch^4 + C\tau^2, \tag{3.23}$$

which gives (3.6a), and (3.6b) follows from (2.4) and (3.6a).

$$\|\theta^n\| \le Ch^2 + C\tau,$$

which completes the proof. □

Now let us start to prove that Assumption 3.1 is true.

In fact, according to (3.1), it easy to check that Assumption 3.1 is true for $n = 0$. Assume that Assumption 3.1 is not always true for all of $0 \le n \le N$. Let N_1 be the minimum of $0 \le n \le N$ which Assumption 3.1 is false, we have

$$\left\|U^{N_1} - u^{N_1}\right\|_{0,\infty} > K_1 \tag{3.24}$$

and Theorem 3.1 is true for $0 \le n \le N_1$.

Combining (2.5), (3.6a) and (2.4) with $\tau = o(h)$ yields

$$\begin{aligned}\left\|U^{N_1} - u^{N_1}\right\|_{0,\infty} &\le \left\|U^{N_1} - P_h u^{N_1}\right\|_{0,\infty} + \left\|u^{N_1} - P_h u^{N_1}\right\|_{0,\infty}\\ &\le Ch^{-1}\left\|U^{N_1} - P_h u^{N_1}\right\| + Ch^{-1}\left\|u^{N_1} - P_h u^{N_1}\right\|\\ &\le Ch^{-1}(h^2+\tau) + Ch\left\|u^{N_1}\right\|_2 \le C_1(h + h^{\alpha_1}),\end{aligned}$$

where C_1 is a constant independent of h, and α_1 is a positive constant. Choose h_0 to make $C_1(h_1 + h_1^{\alpha_1}) \le K_1$. Then, for $0 < h < h_1$, we have

$$\left\|U^{N_1} - u^{N_1}\right\|_{0,\infty} \le K_1, \tag{3.25}$$

which conflicts with (3.24). So Assumption 3.1 is always true for all of $0 \le n \le N$.

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4. The Crank-Nicolson method

In this section, we focus on the time discretization with the Crank-Nicolson scheme. Let $\tau = T/N$ be the time step of the interval $[0,T]$, the time nodes $t_j = j\tau,\ j = 0,1,\cdots,N$,

$$t_{j+\frac{1}{2}} = \frac{1}{2}\left(t_{j+1} + t_j\right)$$

and time elements $I_j = (t_j, t_{j+1}),\ j = 0,1,\cdots,N-1$. For function series $U^n(x,y),\ n = 0,1,\cdots$, let

$$\partial_t U^{n+\frac{1}{2}} = \frac{1}{\tau}\left[U^{n+1}(x,y) - U^n(x,y)\right],$$
$$U^{n+\frac{1}{2}} = \frac{1}{2}\left[U^{n+1}(x,y) + U^n(x,y)\right].$$

To simplify the notation, we denote $u(x,y,t_n)$ by u^n. Then we define the fully discrete finite element solution $U^n(x,y) \in S^h, 0 \le n \le N$ of problem (1.1) satisfying the Crank-Nicolson scheme

$$\begin{cases} i\left(\partial_t U^{n+\frac{1}{2}}, v_h\right) = a\left(U^{n+\frac{1}{2}}, v_h\right) + \left(|U^{n+\frac{1}{2}}|^2 U^{n+\frac{1}{2}}, v_h\right) + \left(f^{n+\frac{1}{2}}, v_h\right), \quad \forall v_h \in S^h, \\ U^0(x,y) = P_h u_0(x,y). \end{cases} \tag{4.1}$$

Lemma 4.1. *Let function series $U^n(x,y), 0 \le n \le N$, be the fully discrete finite element solutions defined in* (4.1)*, then we have*

$$\|U^n\| \le C. \tag{4.2}$$

Proof. Taking $v_h = U^{n+\frac{1}{2}}$ in (4.1), we get

$$\begin{aligned} & i\left(\partial_t U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\right) \\ =& a\left(U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\right) + \left(|U^{n+\frac{1}{2}}|^2 U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\right) + \left(f^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\right). \end{aligned} \tag{4.3}$$

