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Lower bound in the Roth theorem for amenable groups

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Abstract. Let T_1 and T_2 be two commuting probability measure-preserving actions of a countable amenable group such that the group spanned by these actions acts ergodically. We show that $\mu(A \cap T_1^g A \cap T_1^g T_2^g A) > \mu(A)^4 - \epsilon$ on a syndetic set for any measurable set A and any $\epsilon > 0$. The proof uses the concept of a sated system, introduced by Austin.

1. *Introduction*

In this article we are concerned with the correlation sequence $c_g = \mu(A \cap T_1^g A \cap T_1^g T_2^g A)$, where A is a positive measure subset of a regular probability space (X, μ) and T_1, T_2 are measure-preserving actions of a countable amenable group G, which commute in the sense that $T_1^g T_2^h = T_2^h T_1^g$ almost everywhere for any $g, h \in G$. In the case $G = \mathbb{Z}$, $T_1 = T_2$, the ergodic theoretic version [**14**] of the Roth theorem [**21**] states that c_g is positive on a syndetic set. The ergodic theoretic version of the corners theorem [**1**] says that the same continues to hold without the assumption $T_1 = T_2$. This has been extended to general countable amenable groups in [**8**].

In the ergodic case, namely if the only sets that are both T_1- and T_2-invariant are those with measure 0 or 1, one can say more. If $G = \mathbb{Z}$ and $T_1 = T_2$, it is known the correlation function c_g is not only positive on a syndetic set, but in fact bounded below by $\mu(A)^3 - \epsilon$ on a syndetic set, for any $\epsilon > 0$ [**7**, Theorem 1.2]. If $G = \mathbb{Z}$ but T_1 and T_2 are not necessarily equal, then the best result to date is that c_g is bounded below by $\mu(A)^4 - \epsilon$ on a syndetic set for any $\epsilon > 0$ [**9**, Theorem 1.1]. The exponent 4 in the latter result cannot be improved to 3; see Theorem B.1. Our purpose is to obtain a similar lower bound for general countable amenable groups (in fact a similar result holds for locally compact second countable amenable groups, although its formulation is more involved; see Theorem 5.7).

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THEOREM 1.1. *Let G be a countable amenable group and T_1, T_2 be commuting measure-preserving left G-actions on a probability space (X, μ). Suppose that the group spanned by T_1 and T_2 acts ergodically on (X, μ). Then, for every measurable set $A \subset X$ and every $\epsilon > 0$, the set*

$$R_\epsilon := \{g \in G : \mu(A \cap T_1^g A \cap T_1^g T_2^g A) > \mu(A)^4 - \epsilon\}$$

is both left and right syndetic.

By a version of the Furstenberg correspondence principle [**6**, Theorem 4.17], this result has a combinatorial interpretation. Recall that the *upper Banach density* of a subset $A \subset G$ is defined by $\overline{d}(A) := \sup_F \limsup_N |A \cap F_N|/|F_N|$, where the supremum is taken over all left Følner sequences in G.

THEOREM 1.2. *Let G be a countable amenable group and let $E \subset G \times G$. Then, for every $\epsilon > 0$, the set*

$$\{g \in G : \overline{d}(E \cap (g, \mathrm{id})E \cap (g, g)E) \geq \overline{d}(E)^4 - \epsilon\}$$

is both left and right syndetic.

It is known that no result similar to Theorem 1.1 can hold for four commuting measure-preserving actions [**7**, Theorem 1.3]. The question as to whether a power lower bound exists for three commuting actions remains open, even for $G = \mathbb{Z}$. A proof that the corresponding Cesàro limit is positive has been recently made available by Austin [**3**].

At first glance, it might appear that at least non-triviality of R_ϵ in Theorem 1.1 would follow from the previously known case $G = \mathbb{Z}$ upon restriction to a suitable cyclic subgroup of G. However, there are at least two obstructions to such reasoning. First, G can be a torsion group. Secondly, even if there exist copies of $\mathbb{Z}$ in G, the restrictions of (T_1, T_2) to these copies can be non-ergodic, preventing one from applying the $G = \mathbb{Z}$ case.

Our arguments rely on a *magic extension* of the system (X, μ, T_1, T_2), obtained as a *sated extension* in the sense of Austin [**5**]. The concept of a magic extension was introduced by Host [**17**]. In the course of the proof, we obtain new proofs of two further results about commuting actions of amenable groups. The first result is an extension to arbitrary left Følner sequences of the convergence theorem for cubic averages due to Griesmer [**15**] (Corollary 3.8). This has been previously shown by Bergelson and Leibman (private communication). The second result is the $k = 3$ case of the convergence theorem for multiple ergodic averages [**23**, Theorem 1.1(2)] (Proposition A.1). Another proof (for arbitrary k) using sated extensions appeared after the completion of this work, in [**3**, Theorem A].

2. *Preliminaries*

2.1. *Følner and Reiter sequences.* Let G be a locally compact σ-compact (lcsc) group with a left Haar measure m. If the group G is amenable, then, by [**18**, Theorem 4.16], it admits a *left Følner sequence*, that is, a sequence of non-null compact sets $F_N \subset G$ such that $m(gF_N \Delta F_N)/m(F_N) \to 0$ uniformly for g in compact subsets of G.

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The space $M(G)$ of complex Radon measures on G is a Banach $*$-algebra with the convolution $\int f\,d(\mu * \nu) = \int_{g,h} f(gh)\,d\mu(g)\,d\nu(h)$, the involution $\int f\,d\mu^* = \int f(g^{-1})\,d\bar{\mu}(g)$, and the total variation norm (see [**11**] for more details). Let $\mathcal{P}(G) \subset M(G)$ be the set of probability Radon measures. A sequence $(F_N) \subset \mathcal{P}(G)$ is called a *left Reiter sequence* if $\|h * F_N - F_N\| \to 0$ as $N \to \infty$ for every $h \in \mathcal{P}(G)$. Right and two-sided Reiter sequences are defined analogously. If (F_N) is a left Reiter sequence and $h \in \mathcal{P}(G)$, $(F'_N) \subset \mathcal{P}(G)$, then $(h * F_N)$ and $(F_N * F'_N)$ are again left Reiter sequences, as follows from the associativity of convolution. Note that (F_N) is a left Reiter sequence if and only if (F_N^*) is a right Reiter sequence.

Given a left Følner sequence $F = (F_N)$, the sequence $(m(F_N)^{-1} 1_{F_N} m)$ is a left Reiter sequence, which will be denoted by the same symbol F. The main conceptual reason to work with Reiter sequences rather than Følner sequences is that two-sided Reiter sequences also exist in non-unimodular groups, as opposed to two-sided Følner sequences. Since for our purposes both concepts work equally well, we chose to use the more general one.

For a fixed Reiter sequence F, we write $\text{C-lim}_g\, u_g := \lim_N \int u_g\,dF_N(g)$ for the Cesàro limit along F, if this limit exists. If the Cesàro limit exists for every Reiter sequence, then it does not depend on the Reiter sequence. In this case, we call it the uniform Cesàro limit and denote it by $\text{UC-lim}_g\, u_g$.Recall the van der Corput lemma.

LEMMA 2.1. [**8**, Lemma 4.2] *Let G be an lcsc group with a left Reiter sequence F. Suppose that $g \mapsto u_g$ is a bounded measurable function to a Hilbert space. Then*

$$\limsup_m \left\| \int_g u_g\,dF_m(g) \right\|^2 \le \inf_{H \in \mathcal{P}(G)} \limsup_m \int_g \int_l \int_h \langle u_{hg}, u_{lg} \rangle\,dH(h)\,dH(l)\,dF_m(g).$$

Proof. By definition of a Reiter sequence, we have, for every $H \in \mathcal{P}(G)$,

$$\lim_m \left\| \int_g u_g\,d(F_m - H * F_m)(g) \right\| = 0.$$

Hence, it suffices to estimate

$$\begin{aligned}
&\limsup_m \left\| \int_g \int_h u_{hg}\,dH(h)\,dF_m(g) \right\|^2 \\
&\le \limsup_m \left(\int_g \left\| \int_h u_{hg}\,dH(h) \right\| dF_m(g) \right)^2 \\
&\le \limsup_m \int_g \left\| \int_h u_{hg}\,dH(h) \right\|^2 dF_m(g) \\
&= \limsup_m \int_g \int_l \int_h \langle u_{hg}, u_{lg} \rangle\,dH(h)\,dH(l)\,dF_m(g),
\end{aligned}$$

where we used the triangle and the Cauchy–Schwarz inequalities. □

COROLLARY 2.2. *Under the assumptions of Lemma 2.1, we have*

$$\limsup_m \left\| \int_g u_g\,dF_m(g) \right\|^2 \le \inf_n \limsup_m \int_g \int_h \langle u_g, u_{hg} \rangle\,dF'_n(h)\,dF_m(g),$$

for some two-sided Reiter sequence F'.

