Econometric Theory, **25**, 2009, 819–846. Printed in the United States of America.
doi:10.1017/S0266466609090720

COPULA-BASED CHARACTERIZATIONS FOR HIGHER ORDER MARKOV PROCESSES

RUSTAM IBRAGIMOV
Harvard University

In this paper, we obtain characterizations of higher order Markov processes in terms of copulas corresponding to their finite-dimensional distributions. The results are applied to establish necessary and sufficient conditions for Markov processes of a given order to exhibit m-dependence, r-independence, or conditional symmetry. The paper also presents a study of applicability and limitations of different copula families in constructing higher order Markov processes with the preceding dependence properties. We further introduce new classes of copulas that allow one to combine Markovness with m-dependence or r-independence in time series.

1. INTRODUCTION

In recent years, a number of studies in economics, finance, and risk management have focused on the analysis of dependence measures and related concepts for time series. It was observed that the use of correlation as a measure of dependence is problematic in many setups, including the departure from elliptic distributions that is common for real world risks and financial market data (see, among others, the discussion in Embrechts, McNeil, and Straumann, 2002; McNeil, Frey, and Embrechts, 2005, Ch. 5; de la Peña, Ibragimov, and Sharakhmetov, 2006; and references therein). Another problem with the use of correlation is that it is a bivariate measure of dependence and even using its time varying versions, at best, leads to only capturing the pairwise dependence in data sets, failing to measure more complicated dependence structures. Also, correlation is defined only in the case of data with finite second moments, and its reliable estimation is problematic

This paper was previously circulated under the title "Copula-Based Dependence Characterizations and Modeling for Time Series." An extended working paper version of the paper is available as Ibragimov, 2005, "Copula-Based Dependence Characterizations and Modeling for Time Series," Harvard Institute of Economic Research Discussion paper 2094. I thank three referees, Donald Andrews, Brendan Beare, Christian Gourieroux, George Lentzas, Jeremiah Lowin, Andrew Patton, Peter Phillips, Murray Rosenblatt, Yildiray Yildirim, and the participants at seminars at the Departments of Economics at Boston University, Harvard University, and Yale University, Whitman School of Management at Syracuse University, and the Harvard Statistics Summer Retreat on Recent Advances in Computational Finance (June 2006) for helpful comments and suggestions. A part of the paper was completed under financial support from a Yale University Dissertation Fellowship and a Cowles Foundation Prize. Address correspondence to Rustam Ibragimov, Department of Economics, Harvard University, Littauer Center, 1805 Cambridge St., Cambridge, MA 02138, USA; e-mail: ribragim@fas.harvard.edu.

© 2009 Cambridge University Press 0266-4666/09 $15.00

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

in the case of infinite fourth moments (see the discussion in Cont, 2001). However, as discussed in a number of studies (see, e.g., Loretan and Phillips, 1994; Cont, 2001; Ibragimov, 2005b, 2009; and references therein), heavy-tailed behavior with infinite fourth moments is present in many financial and commodity market data sets, and even first moments or variances are infinite for certain time series in finance and economics. Several approaches have been proposed recently to deal with the preceding problems. One of these approaches, which is becoming increasingly popular in dependence modeling and analysis, is based on copulas. Copulas are functions that allow one, by a celebrated theorem due to Sklar (1959), to represent a joint distribution of random variables (r.v.s) as a function of marginal distributions.[1] Copulas, therefore, capture dependence properties of the data generating process (more precisely, they reflect all the dependence properties that are invariant to increasing transformations of data). In recent years, copulas and related concepts have been applied to a wide range of problems in economics, finance, and risk management (see, among others, Cherubini, Luciano, and Vecchiato, 2004, and references therein; Patton, 2004; McNeil et al., 2005; Hu, 2006; the review in de la Peña et al., 2006; Granger, Teräsvirta, and Patton, 2006; Patton, 2006).

One should note that, so far, most of the studies have focused on the analysis of copulas and dependence measures only in the bivariate case and only a few papers have considered the problems in copula theory in the time series context. A drawback of the approach based on bivariate copulas and dependence measures is that, similar to the case of linear correlation, it can capture, at best, only pairwise dependence patterns and cannot be used in the case of more complicated dependence structures. However, dependence characteristics of real data sets can be far more general than those determined by pairs of variables: for example, the behavior of financial indexes across markets is interrelated and is affected by a number of factors common to all of the markets. In addition, estimation procedures for copulas developed in the context of independent observations of random vectors are not directly applicable in the analysis of time series dependence characteristics.