Noticing that

$$\frac{1}{2\tau}\left(\|U^{n+1}\|^2 - \|U^n\|^2\right) = Re\left\{\left(\partial_t U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\right)\right\},$$

and comparing the imaginary parts of (4.3), we have

$$\begin{aligned} & \frac{1}{2\tau}\left(\left\|U^{n+1}\right\|^2 - \left\|U^n\right\|^2\right) = Im\left\{\left(f^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\right)\right\} \\ \le & C\left\|f^{n+\frac{1}{2}}\right\|\left\|U^{n+\frac{1}{2}}\right\| \le C\left\|f^{n+\frac{1}{2}}\right\|\left(\left\|U^{n+1}\right\| + \left\|U^n\right\|\right), \end{aligned} \tag{4.4}$$

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it follows from (4.4) that

$$\left\|U^{n+1}\right\| - \|U^n\| \le C\tau\left\|f^{n+\frac{1}{2}}\right\|. \tag{4.5}$$

Summing up for n in (4.5), we can get

$$\begin{aligned}\|U^n\| &\le \|U^0\| + C\tau\sum_{j=0}^{n-1}\left\|f^{j+\frac{1}{2}}\right\| \\ &= \|P_h u_0\| + C\tau\sum_{j=0}^{n-1}\left\|f^{j+\frac{1}{2}}\right\| \le C\left(\|u_0\| + T\max_{0\le j\le N-1}\left\|f^{j+\frac{1}{2}}\right\|\right),\end{aligned} \tag{4.6}$$

Therefore, (4.2) follows from (4.6). □

Assumption 4.1. *Assume that there exists $0 < h_2 < 1$ such that for $0 < h < h_2$, when $\tau = o(h^{1/2})$, there holds*

$$\|U^n - u^n\|_{0,\infty} < K_2, \quad 0 \le n \le N, \tag{4.7}$$

where $U^n(x,y)$ be the fully discrete finite element solution defined in (4.1), and K_2 is a positive constant.

Theorem 4.1. *Let $u(x,y,t)$ be the solution defined in (2.1), function series $U^n(x,y), 0 \le n \le N$ be the fully discrete finite element solutions defined in (4.1). If $\tau = o(h^{1/2})$ and $\tau < (M_2)^{-1}$, then we have error estimates*

$$\|U^n - P_h u^n\| \le Ch^2 + C\tau^2, \tag{4.8}$$

$$\|u^n - U^n\| \le Ch^2 + C\tau^2, \tag{4.9}$$

where

$$M_2 = \max_{0\le j\le N-1}\left\|u^{j+\frac{1}{2}}\right\|_{0,\infty}\left(\left\|u^{j+\frac{1}{2}}\right\|_{0,\infty} + \left\|U^{j+\frac{1}{2}}\right\|_{0,\infty}\right).$$

Proof. From (2.1) and (4.1), we can get

$$\begin{aligned}&i\left(u_t^{n+\frac{1}{2}} - \partial_t U^{n+\frac{1}{2}}, v_h\right) \\ &= a\left(u^{n+\frac{1}{2}} - U^{n+\frac{1}{2}}, v_h\right) + \left(\left|u^{n+\frac{1}{2}}\right|^2 u^{n+\frac{1}{2}} - \left|U^{n+\frac{1}{2}}\right|^2 U^{n+\frac{1}{2}}, v_h\right).\end{aligned} \tag{4.10}$$

Let $u - U = \rho - \theta$ with

$$\rho = u - P_h u, \quad \theta = U - P_h u. \tag{4.11}$$

Combining (4.10) with (4.11) gives

$$\begin{aligned}&i\left(u_t^{n+\frac{1}{2}} - \partial_t u^{n+\frac{1}{2}}, v_h\right) + i\left(\partial_t\rho^{n+\frac{1}{2}} - \partial_t\theta^{n+\frac{1}{2}}, v_h\right) \\ &= a\left(\rho^{n+\frac{1}{2}} - \theta^{n+\frac{1}{2}}, v_h\right) + \left(\left|U^{n+\frac{1}{2}}\right|^2(\rho^{n+\frac{1}{2}} - \theta^{n+\frac{1}{2}}), v_h\right) + \left((\left|u^{n+\frac{1}{2}}\right|^2 - \left|U^{n+\frac{1}{2}}\right|^2)u^{n+\frac{1}{2}}, v_h\right).\end{aligned} \tag{4.12}$$