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Proof. Substitute $H = F_n$ in Lemma 2.1. By the Fubini theorem and the Reiter property, we have

$$\begin{aligned}
&\limsup_m \iiint \langle u_{hg}, u_{lg}\rangle \, dF_n(h)\, dF_n(l)\, dF_m(g) \\
&= \limsup_m \iiint \langle u_{hg}, u_{lg}\rangle \, d(\delta_{h^{-1}} * F_m)(g)\, dF_n(h)\, dF_n(l) \\
&= \limsup_m \int_g \iint \langle u_g, u_{lh^{-1}g}\rangle \, dF_n(h)\, dF_n(l)\, dF_m(g).
\end{aligned}$$

The inner integrand can be written as

$$\begin{aligned}
\iint \langle u_g, u_{lh^{-1}g}\rangle \, dF_n(l)\, dF_n(h) &= \iint \langle u_g, u_{lhg}\rangle \, dF_n(l)\, dF_n^*(h) \\
&= \int \langle u_g, u_{hg}\rangle \, d(F_n * F_n^*)(h),
\end{aligned}$$

and we obtain the conclusion with $F_n' = F_n * F_n^*$. □

2.2. *Category of measure-preserving systems.* From now on, let G be an lcsc amenable group with a left Reiter sequence F_N.

Definition 2.3. The category $\mathcal{C}_k$ of k-tuples of commuting measure-preserving actions consists of the following data. The objects are the tuples $(X, \mathcal{X}, \mu, T_1, \ldots, T_k)$, where $(X, \mathcal{X}, \mu)$ is a regular Borel probability space and $T_1, \ldots, T_k$ are continuous commuting measure-preserving G-actions. The morphisms are the continuous factor maps, that is, continuous measure-preserving maps that intertwine the respective G-actions.

The restriction to continuous actions on regular Borel spaces is not substantial. Indeed, suppose that we are given a measurable measure-preserving G-action on a separable measure space X. The associated unitary antirepresentation on $L^2(X)$ is weakly measurable, and therefore strongly continuous by [**16**, Theorem 22.20(b)]. In view of [**19**, Theorem 7.5.5], this implies that the action admits a topological model on a compact metric space.

Measure-preserving actions on X induce anti-actions on the spaces $L^p(X)$, which are denoted by the same symbol.

2.3. *Conditionally almost periodic and weakly mixing functions.* The following result is folklore and goes back to Furstenberg in the case $G = \mathbb{Z}$. A proof for Følner sequences in general lcsc amenable groups will appear in [**20**]; the same proof works without further changes for Reiter sequences. The very last assertion is a pure Hilbert space result whose proof may be found in [**10**].

THEOREM 2.4. *Let G be an lcsc amenable group, T be a measure-preserving G-action on a regular Borel probability space X, and $X \to Y$ be a factor map. Then we have*

$$L^2(X) = A(X|Y, T) \oplus W(X|Y, T),$$

where:

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(1) *the space $A(X|Y,T)$ is spanned by the finite rank T-invariant $L^\infty(Y)$-submodules of $L^\infty(X)$†; and*
(2) *the space $W(X|Y,T)$ consists of the functions f such that every $h \in L^\infty(X)$ and some/every left Reiter sequence F, we have*

$$\lim_n \int_g \|\mathbb{E}(hT^g f|Y)\|^2 \, dF_n(g) = 0.$$

Moreover, for any two factor maps $X_1 \to Y$ and $X_2 \to Y$; we have

$$A(X_1 \times_Y X_2|Y,T) = A(X_1|Y,T) \otimes_Y A(X_2|Y,T).$$

Finally, if Y is the trivial factor, then $A(X|Y,T)$ is the closed linear span of the finite-dimensional T-invariant subspaces.

2.4. *Couplings of measure spaces.* We recall how to construct couplings of regular measure spaces.

LEMMA 2.5. *Let $(X_i, \mathcal{X}_i, \mu_i)_{i\in I}$ be (inner and outer) regular Borel probability spaces and let μ be a finitely additive positive function on the semiring of the sets of the form $\prod_{i\in I} A_i$, where $A_i \in \mathcal{X}_i$ and $A_i \neq X_i$ only for finitely many $i \in I$. Suppose that $\mu(\prod_{i\in I} A_i) \le \min_i \mu_i(A_i)$ for any A_i, as above. Then μ admits a unique extension to a Borel probability measure on $(X,\mathcal{X}) := (\prod_{i\in I} X_i, \bigotimes_{i\in I} \mathcal{X}_i)$.*

Proof. We will show that μ is in fact σ-additive. To this end, suppose that $\prod_i A_i = \biguplus_{m\in\mathbb{N}} \prod_i A_{m,i}$ is a disjoint union with $A_i, A_{m,i} \in \mathcal{X}_i$. By finite additivity, we have $\mu(\prod_i A_i) \ge \sum_{m\in\mathbb{N}} \mu(\prod_i A_{m,i})$. In order to prove the converse inequality, we use the regularity of the measures μ_i. Let $\delta > 0$. By inner regularity, there exist compact subsets $C_i \subset A_i$ such that $C_i = X_i$ if $A_i = X_i$ and $\sum_{i\in I} \mu(C_i) > \sum_i \mu(A_i) - \delta$. It follows that $\mu(\prod_i C_i) > \mu(\prod_i A_i) - \delta$. By outer regularity, there exist open subsets $U_{m,i} \supset A_{m,i}$ such that $\sum_i \mu(U_{m,i}) < \sum_i \mu(A_{m,i}) + \delta/2^m$. It follows that $\mu(\prod_i U_{m,i}) < \mu(\prod_i A_{m,i}) + \delta/2^m$.

By construction, the compact set $\prod_i C_i$ is covered by the open sets $\prod_i U_{m,i}$. Hence, there exists a finite subset $M \subset \mathbb{N}$ such that the corresponding open sets cover the whole compact set. By additivity of μ, this implies

$$\begin{aligned}\mu\Big(\prod_i A_i\Big) &< \mu\Big(\prod_i C_i\Big) + \delta \le \sum_{m\in M} \mu\Big(\prod_i U_{m,i}\Big) + \delta \le \sum_{m\in\mathbb{N}} \mu\Big(\prod_i U_{m,i}\Big) + \delta \\ &\le \sum_{m\in\mathbb{N}} \Big(\mu\Big(\prod_i A_{m,i}\Big) + \delta/2^m\Big) + \delta = \sum_{m\in\mathbb{N}} \mu\Big(\prod_i A_{m,i}\Big) + 2\delta.\end{aligned}$$

Since δ was arbitrary, this shows that μ is σ-subadditive. By the Carathéodory theorem, the function μ has a unique extension to a probability measure on the space $(X,\mathcal{X})$. □

† All that we need to know is that both the Y-measurable and the T-invariant functions are in $A(X|Y,T)$.

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3. *Sated systems and cubic averages*

We will use notation and vocabulary from Austin's thesis [**5**]. A subclass of $\mathcal{C}_k$ is called *idempotent* if it contains a trivial system and is closed under measure-theoretic isomorphisms, inverse limits, and joinings.

LEMMA 3.1. [**5**, Definition 2.2.3] *Let $\mathcal{I}_k$ be an idempotent subclass of $\mathcal{C}_k$. Then we have a functor on $\mathcal{C}_k$, which we denote by the same symbol $\mathcal{I}_k$, such that for each object X of $\mathcal{C}_k$, the object $\mathcal{I}_k X$ is the maximal factor of X contained in $\mathcal{I}_k$ (in particular, such a maximal factor exists).*

Let $(X, \mathcal{X}, \mu)$ be a measure space and $\mathcal{B} \subset \mathcal{X}$ be a sub-σ-algebra. Two sub-σ-algebras $\mathcal{B} \subset \mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{X}$ are called *relatively independent over* $\mathcal{B}$ if, for every $f \in L^2(\mathcal{B}_1)$ such that $f \perp \mathcal{B}$, we have $f \perp \mathcal{B}_2$. The notion of relative independence is in fact symmetric in $\mathcal{B}_1$ and $\mathcal{B}_2$; we refer to [**22**, Appendix] for an exposition of several further characterizations of relative independence.

Suppose that $\mathcal{C}$ is a subclass of $\mathcal{C}_k$ and $\mathcal{I}_k$ is an idempotent class. A system X in $\mathcal{C}$ is called *$\mathcal{I}_k$-sated in $\mathcal{C}$* if for every extension $X' \to X$ with X' in $\mathcal{C}$, the factors X and $\mathcal{I}_k X'$ are relatively independent over $\mathcal{I}_k X$. Note that Austin considers satedness only with $\mathcal{C} = \mathcal{C}_k$; we will have to work with a subclass in order to preserve ergodicity. Recall that the *inverse limit* of a sequence $(X_{i\in\mathbb{N}})$ in $\mathcal{C}_k$ with factor maps $\pi_{i+1,i} : X_{i+1} \to X_i$ is a system X in $\mathcal{C}_k$ with factor maps $\pi_i : X \to X_i$ such that $\pi_{i+1,i} \circ \pi_{i+1} = \pi_i$ and $\mathcal{X} = \vee_{i\in\mathbb{N}} \pi_i^{-1}(\mathcal{X}_i)$. The next result is a variant of [**5**, Theorem 2.3.2].