The problems of copula theory and its applications in the multivariate and time series context have been considered, in particular, in the following papers. Joe (1987, 1989) proposed multivariate extensions of Pearson's coefficient and the Kullback–Leibler and Shannon mutual information. Nelsen (1996) considered measures of multivariate association generalizing bivariate Spearman's rho and Kendall's tau. Focusing on finite-dimensional random vectors with dependent components, de la Peña et al. (2006) obtained *U*-statistics-based representations for multivariate joint distributions and copulas. As a corollary of the results, de la Peña et al. (2006) derived similar representations for multivariate dependence measures and obtained sharp complete decoupling moment and probability inequalities for dependent r.v.s in terms of their dependence characteristics.

Darsow, Nguyen, and Olsen (1992) obtained characterizations of first-order Markov processes in terms of copula functions corresponding to their two-dimensional distributions. Chen and Fan (2004, 2006) considered parametric

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

copula estimation procedures for time series based on bivariate copulas and applied the results in the problems of evaluating density forecasts. Fermanian, Radulović, and Wegkamp (2004) established weak convergence of empirical copula processes in the case of independently observed vectors with dependent components. Doukhan, Fermanian, and Lang (2004) focused on the analysis of the asymptotics of empirical copula processes for weakly dependent sequences of random vectors.[2] Ibragimov (2005a) discussed weak convergence of empirical copulas under β-mixing assumptions.

In this paper, we obtain characterizations of Markov processes of an arbitrary order in terms of copulas corresponding to their finite-dimensional distributions (Section 2). These results extend the characterizations of first-order Markov processes in terms of bivariate copulas in Darsow et al. (1992). The results show that a Markov process of order k is fully determined by its $(k+1)$-dimensional copulas and one-dimensional marginal cumulative distribution functions (c.d.f.s). The characterizations thus provide a justification for estimation of finite-dimensional copulas of time series with higher order Markovian dependence structure. Using the results, we obtain necessary and sufficient conditions for higher order Markov processes to exhibit several additional dependence properties, such as m-dependence, r-independence, or conditional symmetry (Section 3). These conditions, in particular, generalize U-statistics-based characterizations of copulas of vectors with r-independent or m-dependent components in de la Peña et al. (2006) to the case of dependent time series. They further show that dependence properties of copula-based time series provide additional nontrivial restrictions on the U-statistics characterizations of copulas for the processes in consideration that can be used in inference on their properties.

The results obtained in the paper provide a copula-based approach to the analysis of higher order Markov processes that is alternative to the conventional one based on transition probabilities. The advantage of the approach based on copulas is that it allows one to separate the study of dependence properties (e.g., r-independence, m-dependence, or conditional symmetry) of the stochastic processes under consideration from the analysis of the effects of marginal distributions (say, unconditional heavy-tailedness or skewness). In particular, the results provide methods for construction of higher order Markov processes with arbitrary one-dimensional margins that, possibly, satisfy additional dependence assumptions. These processes can be used, for instance, in the analysis of the robustness of statistical and econometric procedures to weak dependence. In addition, they provide examples of non-Markovian processes that nevertheless satisfy Chapman–Kolmogorov stochastic equations. Higher order Markov processes with prescribed dependence properties can be constructed, for instance, using inversion of finite-dimensional c.d.f.s of known examples of dependent time series (see the discussion at the end of Section 2).

In Sections 4 and 5, we present an analysis of applicability and limitations of different classes of copulas in constructing higher order Markov processes with prescribed dependence properties. In Section 4, we focus on processes based on

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

expansions by linear functions (bivariate and multivariate Eyraud–Farlie–Gumbel –Morgenstern copulas) and also on more general copulas that involve products of nonlinear functions of the arguments, such as power copulas. We obtain impossibility/reduction-type results that show that time series based on such copulas that simultaneously exhibit Markovness and m-dependence or r-independence properties are, in fact, sequences of independent r.v.s (Theorems 5 and 6 and Corollaries 2–4). In Section 5, we introduce a new class of copulas based on expansions by Fourier polynomials (Fourier copulas) that, in contrast to the copula families considered in Section 4, allow one to combine higher order Markovness with m-dependence or r-independence. In Section 6 of the paper we make some concluding remarks. Appendix A reviews the definition and discusses the main properties of copula functions, together with their examples. Appendix B contains the proofs of the results obtained in the paper.