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From (3.10) and (4.12), we have

$$\begin{aligned} i\left(\partial_t\theta^{n+\frac{1}{2}},v_h\right)=&i\left(\partial_t\rho^{n+\frac{1}{2}},v_h\right)+i\left(u_t^{n+\frac{1}{2}}-\partial_t u^{n+\frac{1}{2}},v_h\right)\\ &+a\left(\theta^{n+\frac{1}{2}},v_h\right)+\left(|U^{n+\frac{1}{2}}|^2\theta^{n+\frac{1}{2}},v_h\right)-\left(|U^{n+\frac{1}{2}}|^2\rho^{n+\frac{1}{2}},v_h\right)\\ &-\left((\left|u^{n+\frac{1}{2}}\right|^2-\left|U^{n+\frac{1}{2}}\right|^2)u^{n+\frac{1}{2}},v_h\right). \end{aligned} \tag{4.13}$$

Taking $v_h=\theta^{n+\frac{1}{2}}$ in (4.13), we can get

$$\begin{aligned} i\left(\partial_t\theta^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}}\right)=&i\left(\partial_t\rho^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}}\right)+i\left(u_t^{n+\frac{1}{2}}-\partial_t u^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}}\right)+a\left(\theta^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}}\right)\\ &+\left(|U^{n+\frac{1}{2}}|^2\theta^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}}\right)-\left(|U^{n+\frac{1}{2}}|^2\rho^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}}\right)\\ &-\left((|u^{n+\frac{1}{2}}|^2-|U^{n+\frac{1}{2}}|^2)u^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}}\right). \end{aligned} \tag{4.14}$$

Noticing that

$$\frac{1}{2\tau}(\|\theta^{n+1}\|^2-\|\theta^n\|^2)=Re\left\{(\partial_t\theta^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}})\right\},$$

and comparing the imaginary parts of (4.14), we have

$$\begin{aligned} &\frac{1}{2\tau}(\|\theta^{n+1}\|^2-\|\theta^n\|^2)\\ =&Re\left\{(\partial_t\rho^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}})+(u_t^{n+\frac{1}{2}}-\partial_t u^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}})\right\}\\ &-Im\left\{(|U^{n+\frac{1}{2}}|^2\rho^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}})+\left((\left|u^{n+\frac{1}{2}}\right|^2-\left|U^{n+\frac{1}{2}}\right|^2)u^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}}\right)\right\}. \end{aligned} \tag{4.15}$$

It follows from (4.15) that

$$\begin{aligned} &\frac{1}{2\tau}(\left\|\theta^{n+1}\right\|^2-\|\theta^n\|^2)\\ \le&\left|(\partial_t\rho^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}})\right|+\left|(u_t^{n+\frac{1}{2}}-\partial_t u^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}})\right|\\ &+\left|(\left|U^{n+\frac{1}{2}}\right|^2\rho^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}})\right|+\left|\left((\left|u^{n+\frac{1}{2}}\right|^2-\left|U^{n+\frac{1}{2}}\right|^2)u^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}}\right)\right|. \end{aligned} \tag{4.16}$$

Thus

$$\begin{aligned} \left\|\theta^{n+1}\right\|^2-\|\theta^n\|^2\le&2\tau\left(\left\|\partial_t\rho^{n+\frac{1}{2}}\right\|+\left\|u_t^{n+\frac{1}{2}}-\partial_t u^{n+\frac{1}{2}}\right\|+\left\|\left|U^{n+\frac{1}{2}}\right|^2\rho^{n+\frac{1}{2}}\right\|\right.\\ &\left.+\left\|(\left|u^{n+\frac{1}{2}}\right|^2-\left|U^{n+\frac{1}{2}}\right|^2)u^{n+\frac{1}{2}}\right\|\right)\left\|\theta^{n+\frac{1}{2}}\right\|\\ \le&\tau\left(\left\|\partial_t\rho^{n+\frac{1}{2}}\right\|+\left\|u_t^{n+\frac{1}{2}}-\partial_t u^{n+\frac{1}{2}}\right\|+\left\|\left|U^{n+\frac{1}{2}}\right|^2\rho^{n+\frac{1}{2}}\right\|\right.\\ &\left.+\left\|(\left|u^{n+\frac{1}{2}}\right|^2-\left|U^{n+\frac{1}{2}}\right|^2)u^{n+\frac{1}{2}}\right\|\right)\left(\|\theta^n\|+\left\|\theta^{n-1}\right\|\right), \end{aligned}$$