THEOREM 3.2. *Suppose that $\mathcal{C}$ is a subclass of $\mathcal{C}_k$ that is closed under inverse limits of sequences, and $\mathcal{I}_k$ is an idempotent subclass of $\mathcal{C}_k$. Then, for every X in $\mathcal{C}$, there exists an extension X' that is $\mathcal{I}_k$-sated in $\mathcal{C}$.*

Here and later, denote by I_i the σ-algebra of T_i-invariant sets. We will use Theorem 3.2 with the subclass $\mathcal{I}_k$ of $\mathcal{C}_k$ consisting of systems X such that $\mathcal{X} = I_1 \vee \cdots \vee I_k$, and either with $\mathcal{C} = \mathcal{C}_k$ or with the subclass $\mathcal{C}_k^{\mathrm{erg}}$ of jointly ergodic tuples of measure-preserving actions. It is clear that the class $\mathcal{I}_k$ is idempotent.

The starting point of our investigation is a weak convergence result for cubic averages of dimension 2. For brevity, we will write $T_\epsilon^g = \prod_{i\in\epsilon} T_i^g$ and $T_\epsilon^{\vec{g}} = \prod_{i\in\epsilon} T_i^{g_i}$ for $\epsilon \subseteq \{1, \ldots, k\}$ and $g \in G$, $\vec{g} = (g_1, \ldots, g_k) \in G^k$. Thus, for instance, $T_{\{1,2\}}^g = T_1^g T_2^g$ and $T_{\{1,2\}}^{\vec{g}} = T_1^{g_1} T_2^{g_2}$. We will omit braces from subscripts if no confusion is possible, for example, $T_{\{1,2\}}^g = T_{1,2}^g$.

LEMMA 3.3. *Suppose that $X \in \mathcal{C}_2$. Then, for any $f_\epsilon \in L^\infty(X)$, $\epsilon \subseteq \{1, 2\}$, and any left Reiter sequences Φ, Ψ the limit*

$$\lim_n \iiint \prod_\epsilon T_\epsilon^{\vec{g}} f_\epsilon \, d\mu \, d\Phi_n(g_1) \, d\Psi_n(g_2) \tag{3.4}$$

exists, and, in particular, it does not depend on Φ, Ψ. If, in addition, X is $\mathcal{I}_2$-sated, then the limit vanishes, provided that $f_{1,2} \perp \mathcal{I}_2 X$.

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Proof. Suppose first that X is $\mathcal{I}_2$-sated and recall that we assume the underlying measure space $(X, \mathcal{X}, \mu)$ to be regular. Let B be a countable dense subalgebra of $\mathcal{X}$ and pick a subsequence of $\Phi \times \Psi$, which we denote by the same symbol, such that

$$\mu^{\square}\Big(\prod_{\epsilon} A_\epsilon\Big) := \lim_n \iiint \prod_{\epsilon} T_\epsilon^{\vec{g}} 1_{A_\epsilon}\, d\mu\, d\Phi_n(g_1)\, d\Psi_n(g_2)$$

exists for any $A_\epsilon \in B$. The limit on the right-hand side of the above display is bounded by $\min_\epsilon \mu(A_\epsilon)$, and it follows that $\mu^{\square}$ extends to a function on the semiring of sets of the form $\prod_\epsilon A_\epsilon$, $A_\epsilon \in \mathcal{X}$, by the same formula. It is easy to see that $\mu^{\square}$ is finitely additive.

By Lemma 2.5, the function $\mu^{\square}$ has a unique extension to a probability measure on X^4. Since $\mu^{\square}$ is clearly invariant under the side transformations $T_{\square,1}$ and $T_{\square,2}$, given by

$$T_{\square,i} = \times_{\epsilon \subseteq \{1,2\}} T_{\epsilon \cap \{i\}},$$

the uniqueness implies that the extension, which we again denote by the symbol $\mu^{\square}$, is invariant with respect to these transformations. Thus, $(X^4, \mathcal{X}^{\otimes 4}, \mu^{\square}, T_{\square,1}, T_{\square,2})$ is an extension of X under the projection $\pi_{1,2}$. Moreover, $\int \otimes_\epsilon f_\epsilon\, d\mu^{\square}$ is given by the formula (3.4).

Suppose now that $f_{1,2} \perp \mathcal{I}_2 X$. By the satedness assumption, the function $f_{1,2} \circ \pi_{1,2}$ on the cube extension is orthogonal to $I_{\square,1} \vee I_{\square,2}$ under $\mu^{\square}$. On the other hand, if $\epsilon \subsetneq \{1, 2\}$, then $f_\epsilon \circ \pi_\epsilon$ is $I_{\square,i}$-measurable for any $i \notin \epsilon$, so that

$$0 = \int \prod_{\epsilon} f_\epsilon \circ \pi_\epsilon\, d\mu^{\square} = \lim_n \iiint \prod_{\epsilon} T_\epsilon^{\vec{g}} f_\epsilon\, d\mu\, d\Phi_n(g_1)\, d\Psi_n(g_2).$$

Since this limit does not depend on the subsequence of $\Phi \times \Psi$ that was chosen at the beginning, a subsubsequence argument shows that this limit in fact exists and vanishes for the original Reiter sequences.

On the other hand, in the case that $f_{1,2}$ is $\mathcal{I}_2 X$-measurable, by density and linearity, it suffices to consider $f_{1,2} = h_1 h_2$, where each h_i is T_i-invariant. In this case, we obtain

$$\begin{aligned}
&\lim_n \iiint \prod_{\epsilon} T_\epsilon^{\vec{g}} f_\epsilon\, d\mu\, d\Phi_n(g_1)\, d\Psi_n(g_2)\\
&\quad = \lim_n \iiint T_1^{g_1}(f_1 h_2) T_2^{g_2}(f_2 h_1) f_\emptyset\, d\mu\, d\Phi_n(g_1)\, d\Psi_n(g_2)\\
&\quad = \int \mathbb{E}(f_1 h_2 | I_1)\mathbb{E}(f_2 h_1 | I_2) f_\emptyset\, d\mu,
\end{aligned}$$

by the mean ergodic theorem. This limit is manifestly independent of the Reiter sequences.

In the general case of a not necessarily $\mathcal{I}_2$-sated system X, we use Theorem 3.2 to pass to a sated extension and note that the existence of the limit equation (3.4) for functions on this extension implies the existence of that limit for functions on X. □

Thus, we obtain a measure $\mu^{\square}$ on X^4 and two measure-preserving G-actions on $(X^4, \mu^{\square})$ such that the resulting measure-preserving system is an extension of X. This explicitly constructed extension allows us to exploit satedness. Specifically, we aim at obtaining systems with the following property.

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Definition 3.5. We call a system $X \in \mathcal{C}_2$ *magic* if $A(X|I_2, T_1) = I_1 \vee I_2$ (recall that $A(X|I_2, T_1)$ was defined in Theorem 2.4).

An equivalent notion was first introduced by Host [**17**] for commutative G (in fact he introduced a corresponding notion for k-tuples of commuting $\mathbb{Z}$-actions for every $k \in \mathbb{N}$). The next proposition is our main tool for exploiting information about characteristic factors.

PROPOSITION 3.6. *Suppose that $X \in \mathcal{C}_2$ is $\mathcal{I}_2$-sated. Then X is magic.*

Proof. The inclusion $A(X|I_2, T_1) \supseteq I_1 \vee I_2$ holds in any measure-preserving system. For the converse, consider $f_1 \perp I_1 \vee I_2$. Then, for every $f_2 \in L^\infty(X)$, we have

$$\limsup_n \int_h \|\mathbb{E}(f_2 T_1^h f_1 | I_2)\|^2 \, dF_n(h)$$
$$= \limsup_n \int_h \lim_m \int_g \int f_2 T_1^h f_1 \cdot T_2^g (f_2 T_1^h f_1) \, d\mu \, dF_m(g) \, dF_n(h).$$

This vanishes by Lemma 3.3. Thus, $f_1 \perp A(X|I_2, T_1)$. □

In the remaining part of this section, we extend Griesmer's cubic convergence result [**15**, Theorem 1.4(1)] to arbitrary left Reiter sequences.