2. COPULA-BASED TIME SERIES MARKOV CHARACTERIZATIONS

This section presents our main results on copula-based characterizations for Markov processes of an arbitrary order. In what follows, we use definitions and the results in copula theory reviewed in Appendix A. To highlight the main concepts and ideas discussed, we assume throughout the paper that all copulas considered are absolutely continuous and the processes under study have continuous univariate c.d.f.s, if not stated otherwise. These assumptions, in particular, imply that the copulas corresponding to the finite-dimensional distributions of the processes under consideration are unique (see Proposition 1 in Appendix A). However, most of the results discussed in the paper can be extended to the case of not necessarily continuous marginal c.d.f.s and not necessarily absolutely continuous copulas.[3]

Let $m, n \geq k \geq 1$. Let A and B be, respectively, m- and n-dimensional copulas such that

$$\begin{aligned} &A(u_1, \ldots, u_{m-k}, \xi_1, \ldots, \xi_k)\Big|_{u_i=1, i=1,\ldots,m-k} \\ &\quad = B(\xi_1, \ldots, \xi_k, u_{k+1}, \ldots, u_n)\Big|_{u_i=1, i=k+1,\ldots,n} \\ &\quad = C(\xi_1, \ldots, \xi_k), \end{aligned} \tag{1}$$

$\xi_i \in [0, 1]$, $i = 1, \ldots, k$, where C is a k-dimensional copula (relation (1) means that a k-dimensional margin of the copula A is the same as a k-dimensional margin of the copula B).

Let $V_1, \ldots, V_m$ and $W_1, \ldots, W_n$ be r.v.s with joint c.d.f.s A and B (see Definition 5 in Appendix A). Denote by $A_{1,\ldots,m|m-k+1,\ldots,m}(u_1, \ldots, u_{m-k}, \xi_1, \ldots, \xi_k) = P(V_1 \leq u_1, \ldots, V_{m-k} \leq u_{m-k} |\ V_{m-k+1} = \xi_1, \ldots, V_m = \xi_k)$ and $B_{1,\ldots,n|1,\ldots,k}(\xi_1, \ldots, \xi_k, u_{m+1}, \ldots, u_{m+n-k}) = P(W_{k+1} \leq u_{m+1}, \ldots,\ W_n \leq u_{m+n-k} |\ W_1 =$

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

$\xi_1, \ldots, W_k = \xi_k)$ the conditional analogues of the copulas A and B. One has

$$A_{1,\ldots,m|m-k+1,\ldots,m}(u_1, \ldots, u_{m-k}, \xi_1, \ldots, \xi_k) = \frac{\partial^k A(u_1, \ldots, u_{m-k}, \xi_1, \ldots, \xi_k)}{\partial v_{m-k+1} \ldots \partial v_m} \Big/ \frac{\partial^k C(\xi_1, \ldots, \xi_k)}{\partial v_1 \ldots \partial v_k},$$

$$B_{1,\ldots,n|1,\ldots,k}(\xi_1, \ldots, \xi_k, u_{m+1}, \ldots, u_{m+n-k}) = \frac{\partial^k B(\xi_1, \ldots, \xi_k, u_{m+1}, \ldots, u_{m+n-k})}{\partial v_1 \ldots \partial v_k} \Big/ \frac{\partial^k C(\xi_1, \ldots, \xi_k)}{\partial v_1 \ldots \partial v_k}, \quad \textbf{(2)}$$

where $\partial^k A(v_1, \ldots, v_m)/\partial v_{m-k+1} \ldots \partial v_m$, $\partial^k B(v_1, \ldots, v_n)/\partial v_1 \ldots \partial v_k$, and $\partial^k C(v_1, \ldots, v_k)/\partial v_1 \ldots \partial v_k$ denote the partial derivatives of the copulas A, B, and C.

Further, define the $\star^k$-product of the copulas A and B, $D = A \star^k B : [0,1]^{m+n-k} \to [0,1]$ via the relation

$$\begin{aligned} D(u_1, \ldots, u_{m+n-k}) = \int_0^{u_{m-k+1}} \ldots \int_0^{u_m} & A_{1,\ldots,m|m-k+1,\ldots,m}(u_1, \ldots, u_{m-k}, \xi_1, \ldots, \xi_k) \\ & \times B_{1,\ldots,n|1,\ldots,k}(\xi_1, \ldots, \xi_k, u_{m+1}, \ldots, u_{m+n-k}) \\ & \times C(d\xi_1, \ldots, d\xi_k). \end{aligned} \quad \textbf{(3)}$$

The $\star^k$-operator is a generalization of the star $\star$-operator considered in Darsow et al. (1992); the $\star$-operator in Darsow et al. (1992) is a particular case of its preceding $\star^k$-analogue with $k = 1$ (see Appendix A). Similar to the case of $k = 1$ in Darsow et al. (1992) one can show that the operator $\star^k$ is associative, distributive over convex combinations, and continuous in each place (but not jointly continuous).