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which yields

$$\left\|\theta^{n+1}\right\| - \|\theta^n\| \le \tau \left(\left\|\partial_t \rho^{n+\frac{1}{2}}\right\| + \left\|u_t^{n+\frac{1}{2}} - \partial_t u^{n+\frac{1}{2}}\right\| + \left\|\left|U^{n+\frac{1}{2}}\right|^2 \rho^{n+\frac{1}{2}}\right\| + \left\|\left(\left|u^{n+\frac{1}{2}}\right|^2 - \left|U^{n+\frac{1}{2}}\right|^2\right) u^{n+\frac{1}{2}}\right\| \right). \tag{4.17}$$

Combining (4.11) with (2.4), we have

$$\begin{aligned} &\left\|\partial_t \rho^{n+\frac{1}{2}}\right\| = \left\|\partial_t u^{n+\frac{1}{2}} - \partial_t P_h u^{n+\frac{1}{2}}\right\| = \tau^{-1} \left\|(u^{n+1} - u^n) - P_h(u^{n+1} - u^n)\right\| \\ \le & C\tau^{-1} h^2 \left\|u^{n+1} - u^n\right\|_2 \le C\tau^{-1} h^2 \int_{t_n}^{t_{n+1}} \|u_t(\cdot, t)\|_2 dt. \end{aligned} \tag{4.18}$$

Furthermore, we have

$$\begin{aligned} &\left\|u_t^{n+\frac{1}{2}} - \partial_t u^{n+\frac{1}{2}}\right\| = \left\|u_t(\cdot, t_{n+\frac{1}{2}}) - \partial_t u(\cdot, t_{n+\frac{1}{2}})\right\| \\ =& \frac{1}{2\tau} \nu \left\| \int_{t_n}^{t_{n+\frac{1}{2}}} (t - t_n)^2 u_{ttt}(\cdot, t) dt + \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} (t - t_{n+1})^2 u_{ttt}(\cdot, t) dt \right\| \\ \le & \frac{1}{2\tau} \left\| \int_{t_n}^{t_{n+\frac{1}{2}}} (\frac{\tau}{2})^2 u_{ttt}(\cdot, t) dt + \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} (\frac{\tau}{2})^2 u_{ttt}(\cdot, t) dt \right\| \\ =& \frac{\tau}{8} \left\| \int_{t_n}^{t_{n+1}} (u_{ttt}(\cdot, t) dt \right\| \le C\tau \int_{t_n}^{t_{n+1}} \left\|u_{ttt}(\cdot, t)\right\| dt, \end{aligned} \tag{4.19}$$

Combining (4.7) with (4.11) and (2.4), gives

$$\begin{aligned} &\left\||U^{n+\frac{1}{2}}|^2 \rho^{n+\frac{1}{2}}\right\| \le \left\|U^{n+\frac{1}{2}}\right\|_{0,\infty}^2 \left\|\rho^{n+\frac{1}{2}}\right\| \le C \left\|\rho^{n+\frac{1}{2}}\right\| \\ =& C \left\|u^{n+\frac{1}{2}} - P_h u^{n+\frac{1}{2}}\right\| \le Ch^2 \left\|u^{n+\frac{1}{2}}\right\|_2, \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} &\left\|(|u^{n+\frac{1}{2}}|^2 - |U^{n+\frac{1}{2}}|^2) u^{n+\frac{1}{2}}\right\| \le \left\|u^{n+\frac{1}{2}}\right\|_{0,\infty} \left\||u^{n+\frac{1}{2}}|^2 - |U^{n+\frac{1}{2}}|^2\right\| \\ =& \left\|u^{n+\frac{1}{2}}\right\|_{0,\infty} \left\|(|u^{n+\frac{1}{2}}| + |U^{n+\frac{1}{2}}|) \left(|u^{n+\frac{1}{2}}| - |U^{n+\frac{1}{2}}|\right)\right\| \\ \le & \left\|u^{n+\frac{1}{2}}\right\|_{0,\infty} (\left\|u^{n+\frac{1}{2}}\right\|_{0,\infty} + \left\|U^{n+\frac{1}{2}}\right\|_{0,\infty}) \left\|u^{n+\frac{1}{2}} - U^{n+\frac{1}{2}}\right\| \\ \le & M_2 \left\|u^{n+\frac{1}{2}} - U^{n+\frac{1}{2}}\right\| \le M_2 (\left\|\theta^{n+\frac{1}{2}}\right\| + \left\|\rho^{n+\frac{1}{2}}\right\|) \\ \le & M_2 \left\|\theta^{n+\frac{1}{2}}\right\| + Ch^2 \left\|u^{n+\frac{1}{2}}\right\|_2. \end{aligned} \tag{4.21}$$