LEMMA 3.7. *Let $X \in \mathcal{C}_2$ and $f_\epsilon \in L^\infty(X)$, $\epsilon \subseteq \{1, 2\}$. Suppose that $f_1 \perp A(X|I_2, T_1)$ or $f_2 \perp A(X|I_1, T_2)$. Then, for any left Reiter sequences Φ, Ψ on G, we have*

$$\lim_n \iint \prod_{\epsilon \subseteq \{1,2\}} T_\epsilon^{\vec{g}} f_\epsilon \, d\Phi_n(g_1) \, d\Psi_n(g_2) = 0 \quad \text{in } L^2(X).$$

Proof. Since the $\epsilon = \emptyset$ term is a bounded function that does not depend on g_1, g_2, we may discard it. We apply Corollary 2.2 to the map $(g_1, g_2) \mapsto \prod_{\epsilon \neq \emptyset} T_\epsilon^{\vec{g}} f_\epsilon$. To show that it converges to zero in the Cesàro sense along $\Phi \times \Psi$, it thus suffices to show that

$$\liminf_n \limsup_m \iiint \prod_{\epsilon \neq \emptyset} T_\epsilon^{\vec{g}} (f_\epsilon T_\epsilon^{\vec{h}} f_\epsilon) \, d\mu \, d(\Phi_m \times \Psi_m)(g_1, g_2) \, d(\Phi'_n \times \Psi'_n)(h_1, h_2) = 0.$$

By Lemma 3.3, the limit superior in m is actually a limit and it does not depend on Φ, Ψ. Thus, we may replace Φ, Ψ by two-sided Reiter sequences (this is how we remove the two-sidedness assumption from Griesmer's convergence result). The double limit equals

$$\liminf_n \lim_m \iiint T_1^{g_1^{-1}} T_2^{g_2^{-1}}$$
$$\times \prod_{\epsilon \neq \emptyset} T_\epsilon^{\vec{g}} (f_\epsilon T_\epsilon^{\vec{h}} f_\epsilon) \, d\mu \, d(\Phi_m \times \Psi_m)(g_1, g_2) \, d(\Phi'_n \times \Psi'_n)(h_1, h_2),$$

and, by the mean ergodic theorem, this equals

$$\liminf_n \iint \mathbb{E}(f_1 T_1^{h_1} f_1 | I_2) \mathbb{E}(f_2 T_2^{h_2} f_2 | I_1) f_{1,2} T_{1,2}^{\vec{h}} f_{1,2} \, d\mu \, d(\Phi'_n \times \Psi'_n)(h_1, h_2).$$

By the Cauchy–Schwarz inequality, this is bounded by

$$\liminf_n \|f_{1,2}\|_\infty^2 \int \|\mathbb{E}(f_1 T_1^{h_1} f_1 | I_2)\|_2 \, d\Phi'_n(h_1) \int \|\mathbb{E}(f_2 T_2^{h_2} f_2 | I_1)\|_2 \, d\Psi'_n(h_2),$$

and this vanishes by the assumption. □

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COROLLARY 3.8. *Suppose that $X \in \mathcal{C}_2$. Then, for any $f_\epsilon \in L^\infty(X)$, $\epsilon \subseteq \{1, 2\}$, and any left Reiter sequences Φ, Ψ, the limit*

$$\lim_n \iint \prod_{\epsilon \subseteq \{1,2\}} T_\epsilon^{\vec{g}} f_\epsilon \, d\Phi_n(g_1) \, d\Psi_n(g_2)$$

exists in $L^2(X)$ and does not depend on Φ, Ψ.

Proof. By Theorem 3.2, we may assume that X is $\mathcal{I}_2$-sated. Then the system (X, μ, T_2, T_1) (in which the roles of the two actions were interchanged) is also $\mathcal{I}_2$-sated. By Proposition 3.6, it follows that $I_1 \vee I_2 = A(X|I_1, T_2) = A(X|I_2, T_1)$. By Lemma 3.7, we may assume that the functions f_1 and f_2 are measurable with respect to the σ-algebra $I_1 \vee I_2$. By density and linearity, we may assume that $f_2 = h_2^1 h_2^2$ and $f_1 = h_1^1 h_1^2$, with h_ϵ^i being T_i-invariant. In this case, we have

$$\iint \prod_{\epsilon \subseteq \{1,2\}} T_\epsilon^{\vec{g}} f_\epsilon \, d\Phi_n(g_1) \, d\Psi_n(g_2) = \iint f_\emptyset h_1^1 h_2^2 T_{1,2}^{\vec{g}}(f_{1,2} h_1^2 h_2^1) \, d\Phi_n(g_1) \, d\Psi_n(g_2),$$

and the conclusion follows from the mean ergodic theorem. □

4. *Furstenberg averages with almost periodic weights*

Recall that the right shift of a function f on G by an element g is defined by $R_g f(h) := f(hg)$ and the left shift by $L_g f(h) := f(g^{-1}h)$. The left and the right shifts are commuting G-actions. A continuous function $f : G \to \mathbb{C}$ is called *almost periodic* if the set $L_G R_G f = \{L_g R_{g'} f : g, g' \in G\}$ is totally bounded with respect to the metric induced by the supremum norm. The set of continuous almost periodic functions is denoted by $\mathrm{AP}(G)$; it is a closed conjugation invariant subalgebra of the space of bounded continuous functions on G.

LEMMA 4.1. *The left and the right shift are jointly continuous on $G \times \mathrm{AP}(G)$.*

Proof. We will show that R is jointly continuous at every point $(g_0, f_0) \in G \times \mathrm{AP}(G)$; the proof for L is nearly identical. Let $\epsilon > 0$. By definition of $\mathrm{AP}(G)$, there exists a finite ϵ-dense subset $F \subset L_G f_0$. Since $F \subset C(X)$, there exists a neighborhood U of g_0 such that $|f(g) - f(g_0)| < \epsilon$, for every $g \in U$ and every $f \in F$. Let $f' \in \mathrm{AP}(G)$ be such that $\|f' - f_0\|_\infty < \epsilon$. Then, for every $g' \in G$ and $g \in U$, we have

$$\begin{aligned} |R_g f'(g') - R_{g_0} f_0(g')| &\le |R_g f'(g') - R_g f_0(g')| + |R_g f_0(g') - R_{g_0} f_0(g')| \\ &\le \epsilon + |L_{(g')^{-1}} f_0(g) - L_{(g')^{-1}} f_0(g_0)| \le 3\epsilon + |f(g) - f(g_0)| \le 4\epsilon, \end{aligned}$$

for some $f \in F$. □

We recall a consequence of the Peter–Weyl theorem.

THEOREM 4.2. *Let $f \in C(G)$. Then the following conditions are equivalent.*

(1) $f \in \mathrm{AP}(G)$.
(2) *There exists a compact group K and a continuous homomorphism $\iota : G \to K$ such that $f = f' \circ \iota$ for some $f' \in C(K)$.*

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(3) *f is a uniform limit of matrix coefficients, that is, functions of the form $g \mapsto \langle \pi(g)v, w\rangle$, where $\pi : G \to U(d)$ is a continuous finite-dimensional representation and $v, w \in \mathbb{C}^d$.*

(4) *f is a uniform limit of functions of the form $\chi(g) = \langle w, \pi(g)v\rangle$, where $\pi : G \to U(d)$ is a measurable antihomomorphism and $v, w \in \mathbb{C}^d$.*

Proof. If (2) holds, then $L_G R_G f$ is totally bounded as an isometric image of a subset of $L_K R_K f'$, and this shows (1). Conversely, if (1) holds, then $X := \overline{R_G f} \subset \mathrm{AP}(G)$ is a compact metric space, and, by Lemma 4.1, the G-action R is jointly continuous on X. Since R_g is isometric for each $g \in G$, and by [**2**, §3, Theorem 2], we obtain a compactification $\iota : G \to K$ and a K-action R' on X such that $R_g = R'_{\iota(g)}$, for all $g \in G$. Let $e : X \to \mathbb{C}$ be the evaluation at the identity. Then $f'(k) := e(R'_k f)$ is a continuous function on K that extends f.

Suppose now that f is a matrix coefficient associated to a representation π. Then $f \in \mathrm{AP}(G)$, since (2) is satisfied with $\iota = \pi$. Since $\mathrm{AP}(G)$ is closed, this shows that (3) implies (1). Conversely, suppose that (2) holds. Then f' is a uniform limit of matrix coefficients on K, by [**11**, Theorem 5.11]. On the other hand, if χ is a matrix coefficient on K, then $\chi \circ \iota$ is a matrix coefficient on G, so we obtain (3).

Finally, it is clear that (3) implies (4). Conversely, every measurable homomorphism $G \to U(d)$ is continuous [**16**, Theorem 22.18], so that (4) implies (3). □

COROLLARY 4.3. *For every $\chi \in AP(G)$, the uniform Cesàro limit* $\mathrm{UC\text{-}lim}_g\, \chi(g)$ *exists. If χ is positive and not identically zero, then* $\mathrm{UC\text{-}lim}_g\, \chi(g) > 0$.

Proof. By Theorem 4.2, we have $\chi = f \circ \iota$ for some compactification $\iota : G \to K$ and some $f \in C(K)$. Since $\iota(G)$ is dense in K, the only G-invariant measure on K is the Haar measure ν. Hence, $\mathrm{UC\text{-}lim}_g\, \chi_g = \mathrm{UC\text{-}lim}_g\, f \circ \iota(g)$ exists and equals $\int_K f\, d\nu$, by the ergodic theorem for uniquely ergodic actions.