In terms of the densities $\partial^m A(v_1, \ldots, v_m)/\partial v_1 \ldots \partial v_m$, $\partial^n B(v_1, \ldots, v_n)/\partial v_1 \ldots \partial v_n$ and $\partial^{m+n-k} D(v_1, \ldots, v_{m+n-k})/\partial v_1 \ldots \partial v_{m+n-k}$ of the copulas A and B and $D = A \star^k B$, relation (3) is equivalent to the following relation:

$$\begin{aligned} \frac{\partial^{m+n-k} D(u_1, \ldots, u_{m+n-k})}{\partial v_1 \ldots \partial v_{m+n-k}} &= \frac{\partial^m A(u_1, \ldots, u_{m-k}, u_{m-k+1}, \ldots, u_m)}{\partial v_1 \ldots \partial v_m} \\ &\quad \times \frac{\partial^n B(u_{m-k+1}, \ldots, u_m, u_{m+1}, \ldots, u_{m+n-k})}{\partial v_1 \ldots \partial v_n} \Big/ \frac{\partial^k C(u_{m-k+1}, \ldots, u_m)}{\partial v_1 \ldots \partial v_k} \end{aligned}$$

or, equivalently,

$$\frac{\partial^{m+n-k} D(u_1, \ldots, u_{m+n-k})}{\partial v_1 \ldots \partial v_{m+n-k}} \cdot \frac{\partial^k C(u_{m-k+1}, \ldots, u_m)}{\partial v_1 \ldots \partial v_k} = \frac{\partial^m A(u_1, \ldots, u_{m-k}, u_{m-k+1}, \ldots, u_m)}{\partial v_1 \ldots \partial v_m} \cdot \frac{\partial^n B(u_{m-k+1}, \ldots, u_m, u_{m+1}, \ldots, u_{m+n-k})}{\partial v_1 \ldots \partial v_n}.$$

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

Let $T \subseteq \mathbf{R}$. The processes considered throughout the paper are assumed to be real-valued and continuous and to be defined on the same probability space $(\Omega, \Im, P)$. Stationarity refers to strict stationarity. For a r.v. X on $(\Omega, \Im, P)$ and $x \in \mathbf{R}$, $I_{X<x}$ denotes the indicator of the event $\{X < x\}$. In addition, as usual, for r.v.s $Y_1, \ldots, Y_s, X_1, \ldots, X_l$ on $(\Omega, \Im, P)$ and $x_1, \ldots, x_l \in \mathbf{R}$, $P(X_1 < x_1, \ldots, X_l < x_l \big| Y_1, \ldots, Y_s)$ stands for $\mathrm{E}\big(I_{X_1<x_1} \ldots I_{X_l<x_l} \big| Y_1, \ldots, Y_s\big)$. For two r.v.s X and Y on $(\Omega, \Im, P)$ we write $X = Y$ if $X = Y$ (a.s.).

DEFINITION 1. *A process* $\{X_t\}_{t\in T}$ *is called a* Markov process of order $k \geq 1$ *if, for all* t, $t_i \in T$, $i = 1, \ldots, n$, *such that* $t_1 < \cdots < t_{n-k} < t_{n-k+1} < \cdots < t_n < t$ *and all* $x \in \mathbf{R}$,

$$P\big(X_t < x \big| X_{t_1}, \ldots, X_{t_{n-k}}, X_{t_{n-k+1}}, \ldots, X_{t_n}\big) = P\big(X_t < x \big| X_{t_{n-k+1}}, \ldots, X_{t_n}\big). \quad \textbf{(4)}$$