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It follows from (4.17) - (4.21) that

$$\|\theta^{n+1}\| - \|\theta^n\| \le Ch^2 \int_{t_n}^{t_{n+1}} \|u_t(\cdot,t)\|_2 dt + C\tau^2 \int_{t_n}^{t_{n+1}} \|u_{ttt}(\cdot,t)\| dt + C\tau h^2 \left\| u^{n+\frac{1}{2}} \right\|_2 + M_2 \tau \left\| \theta^{n+\frac{1}{2}} \right\|. \tag{4.22}$$

Note that

$$\theta^0 = 0, \tag{4.23}$$

summing up (4.22) for n and combining (4.23), we have

$$\begin{aligned} \|\theta^n\| &\le Ch^2 \int_0^{t_n} \|u_t(\cdot,t)\|_2 dt + C\tau^2 \int_0^{t_n} \|u_{ttt}(\cdot,t)\| dt \\ &\quad + C\tau h^2 \sum_{j=1}^{n} \left\| u^{j+\frac{1}{2}} \|_2 + M_2 \tau \sum_{j=1}^{n} \|\theta^j\| \right\| \\ &\le Ch^2 \int_0^{t_n} \|u_t(\cdot,t)\|_2 dt + C\tau^2 \int_0^{t_n} \|u_{ttt}(\cdot,t)\| dt \\ &\quad + CTh^2 \max_{1\le j\le N} \left\| u^{j+\frac{1}{2}} \right\|_2 + M_2 \tau \sum_{j=1}^{n} \|\theta^j\|. \end{aligned}$$

Consequently,

$$\|\theta^n\| \le Ch^2 + C\tau^2 + M_2 \tau \sum_{j=1}^{n} \|\theta^j\|,$$

that is

$$(1 - M_2\tau)\|\theta^n\| \le Ch^2 + C\tau^2 + M_2 \tau \sum_{j=1}^{n-1} \|\theta^j\|. \tag{4.24}$$

When $\tau < (M_2)^{-1}$, using discreted Gronwall inequality in (4.24), we can obtain

$$\|\theta^n\| \le Ch^2 + C\tau^2. \tag{4.25}$$

Therefore, (4.8) follows from (4.25), and (4.9) follows from (2.4) and (4.8). □

Now let us start to prove that Assumption 4.1 is true. It is easy to check that Assumption 4.1 is true for $n = 0$ according to (4.1). Assume that Assumption 4.1 is not always true for all of $0 \le n \le N$. Let N_2 be the minimum of $0 \le n \le N$ which Assumption 4.1 is false, we have

$$\|U^{N_2} - u^{N_2}\|_{0,\infty} > K_2, \tag{4.26}$$

and Theorem 4.1 is true for $0 \le n \le N_2$.

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Combining (2.5), (4.8) and (2.4) with $\tau = o(h^{1/2})$ yields

$$\begin{aligned}\|U^{N_2} - u^{N_2}\|_{0,\infty} &\le \left\|U^{N_2} - P_h u^{N_2}\right\|_{0,\infty} + \left\|u^{N_2} - P_h u^{N_2}\right\|_{0,\infty} \\ &\le Ch^{-1}\left\|U^{N_2} - P_h u^{N_2}\right\| + Ch^{-1}\left\|u^{N_2} - P_h u^{N_2}\right\| \\ &\le Ch^{-1}(h^2+\tau^2) + Ch\left\|u^{N_2}\right\|_2 \\ &\le C_2(h + h^{\alpha_2}),\end{aligned}$$

where C_2 is a constant independent of h, and α_2 is a positive constant. Choosing h_2 to make

$$C_2(h_2 + h_2^{\alpha_2}) \le K_2.$$

Then, for $0 < h < h_2$, we have

$$\|U^{N_2} - u^{N_2}\|_{0,\infty} \le K_2, \tag{4.27}$$

which conflicts with (4.26). So Assumption 4.1 is always true for all of $0 \le n \le N$.