If χ is positive and not identically zero, then the same is true of f, so $\int_X f\, d\nu > 0$, since ν has full support. □

PROPOSITION 4.4. *Let $X \in \mathcal{C}_2$, $f_1, f_2 \in L^\infty(X)$, and let $\chi \in \mathrm{AP}(G)$. Assume that $f_1 \perp A(X|I_2, T_1)$. Then we have*

$$\mathrm{UC\text{-}lim}_g\, \chi(g) T_1^g f_1 T_{1,2}^g f_2 = 0 \qquad \textit{in norm of } L^2(X).$$

Proof. Fix a left Reiter sequence F. By Theorem 4.2, we may assume that $\chi(g) = \langle w, \pi(g)v\rangle$, where $\pi : G \to U(d)$ is a measurable antihomomorphism and $v, w \in \mathbb{C}^d$. In this case, it suffices to prove $\mathrm{C\text{-}lim}_g\, \pi(g)v \otimes u_g = 0$ in $\mathbb{C}^d \otimes L^2(X)$, where $u_g := T_1^g f_1 T_{1,2}^g f_2$. By the van der Corput inequality (Corollary 2.2), it suffices to show that

$$\liminf_H \int_h \limsup_N \int_g \langle \pi(g)v, \pi(hg)v\rangle \int u_g u_{hg}\, d\mu\, dF_N(g)\, dF'_H(h) = 0,$$

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for a certain two-sided Reiter sequence F'. Since π is an antihomomorphism, this can be written as

$$\begin{aligned}&\liminf_H \int_h \limsup_N \int_g \langle v, \pi(h)v\rangle \int f_1 T_2^g f_2 T_1^h f_1 T_2^g T_{1,2}^h f_2 \, d\mu \, dF_N(g) \, dF'_H(h)\\ &\le \|v\|^2 \liminf_H \int_h \limsup_N \int_g \int f_1 T_2^g f_2 T_1^h f_1 T_2^g T_{1,2}^h f_2 \, d\mu \, dF_N(g) \, dF'_H(h)\\ &= \|v\|^2 \liminf_H \int_h \int f_1 T_1^h f_1 \mathbb{E}(f_2 T_{1,2}^h f_2 | I_2) \, d\mu \, dF'_H(h).\end{aligned}$$

Since the conditional expectation is an orthogonal projection, this is bounded by

$$\|v\|^2 \|f_2\|_\infty^2 \liminf_H \int_h \|\mathbb{E}(f_1 T_1^h f_1 | I_2)\|_2 \, dF'_H(h).$$

This vanishes by the assumption. □

COROLLARY 4.5. *Let $X \in \mathcal{C}_2$, $f_1, f_2 \in L^\infty(X)$, and let $\chi \in \mathrm{AP}(G)$. Then* $\mathrm{UC\text{-}lim}_g \chi(g) T_1^g f_1 T_{1,2}^g f_2$ *exists in the norm of* $L^2(X)$.

Proof. By Theorem 3.2 and Proposition 3.6, we may assume that X is magic. By Proposition 4.4, the above limit vanishes if $f_1 \perp I_1 \vee I_2$. Hence, by density and linearity, it suffices to consider $f_1 = h_1 h_2$, where h_i is T_i-invariant. In this case, we have

$$\chi(g) T_1^g f_1 T_{1,2}^g f_2 = h_1 \chi(g) T_{1,2}^g (f_2 h_2).$$

By Theorem 4.2, we may assume that $\chi(g) = \langle w, \pi(g)v\rangle$ for a measurable antihomomorphism $\pi : G \to U(d)$. The conclusion follows because the map $g \mapsto \pi(g)v \otimes T_{1,2}^g(f_2 h_2)$ converges in the uniform Cesàro sense in $\mathbb{C}^d \otimes L^2(X)$, by the mean ergodic theorem applied to the antirepresentation $\pi \otimes T_{1,2}$. □

Here and later, write $K_\epsilon = K(T_\epsilon) = A(X|\text{trivial}, T_\epsilon)$ for the factor spanned by the finite-dimensional T_ϵ-invariant subspaces of $L^2(X)$ for $\epsilon \subseteq \{1, 2\}$. These factors are used as building blocks for characteristic factors for weighted Furstenberg averages.

COROLLARY 4.6. *Let $X \in \mathcal{C}_2$, $f_0, f_1, f_2 \in L^\infty(X)$, and $\chi \in \mathrm{AP}(G)$. Suppose that X is magic. Then*

$$\begin{aligned}&\mathrm{UC\text{-}lim}_g \chi(g) f_0 T_1^g f_1 T_{1,2}^g f_2\\ &\qquad = \mathrm{UC\text{-}lim}_g \chi(g) \mathbb{E}(f_0 | I_1 \vee K_{1,2}) T_1^g \mathbb{E}(f_1 | I_1 \vee I_2) T_{1,2}^g \mathbb{E}(f_2 | I_2 \vee K_{1,2}).\end{aligned}$$

Proof. By Corollary 4.5, the uniform Cesàro limits on both sides exist in $L^2(X)$. By Proposition 4.4, we may assume that f_1 is $A(X|I_1, T_2)$-measurable, and hence $I_1 \vee I_2$-measurable by definition of a magic system. By density and linearity, it suffices to consider $f_1 = h_1 h_2$, where h_j is T_j-invariant. In this case, we have

$$\chi(g) f_0 T_1^g (h_1 h_2) T_{1,2}^g f_2 = \chi(g) f_0 h_1 T_{1,2}^g (f_2 h_2).$$

Suppose now that $f_2 \perp I_2 \vee K_{1,2}$, so that $f_2 h_2 \perp K_{1,2}$, and fix a left Reiter sequence F. By Theorem 4.2, we have $\chi(g) = \kappa(\iota(g))$ for some compactification $\iota : G \to K$ and some

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$\kappa \in C(K)$. We have a G-action on K by left translation by ι. By Theorem 2.4, we obtain $\kappa \otimes f_2 h_2 \perp A(K \times X|\text{trivial}, \iota \times T_{1,2})$. In particular,

$$\operatorname*{C-lim}_g \kappa(\iota(g)\cdot) T_{1,2}^g(f_2 h_2) = 0$$

in $L^2(K \times X)$. Passing to a subsequence of our Reiter sequence, we obtain

$$\operatorname*{C-lim}_g \kappa(\iota(g)k) T_{1,2}^g(f_2 h_2) = 0$$

in $L^2(X)$ for almost every $k \in K$.By uniform continuity, the same actually holds for every $k \in K$, and, substituting $k = \mathrm{id}_K$, we obtain $\text{C-lim}_g\, \chi(g) T_{1,2}^g(f_2 h_2) = 0$. Since this limit does not depend on the subsequence, a subsubsequence argument shows that $\text{UC-lim}_g\, \chi(g) T_{1,2}^g(f_2 h_2) = 0$.

The remaining case $f_0 \perp I_1 \vee K_{1,2}$ can be handled similarly. □

5. *An almost periodic correlation function and the recurrence theorem*

Before embarking on the proof of our recurrence theorem, we state two lemmas that facilitate calculation of integrals. The first of them concerns relative independence, while the second deals with a certain trilinear form.

LEMMA 5.1. *Suppose that $X \in \mathcal{C}_2$. Then the σ-algebras I_1, I_2 are relatively independent over $I_1 \wedge I_2$.*

Proof. Let f_i be T_i-invariant and fix a left Reiter sequence F. By the mean ergodic theorem, for the $G \times G$-action (T_1, T_2), we have

$$\begin{aligned}\mathbb{E}(f_1 f_2 | I_1 \wedge I_2) &= \lim_n \iint T_1^g T_2^h f_1 f_2 \, dF_n(g)\, dF_n(h) \\ &= \lim_n \int T_2^h f_1 \, dF_n(h) \int T_1^g f_2 \, dF_n(g) = \mathbb{E}(f_1|I_2)\mathbb{E}(f_2|I_1).\end{aligned}$$

Since T_1 and T_2 commute, the conditional expectation operators onto I_1 and I_2 commute as well, so the above conditional expectations equal those on $I_1 \wedge I_2$. □

Similarly, one can show relative independence over $I_1 \wedge I_2$ for the pairs I_1, $I_{1,2}$ and I_2, $I_{1,2}$. For instance, for the first pair we obtain

$$\begin{aligned}\mathbb{E}(f_1 f_{1,2} | I_1 \wedge I_2) &= \lim_{n,m} \iint T_1^g T_2^h f_1 f_{1,2} \, dF_n(g)\, dF_m(h) \\ &= \lim_{n,m} \int T_2^h f_1 \int T_1^g T_2^h T_{1,2}^{h^{-1}} f_{1,2} \, dF_m(g)\, dF_n(h) \\ &= \lim_{n,m} \int T_2^h f_1 \int T_1^{h^{-1}g} f_{1,2} \, dF_m(g)\, dF_n(h) \\ &= \lim_n \int T_2^h f_1 \mathbb{E}(f_{1,2}|I_1) \, dF_n(h) \\ &= \mathbb{E}(f_1|I_2)\mathbb{E}(f_{1,2}|I_1).\end{aligned}$$

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Definition 5.2. The subcategory $\mathcal{C}_k^{\mathrm{erg}}$ of $\mathcal{C}_k$ consists of ergodic systems, that is, systems for which the σ-algebra $I_1 \wedge \cdots \wedge I_k$ is trivial.

LEMMA 5.3. *Let $X \in \mathcal{C}_2^{\mathrm{erg}}$ and $f_0, f_1, f_2 \in L^\infty(X)$. Suppose that $f_0 \in L^\infty(K_{1,2})$, $f_1 \in L^\infty(I_1)$, and $f_2 \in L^\infty(I_2)$. Then*

$$\int f_0 f_1 f_2 \, d\mu = \int f_0 \mathbb{E}(f_1|I_1 \wedge K_2)\mathbb{E}(f_2|I_2 \wedge K_1) \, d\mu.$$

Proof. We can clearly replace f_0 by $\mathbb{E}(f_0|I_1 \vee I_2)$ on both sides. Since $I_1 \vee I_2$ is $T_{1,2}$-invariant, the conditional expectation $\mathbb{E}(\cdot, I_1 \vee I_2)$ commutes with $T_{1,2}$. Therefore this conditional expectation maps finite-dimensional $T_{1,2}$-invariant subspaces to finite-dimensional $T_{1,2}$-invariant subspaces. It follows that $\mathbb{E}(f_0|I_1 \vee I_2)$ is $K_{1,2}$-measurable, so we may assume that the function f_0 is measurable with respect to the σ-algebra $(I_1 \vee I_2) \wedge K_{1,2} = A(I_1 \vee I_2|\text{trivial}, T_{1,2})$.

Since I_1 and I_2 are independent by Lemma 5.1 we have

$$(X, I_1 \vee I_2, T_{1,2}) \cong (X, I_1, T_{1,2}) \times (X, I_2, T_{1,2}) = (X, I_1, T_2) \times (X, I_2, T_1),$$

so that

$$A(I_1 \vee I_2|\text{trivial}, T_{1,2}) = A(I_1|\text{trivial}, T_2) \otimes A(I_2|\text{trivial}, T_1),$$

by Theorem 2.4. Therefore, we obtain $\int f_0 f_1 f_2 \, d\mu = 0$ if $f_1 \perp I_1 \wedge K_2 = A(I_1|\text{trivial}, T_2)$ or $f_2 \perp I_2 \wedge K_1 = A(I_2|\text{trivial}, T_1)$. □

The next result is central to our approach of establishing a lower bound for weighted ergodic averages. This is the place where the almost periodic function that will be used to construct the appropriate weight first arises.

LEMMA 5.4. *Let $X \in \mathcal{C}_2^{\mathrm{erg}}$ and $f_0, f_1, f_2 \in L^\infty(X)$. Suppose $f_0 \in L^\infty(I_1 \vee K_{1,2})$, $f_1 \in L^\infty(I_1 \vee I_2)$, and $f_2 \in L^\infty(I_2 \vee K_{1,2})$. Then the function*

$$g \mapsto \int f_0 T_1^g f_1 T_{1,2}^g f_2 \, d\mu$$

is almost periodic.

Proof. By density and linearity, we may assume that $f_0 = r_1 r_{1,2}$, $f_1 = s_1 s_2$, $f_2 = t_2 t_{1,2}$, where $r_1, s_1 \in L^\infty(I_1)$, $s_2, t_2 \in L^\infty(I_2)$, and $r_{1,2}, t_{1,2} \in L^\infty(K_{1,2})$. We have

$$\int r_1 r_{1,2} T_1^g (s_1 s_2) T_{1,2}^g (t_2 t_{1,2}) \, d\mu = \int r_1 s_1 T_1^g (s_2 t_2) \cdot r_{1,2} T_{1,2}^g (t_{1,2}) \, d\mu.$$

By Lemma 5.3, this equals

$$\int \mathbb{E}(r_1 s_1|I_1 \wedge K_2) \cdot T_1^g \mathbb{E}(s_2 t_2|I_2 \wedge K_1) \cdot r_{1,2} T_{1,2}^g (t_{1,2}) \, d\mu = \int h_0 T_1^g h_1 T_{1,2}^g h_2 \, d\mu,$$

for some $h_0 \in L^\infty(X)$, $h_1 \in L^\infty(K_1)$, and $h_2 \in L^\infty(K_{1,2})$. In view of Theorem 2.4, by density and linearity, we may assume that $h_1 \in H_1$ and $h_2 \in H_{1,2}$, where H_i are finite-dimensional T_i-invariant subspaces of $L^2(X)$. Let $\{b_i^j\}_j$ be orthonormal bases

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of H_i. Then we have $T_i^g b_i^j = \sum_l \pi_i(g)_l^j b_i^l$ with measurable antihomomorphisms $\pi_i : G \to U(\dim H_i)$. Writing $h_i = \sum_j a_i^j b_i^j$, we obtain

$$\begin{aligned}\int h_0 T_1^g h_1 T_{1,2}^g h_2 \, d\mu &= \int h_0 \sum_{l,j} \pi_1(g)_l^j a_1^j b_1^l \sum_{l',j'} \pi_{1,2}(g)_{l'}^{j'} a_{1,2}^{j'} b_{1,2}^{l'} \, d\mu \\ &= \sum_{l,j,l',j'} \pi_1(g)_l^j \pi_{1,2}(g)_{l'}^{j'} a_1^j a_{1,2}^{j'} \int h_0 b_1^l b_{1,2}^{l'} \, d\mu,\end{aligned}$$

and this is a matrix coefficient function. □

Finally, we need to ensure the existence of ergodic magic extensions.

LEMMA 5.5. *Suppose that $X \in \mathcal{C}_k^{\mathrm{erg}}$ is $\mathcal{I}_k$-sated in $\mathcal{C}_k^{\mathrm{erg}}$. Then X is also $\mathcal{I}_k$-sated in $\mathcal{C}_k$.*

Proof. Let $(Y, \nu) \to (X, \mu)$ be an extension and let $f' \in L^\infty(\mathcal{I}_k Y)$, $f \in L^\infty(X)$ be bounded. Also let $\nu = \int \nu_y \, d\nu(y)$ be the ergodic decomposition of ν. We have $f' = \lim_{n\to\infty} f'_n$ in $L^2(\nu)$, where each f'_n is a finite linear combination of products of bounded T_i-invariant functions for $i = 1, \ldots, k$. Passing to a subsequence, we may assume that the same is true in $L^2(\nu_y)$ for almost every y, so that $f' \in L^\infty(\mathcal{I}_k(Y, \nu_y))$ for almost every y.

Since almost every (Y, ν_y) is an (ergodic) extension of (X, μ), by satedness of X in $\mathcal{C}_k^{\mathrm{erg}}$, this implies that

$$\int f' f \, d\nu_y = \int f' \mathbb{E}(f|\mathcal{I}_k X) \, d\nu_y.$$

Integrating over $y \in Y$, we obtain

$$\int f' f \, d\nu = \int f' \mathbb{E}(f|\mathcal{I}_k X) \, d\nu,$$

and, since f' and f were arbitrary, this shows that $\mathcal{I}_k Y$ and X are relatively independent over $\mathcal{I}_k X$. □

COROLLARY 5.6. *Every $X \in \mathcal{C}_2^{\mathrm{erg}}$ admits an ergodic magic extension.*

Proof. The class $\mathcal{C}_2^{\mathrm{erg}}$ is clearly closed under inverse limits, so we may apply Theorem 3.2 with $\mathcal{C} = \mathcal{C}_2^{\mathrm{erg}}$. The resulting system is $\mathcal{I}_2$-sated, by Lemma 5.5, so it is magic, by Proposition 3.6. □

Now all the tools required for the proof of our main result have been made available. We proceed with its formulation and proof.

THEOREM 5.7. *Let $X \in \mathcal{C}_2^{\mathrm{erg}}$ and let $f : X \to [0, 1]$ be a measurable function. Then, for every $\epsilon > 0$, there exists an almost periodic function $\chi : G \to \mathbb{R}_{\geq 0}$ such that* $\mathrm{UC\text{-}lim}_g \chi(g) = 1$ *and*

$$\mathrm{UC\text{-}lim}_g \chi(g) \int f T_1^g f T_{1,2}^g f \, d\mu \geq \left(\int f \, d\mu \right)^4 - \epsilon.$$

The idea to use an almost periodic weight in order to obtain a lower bound for multiple ergodic averages first appeared in the work of Frantzikinakis [**12**]. Since the almost periodic function χ is necessarily bounded, Theorem 5.7 implies, in particular, that the uniform Cesàro limit of $\int f T_1^g f T_{1,2}^g f \, d\mu$ is positive if $f \not\equiv 0$.

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Proof. By Corollary 5.6, we may assume that X is magic. Let $\epsilon > 0$ be arbitrary. By Lemma 5.4, the function

$$\kappa(g) := \int \mathbb{E}(f|I_1 \vee K_{1,2}) T_1^g \mathbb{E}(f|I_1 \vee I_2) T_{1,2}^g \mathbb{E}(f|I_2 \vee K_{1,2})\, d\mu$$

is almost periodic. Note that we have

$$\kappa(\mathrm{id}_G) \geq \int f \mathbb{E}(f|I_1 \vee K_{1,2}) \mathbb{E}(f|I_1 \vee I_2) \mathbb{E}(f|I_2 \vee K_{1,2})\, d\mu \geq \left(\int f\, d\mu\right)^4 =: B,$$

by [**9**, Lemma 1.6]. Let $\phi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a continuous function such that $\phi(x) = 0$ if $x \leq B - \epsilon$, and $\phi(x) = 1$ if $x \geq B$. Since $\mathrm{AP}(G)$ is a closed algebra, and by the Weierstraß approximation theorem, the function $\phi \circ \kappa$ is in $\mathrm{AP}(G)$. Moreover, this function is positive and equals 1 at the identity, so $\text{UC-lim}_g\, \phi(\kappa(g)) > 0$, by Corollary 4.3. Let $\chi := (\text{UC-lim}_g\, \phi(\kappa(g)))^{-1} \phi \circ \kappa$. Note that $\chi \cdot \kappa \geq \chi \cdot (B - \epsilon)$. By Corollary 4.6, this implies

$$\begin{aligned} \operatorname*{UC-lim}_g \chi(g) \int f T_1^g f T_{1,2}^g f\, d\mu &= \operatorname*{UC-lim}_g \chi(g)\kappa(g) \geq \operatorname*{UC-lim}_g \chi(g)(B - \epsilon) \\ &= B - \epsilon. \end{aligned}$$

□

We note that the above proof also yields a generalization of [**9**, Theorem 1.3] to actions of amenable groups. Indeed, if $\mathcal{X} = I_1 \vee I_2$ and $f \in L^\infty(X)$, then $\mathbb{E}(f|I_1 \vee I_2) = f$ on any extension of X. Therefore, [**9**, Lemma 1.6] in fact gives $\kappa(\mathrm{id}) \geq (\int f\, d\mu)^3$ in that case.