5. Numerical examples

In this section, we present a numerical example to confirm our theoretical analysis. We consider the following cubic nonlinear Schrödinger equation:

$$\begin{cases} iu_t = -\Delta u + u + |u|^2 u + f, & (x,y,t) \in \Omega \times [0,T], \\ u(x,y,t) = 0, & (x,y,t) \in \partial\Omega \times [0,T], \\ u(x,y,0) = u_0(x,y), & (x,y) \in \Omega, \end{cases}$$

where $\Omega = [0,1] \times [0,1]$, the time interval $[0,T] = [0,1]$, and the function $f(x,y,t)$ is chosen corresponding to the exact solution

$$u = e^t(1-x)(1-y)\sin x \sin y + ie^t xy \sin(1-x)\sin(1-y).$$

We solved the problem on uniformly refined rectangular meshes with mesh size h in the bilinear finite element space S^h. First, we fix $\tau = 10^{-4}$ and compute the L^2 errors $\|u^n - U^n\|$ by varying the mesh size h. The obtained convergence results are presented in Tables 1 and 2 by backward Euler scheme and in Tables 4 and 5 by Crank-Nicolson scheme. Results in all tables show $\mathcal{O}(h^2)$ convergence rate clearly by both schemes.

To test the convergence rate in terms of τ, we take the time step $\tau = h^2$ under backward Euler scheme and $\tau = h$ under Crank-Nicolson scheme. The obtained convergence results are presented in Table 3 and Table 6, respectively. Results show the convergence rate $\mathcal{O}(\tau)$ and $\mathcal{O}(\tau^2)$ clearly, respectively, which are consistent with our theoretical analysis in the paper. In Fig. 1, the profile of the exact solution at $t = 1.0$ on the 64×64 mesh grid is plotted. The profiles of the numerical solutions by backward Euler and Crank-Nicolson method at $t = 1.0$ on the 64×64 mesh grid are plotted in Figs. 2 and 3, respectively.

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Table 1: Errors at t = 0.1 obtained with $\tau = 10^{-4}$ by backward Euler scheme.

mesh	$h=\frac{1}{8}$	$h=\frac{1}{16}$	$h=\frac{1}{32}$	$h=\frac{1}{64}$
$\|u^n - U^n\|$	1.0555e-03	2.6155e-04	6.5273e-05	1.6205e-05
rate	/	4.0355	4.0070	4.0280

Table 2: Errors at t = 1.0 obtained with $\tau = 10^{-4}$ by backward Euler scheme.

mesh	$h=\frac{1}{8}$	$h=\frac{1}{16}$	$h=\frac{1}{32}$	$h=\frac{1}{64}$
$\|u^n - U^n\|$	2.4641e-03	6.3870e-04	1.6196e-04	4.0439e-05
rate	/	3.8581	3.9436	4.0050

Table 3: Errors at t = 1.0 obtained with $\tau = h^2$ by backward Euler scheme.

mesh	$h=\frac{1}{8}$	$h=\frac{1}{16}$	$h=\frac{1}{32}$	$h=\frac{1}{64}$
$\|u^n - U^n\|$	2.7587e-03	6.6144e-04	1.6104e-04	4.0157e-05
rate	/	4.1707	4.1073	4.0103

Table 4: Errors at t = 0.1 obtained with $\tau = 10^{-4}$ by Crank-Nicolson scheme.

mesh	$h=\frac{1}{8}$	$h=116$	$h=\frac{1}{32}$	$h=\frac{1}{64}$
$\|u^n - U^n\|$	1.0563e-03	2.6161e-04	6.5477e-05	1.6386e-05
rate	/	4.0376	3.9954	3.9960

Table 5: Errors at t = 1.0 obtained with $\tau = 10^{-4}$ by Crank-Nicolson scheme.

mesh	$h=\frac{1}{8}$	$h=\frac{1}{16}$	$h=\frac{1}{32}$	$h=\frac{1}{64}$
$\|u^n - U^n\|$	2.4557e-03	6.3757e-04	1.6265e-04	4.0705e-05
rate	/	3.8516	3.9199	3.9958