Proof of Theorem 1.1. Suppose first that, for some $\epsilon > 0$, the set R_ϵ is not left syndetic. Then there exists a left Følner sequence F in G none of whose members intersects R_ϵ. Consider the matrix coefficient function χ, given by Theorem 5.7, with $f = 1_A$ and $\epsilon/2$ in place of ϵ. By the assumption, we have

$$\chi(g) \int f T_1^g f T_{1,2}^g f \leq \chi(g)\left(\left(\int f\, d\mu\right)^4 - \epsilon\right),$$

for every $g \in F_N$ for every N, contradicting the conclusion of Theorem 5.7.

In order to see that R_ϵ is also right syndetic, it suffices to note that

$$\begin{aligned} R_\epsilon^{-1} &= \{g : \mu(A \cap T_1^{g^{-1}} A \cap T_{1,2}^{g^{-1}} A) \geq \mu(A)^4 - \epsilon\} \\ &= \{g : \mu(T_{1,2}^g A \cap T_2^g A \cap A) \geq \mu(A)^4 - \epsilon\}, \end{aligned}$$

and this set is left syndetic by the above argument, with the roles of T_1 and T_2 reversed. □

6. *Combinatorial application*

Since the lower bound in our multiple recurrence theorem only holds for ergodic systems, we need an appropriate version of the Furstenberg correspondence principle.

Recall that a point x of a compact metric space X is called *quasi-generic* with respect to a probability measure μ for a continuous action T of G on X if there exists a left Følner sequence such that, for every $f \in C(X)$, we have $\int f\, d\mu = \lim_N \int f(T^g x)\, dF_N(g)$. It follows from the mean ergodic theorem that if μ is ergodic, then μ-almost every point is quasi-generic. As pointed out by Furstenberg, this implies the following version of [**13**, Proposition 3.9], with an identical proof.

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PROPOSITION 6.1. *Let X be a compact metric space and T a continuous G-action. Then every point $x_0 \in X$ such that $\overline{T^G x_0} = X$ is quasi-generic for every ergodic invariant probability measure on X.*

This can be used to prove a version of the Furstenberg correspondence principle [**6**, Theorem 4.17], which provides an ergodic measure-preserving system.

LEMMA 6.2. *Let G be a countable amenable group and $E \subset G$. Then there exists an ergodic measure-preserving system (X, μ, T) and a measurable subset $A \subset X$ with $\mu(A) \geq \overline{d}(E)$ such that*

$$\mu(T^{g_1}A \cap \cdots \cap T^{g_k}A) \leq \overline{d}(g_1E \cap \cdots \cap g_kE),$$

for any $k \in \mathbb{N}$ and $g_1, \ldots, g_k \in G$.

The $G = \mathbb{Z}$ case of Lemma 6.2 was first used by Bergelson, Host, and Kra [**7**, Proposition 3.1], who got the idea from Lesigne.

Proof. If $\overline{d}(E) = 0$, then we can consider a one-point system X and take A to be the empty set. Thus, we may assume that E has positive upper Banach density.

Consider $X' = \{0, 1\}^G$ with the product topology and the left G-action $(T^g x)_h = x_{hg}$. Let $e \in X'$ be the indicator function of E, and set $X := \overline{T^G e}$. Also set $A := \{x \in X : x_{\mathrm{id}} = 1\}$. Let F be a left Følner sequence in G such that $\lim_N (|E \cap F_N|/|F_N|) = \overline{d}(E)$. Passing to a subsequence, we may assume that the sequence of measures $(\int \delta_{T^g e}\, dF_N(g))_N$ converges weakly. Its limit ν is a T-invariant probability measure supported on X such that $\nu(A) = \overline{d}(E)$. By the ergodic decomposition, there exists an ergodic T-invariant probability measure μ on X such that $\mu(A) \geq \nu(A)$.

By Proposition 6.1, the point e is quasi-generic for μ. Let Φ be a left Følner sequence that witnesses the quasi-genericity. Since the set $T^{g_1}A \cap \cdots \cap T^{g_k}A$ is clopen, we have

$$\begin{aligned}
\mu(T^{g_1}A \cap \cdots \cap T^{g_k}A) &= \lim_N \int 1_{T^{g_1}A \cap \cdots \cap T^{g_k}A}(T^g e)\, d\Phi_N(g) \\
&= \lim_N \int \prod_{i=1}^{k} 1_A(T^{g_i^{-1}g} e)\, d\Phi_N(g) \\
&= \lim_N |\Phi_N|^{-1} |\{g \in \Phi_N : g_i^{-1} g \in E,\ i \leq k\}| \\
&= \lim_N |\Phi_N|^{-1} |\Phi_N \cap g_1E \cap \cdots \cap g_kE| \\
&\leq \overline{d}(g_1E \cap \cdots \cap g_kE). \qquad \square
\end{aligned}$$

Proof of Theorem 1.2. Let (X, μ, T) and $A \subset X$ be the ergodic system and the measurable subset obtained by applying Lemma 6.2 to $E \subset G \times G$. Writing $T_1^g := T^{(g,\mathrm{id})}$ and $T_2^g := T^{(\mathrm{id},g)}$, we have $\overline{d}(E \cap (g, \mathrm{id})E \cap (g, g)E) \geq \mu(A \cap T_1^g A \cap T_{1,2}^g A)$. By Theorem 1.1, the latter quantity is bounded below by $\mu(A)^4 - \epsilon$ for a set of g that is both left and right syndetic. On the other hand, $\mu(A)^4 - \epsilon \geq \overline{d}(E)^4 - \epsilon$, and we obtain the claim. $\square$

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A. *Appendix. Mean convergence for $k = 3$*

In this appendix, we give a new proof of the $k = 3$ case of [**23**, Theorem 1.1(2)], using the machinery of sated extensions.

PROPOSITION A.1. *Suppose that $X \in \mathcal{C}_3$ and $f_1,\ f_2,\ f_3 \in L^\infty(X)$. Then the limit*

$$\operatorname*{UC-lim}_g T_1^g f_1 T_{1,2}^g f_2 T_{1,2,3}^g f_3$$

exists in $L^2(X)$.

In order to use a satedness argument, we have to construct an interesting extension of a system $X \in \mathcal{C}_3$. We consider the *Furstenberg coupling* X^F, which consists of the following data. The base space is X^3 and the measure is given by

$$\int f_0 \otimes f_1 \otimes f_2 \, d\mu^F := \operatorname*{UC-lim}_g \int f_0 T_2^g f_1 T_{2,3}^g f_2 \, d\mu.$$

This limit exists by Corollary 4.5, and Lemma 2.5 shows that this does determine a unique measure on X^{k+1}. We have the following G-actions on X^F:

$$T_{F1} = T_1 \times T_1 \times T_1, \quad T_{F2} = \mathrm{Id} \times T_2 \times T_{2,3}, \quad T_{F3} = T_{2,3} \times T_3 \times \mathrm{Id}.$$

The action T_{F1} preserves the measure μ^F, since T_1 commutes with T_2 and T_3. The action T_{F2} preserves the measure μ^F by left invariance of the uniform Cesàro limit and the action T_{F3} by right invariance of the uniform Cesàro limit. Moreover, these actions clearly commute. Hence, we see that X^F is an extension of X under the projection π onto the second coordinate.

We also need a special case of a lemma from [**20**], which in turn generalizes [**4**, Lemma 4.7].

LEMMA A.2. *Let $X \in \mathcal{C}_3$ and $f_1,\ f_2,\ f_3$ be bounded functions on X. Suppose that*

$$\mathbb{E}_{\mu^F}(f_1 \otimes f_2 \otimes f_3 | I_{F1,F2}) = 0.$$

Then

$$\operatorname*{UC-lim}_g T_1^g f_1 T_{1,2}^g f_2 T_{1,2,3}^g f_3 = 0 \qquad \text{in norm of } L^2(X).$$

Proof. Fix a left Reiter sequence F. By Corollary 2.2, it suffices to show that

$$0 = \liminf_n \limsup_m \iiint T_1^g f_1 T_{1,2}^g f_2 T_{1,2,3}^g f_3 \cdot T_1^{hg} f_1 T_{1,2}^{hg} f_2 T_{1,2,3}^{hg} f_3 \, d\mu \, dF_n'(h) \, dF_m(g).$$

This expression equals

$$\begin{aligned}
&\liminf_n \limsup_m \iiint T_1^g f_1 T_{1,2}^g f_2 T_{1,2,3}^g f_3 \\
&\quad \times T_1^g T_1^h f_1 T_{1,2}^g T_{1,2}^h f_2 T_{1,2,3}^g T_{1,2,3}^h f_3 \, d\mu \, dF_n'(h) \, dF_m(g) \\
&= \liminf_n \limsup_m \iiint f_1 T_2^g f_2 T_{2,3}^g f_3 \\
&\quad \times T_1^h f_1 T_2^g T_{1,2}^h f_2 T_{2,3}^g T_{1,2,3}^h f_3 \, d\mu \, dF_n'(h) \, dF_m(g) \\
&= \liminf_n \iint f_1 T_1^h f_1 \otimes f_2 T_{1,2}^h f_2 \otimes f_3 T_{1,2,3}^h f_3 \, d\mu^F \, dF_n'(h).
\end{aligned}$$

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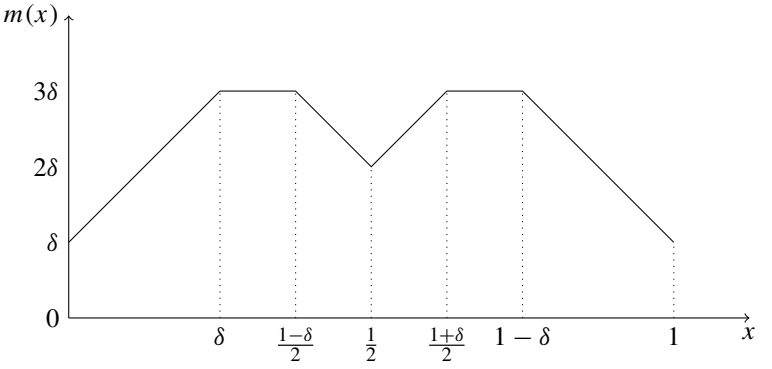


FIGURE B.1. The multicorrelation function $m(x)$.

By the mean ergodic theorem, this equals

$$\|\mathbb{E}_{\mu^F}(f_1 \otimes f_2 \otimes f_3 | I_{F1,F2})\|^2_{L^2(\mu^F)} = 0. \qquad \square$$

Proof of Proposition A.1. Consider the idempotent subclass $\mathcal{J}$ of $\mathcal{C}_3$ that consists of the systems with $\mathcal{X} = I_2 \vee I_3 \vee I_{1,2}$. By Theorem 3.2, we may assume that X is $\mathcal{J}$-sated. Assume now that $X \in \mathcal{C}_3$ is $\mathcal{J}$-sated and $f_2 \perp I_2 \vee I_3 \vee I_{1,2}$. By satedness, we have $1 \otimes f_2 \otimes 1 \perp I_{F2} \vee I_{F3} \vee I_{F1,F2}$. Moreover, for any f_1, $f_3 \in L^\infty(X)$, the function $f_1 \otimes 1 \otimes 1$ is I_{F2}-measurable and the function $1 \otimes 1 \otimes f_3$ is I_{F3}-measurable. Therefore, $f_1 \otimes f_2 \otimes f_3 \perp I_{F1,F2}$. By Lemma A.2, this implies

$$\operatorname*{UC-lim}_g T_1^g f_1 T_{1,2}^g f_2 T_{1,2,3}^g f_3 = 0.$$

Hence, we may assume that f_2 is $I_2 \vee I_3 \vee I_{1,2}$-measurable. By density and linearity, it suffices to consider $f_2 = h_2 h_3 h_{1,2}$, where h_i is T_i-invariant. In this case, we have

$$T_1^g f_1 T_{1,2}^g f_2 T_{1,2,3}^g f_3 = h_{1,2} T_1^g (f_1 h_2) T_{1,2,3}^g (f_3 h_3),$$

and this converges in the uniform Cesàro sense, by Corollary 4.5 applied to the actions T_1 and $T_{2,3}$. $\square$

The main difficulty in extending this approach to the multiple ergodic theorem to $k > 3$ commuting actions consists in explicitly constructing a suitable extension, which can be used to exploit satedness, of a given action. Note that for discrete groups, a new tool for constructing such extensions appeared in Austin's proof; see [**3**, Theorem 2.1].

B. *Appendix. Small correlation sequences*

The lower bound in the ergodic Roth theorem [**7**, Theorem 1.2] might suggest that $\text{C-lim}_n\, \mu(A \cap T^n A \cap T^{2n} A) \geq \mu(A)^3$, for any measure-preserving transformation (X, T) and any $A \subset X$. The following counterexample shows that this is not the case.

Consider an irrational rotation T by α on the circle $X = \mathbb{R} \mod 1$, and let $B \subset X$ be an interval of length $\delta \leq 1/3$. Then the function $m(x) := \mu(B - x \cup B \cup B + x)$ has the form shown in Figure B.1.

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Now let $A := X \setminus B$. Then

$$\lim_N \frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T^n A \cap T^{2n} A) = \lim_N \frac{1}{N}\sum_{n=1}^{N}(1 - m(n\alpha)) = 1 - \int_0^1 m(x)\,dx$$
$$= 1 - \left(3\delta - \frac{5}{2}\delta^2\right) = (1-\delta)^3 - \delta^2/2 + \delta^3 < \mu(A)^3.$$

We will now describe an example that shows that the exponent in Theorem 5.7 cannot be improved to 3. The construction is based on [**9**, Theorem 1.2], but the function f that appears in the proof has been optimized numerically to maximize the exponent (however, we do not claim that this exponent is the best possible). As a pleasant side effect, the optimal function has a particularly simple form.

THEOREM B.1. *For every countable amenable group G, there exist a system $X \in \mathcal{C}_2^{\mathrm{erg}}$ such that the group generated by T_1 and T_2 acts weakly mixingly and a measurable set A with $0 < \mu(A) < 1$ such that*

$$\mu(A \cap T_1^g A \cap T_{1,2}^g A) < \mu(A)^{3.19},$$

for all $g \neq \mathrm{id}_G$.

Proof. Consider $Y := \{0, 1, 2\}^G$ and denote the $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$-Bernoulli measure on Y by ν. The two natural left G-actions on Y are the right Bernoulli shift $(R^g y)_h = y_{hg}$ and the left Bernoulli shift $(L^g y)_h = y_{g^{-1}h}$. Note that these actions preserve ν and commute. Therefore,

$$(X, \mu, T_1, T_2) := (Y \times Y \times Y, \nu \times \nu \times \nu, R \times \mathrm{Id} \times R, L \times R \times \mathrm{Id})$$

is a system in $\mathcal{C}_2$. Since the actions L and R are weakly mixing, it follows that T_1 and T_2 span a weakly mixing group action on X, and, in particular, we obtain $X \in \mathcal{C}_2^{\mathrm{erg}}$.

For $i, j, k \in \{0, 1, 2\}$, let

$$f(i, j, k) := \begin{cases} 1 & \text{if } i, j, k \text{ are pairwise different,} \\ 0 & \text{otherwise.} \end{cases}$$

Let F be the function on X defined by $F(y, z, w) = f(y_{\mathrm{id}_G}, z_{\mathrm{id}_G}, w_{\mathrm{id}_G})$. The function F is the indicator function of a measurable subset $A \subset X$ with $\mu(A) = 3!/3^3$. For every $g \neq \mathrm{id}_G$, we have

$$\begin{aligned}\mu(A \cap T_1^g A \cap T_{1,2}^g A) &= \int F \cdot T_1^{g^{-1}} F \cdot T_{1,2}^{g^{-1}} F \, d\mu \\ &= \int F(y, z, w) \cdot F(R^{g^{-1}} y, z, R^{g^{-1}} w) \\ &\quad \times F(R^{g^{-1}} L^{g^{-1}} y, R^{g^{-1}} z, R^{g^{-1}} w)\, d\mu(y, z, w) \\ &= \int f(y_{\mathrm{id}_G}, z_{\mathrm{id}_G}, w_{\mathrm{id}_G}) \cdot f(y_{g^{-1}}, z_{\mathrm{id}_G}, w_{g^{-1}}) \\ &\quad \times f(y_{\mathrm{id}_G}, z_{g^{-1}}, w_{g^{-1}})\, d\nu(y)\, d\nu(z)\, d\nu(w).\end{aligned}$$

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Since ν is a product measure, this equals

$$\begin{aligned}
&\frac{1}{3^6}\sum_{i,j,k,i',j'k'} f(i,\,j,\,k')f(i',\,j,\,k)f(i,\,j',\,k)\\
&\quad=\frac{1}{3^6}\sum_{i,j,k}\Bigg(\sum_{k':\{i,j,k'\}=\{0,1,2\}}1\Bigg)\Bigg(\sum_{i':\{i',j,k\}=\{0,1,2\}}1\Bigg)\Bigg(\sum_{j':\{i,j',k\}=\{0,1,2\}}1\Bigg)\\
&\quad=\frac{1}{3^6}\sum_{i,j,k}\delta_{i\neq j}\delta_{j\neq k}\delta_{i\neq k}=\frac{3!}{3^6}<\left(\frac{3!}{3^3}\right)^{3.19}=\mu(A)^{3.19}.
\end{aligned}$$

□

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