Table 6: Errors at t = 1.0 obtained with $\tau = h$ by Crank-Nicolson scheme.

mesh	$h=\frac{1}{16}$	$h=\frac{1}{32}$	$h=\frac{1}{64}$	$h=\frac{1}{128}$
$\|u^n - U^n\|$	8.3673e-004	1.8440e-04	4.1395e-05	1.0218e-05
rate	/	4.5375	4.4547	4.0511

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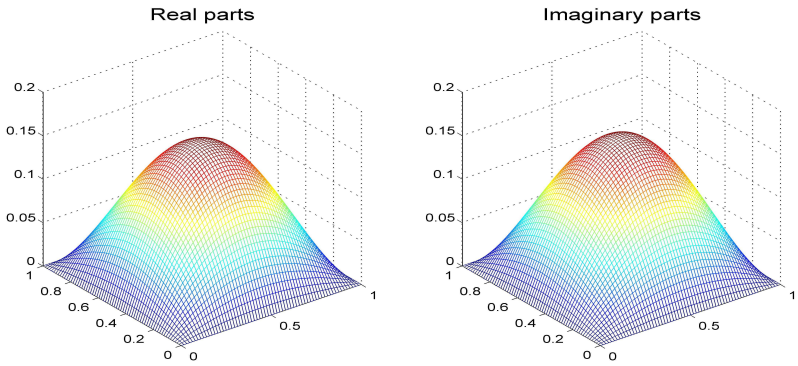


Figure 1: The exact solution at t=1.0 on 64×64 mesh.

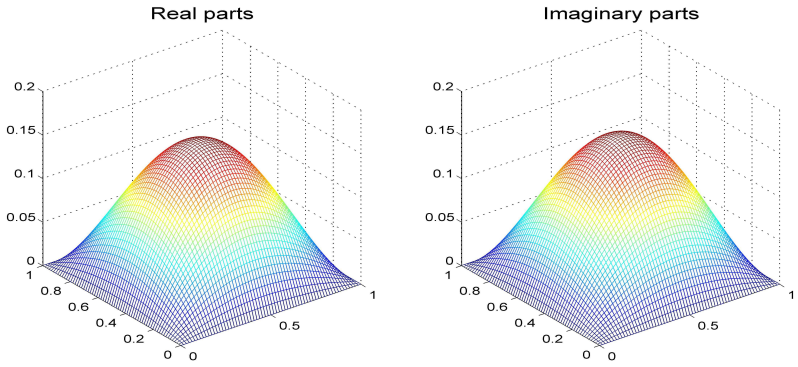


Figure 2: The numerical solution at t=1.0 by Euler scheme on 64×64 mesh.

6. Conclusions

In this paper, we have considered a two-dimensional time-dependent cubic nonlinear Schrödinger equation with the bilinear finite element method. We have presented fully discrete schemes of the equation by Galerkin finite element method in space in conjunction with the backward Euler method and the Crank-Nicolson method in time, respectively. We have further presented the error analysis and obtained the optimal L^2 error estimates of numerical solutions in two schemes. Numerical results have confirmed our theoretical analysis and showed that the time step restriction is not necessary.

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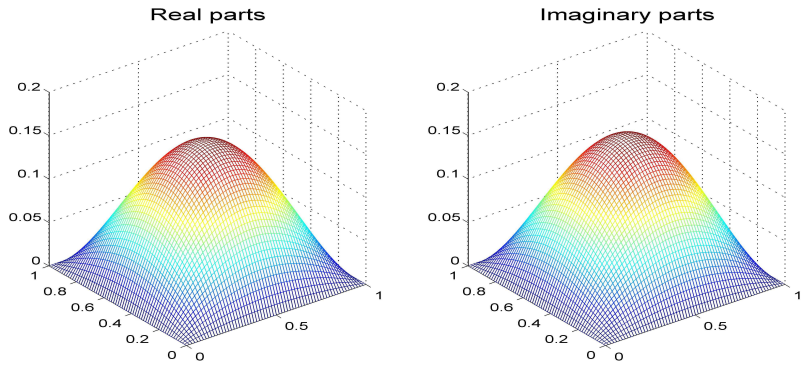


Figure 3: The numerical solution at t=1.0 by C-N scheme on 64×64 mesh.

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