Throughout the rest of the section, $C_{t_1,\ldots,t_k}$, $t_i \in T$, $i = 1, \ldots, k$, $t_1 < \cdots < t_k$, stand for copulas corresponding to the joint distribution of the r.v.s $X_{t_1}, \ldots, X_{t_k}$ in the process $\{X_t\}_{t\in T}$ under consideration. In addition, throughout the paper, formulated equalities and inequalities for two functions f and g defined on $[a, b]^n \subseteq \mathbf{R}^n$ are understood to hold almost everywhere on $[a, b]^n$. That is, we write $f = g$ (or $f(u) = g(u)$) if f and g coincide almost everywhere on $[a, b]^n$: $f(u) = g(u)$ for all $u \in [a, b]^n \backslash \mathcal{A}$, where $\mathcal{A}$ is a subset of $[a, b]^n$ with the Lebesgue measure zero. The meaning of the inequalities $f \geq g$ and $f \leq g$ (or $f(u) \geq g(u)$ and $f(u) \leq g(u)$) is similar.

The following theorem provides a characterization of Markov processes of an arbitrary order in terms of their $(k+1)$-dimensional copulas.

THEOREM 1. *A real-valued stochastic process* $\{X_t\}_{t\in T}$ *is a Markov process of order* k, $k \geq 1$, *if and only if for all* $t_i \in T$, $i = 1, \ldots, n$, $n \geq k+1$, *such that* $t_1 < \cdots < t_n$,

$$C_{t_1,\ldots,t_n} = C_{t_1,\ldots,t_{k+1}} \star^k C_{t_2,\ldots,t_{k+2}} \star^k \ldots \star^k C_{t_{n-k},\ldots,t_n}. \quad \textbf{(5)}$$

Let $n \geq k+1$ and $s \geq 1$. For an n-dimensional copula C denote by C^s the s-fold product $\star^k$ of C with itself.

The following Corollary 1 is an immediate consequance of Theorem 1.

COROLLARY 1. *A sequence of identically distributed r.v.s* $\{X_t\}_{t=1}^{\infty}$ *is a stationary Markov process of order* k, $k \geq 1$, *if and only if for all* $n \geq k+1$,

$$C_{1,\ldots,n}(u_1, \ldots, u_n) = \underbrace{C \star^k C \star^k \ldots \star^k C}_{n-k+1}(u_1, \ldots, u_n) = C^{n-k+1}(u_1, \ldots, u_n), \quad \textbf{(6)}$$

where C *is a* $k+1$*-dimensional copula such that* $C_{i_1+h,\ldots,i_l+h} = C_{i_1,\ldots,i_l}$, $1 \leq h \leq k+1-i_l$, $1 \leq i_1 < \cdots < i_l \leq k+1$, $l = 2, \ldots, k$, *and* $C_{j_1,\ldots,j_l}$, $1 \leq j_1 < \cdots < j_l \leq k+1$, *denote the corresponding marginals of* C: $C_{j_1,\ldots,j_l} = C|_{u_i=1, i\neq j_1,\ldots,j_l}$.

Let, as before, $\{X_t\}_{t\in T}$ be a Markov process of order k with finite-dimensional copulas $C_{t_1,\ldots,t_n}(u_1, \ldots, u_n)$, $t_i \in T$, $i = 1, \ldots, n$, $t_1 < \cdots < t_n$. For $t \in T$, denote

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

by F_t the c.d.f. of X_t. Let, for $t_1 < \cdots < t_k$, $x_1, \ldots, x_k \in \mathbf{R}$ and Borel sets $A \in \mathcal{B}(\mathbf{R})$, $p(t_1, \ldots, t_k, t_{k+1}, x_1, \ldots, x_k, A) = P(X_{t_{k+1}} \in A | X_{t_1} = x_1, \ldots, X_{t_k} = x_k)$ stand for the transition probabilities of $\{X_t\}_{t \in T}$. Using (2), it is not difficult to see, similar to the case of first-order Markov processes in the proof of Theorem 3.2 in Darsow et al. (1992), that, for all Borel sets $A \in \mathcal{B}(\mathbf{R})$ of the form $A = (-\infty, x_{k+1})$, $x_{k+1} \in \mathbf{R}$, one has

$$p(t_1, \ldots, t_k, t_{k+1}, x_1, \ldots, x_k, A)$$
$$= \frac{\partial^k C_{t_1,\ldots,t_k,t_{k+1}}\left(F_{t_1}(x_1), \ldots, F_{t_k}(x_k), F_{t_{k+1}}(x_{k+1})\right)}{\partial u_1 \ldots \partial u_k}$$
$$\Big/ \frac{\partial^k C_{t_1,\ldots,t_k}\left(F_{t_1}(x_1), \ldots, F_{t_k}(x_k)\right)}{\partial u_1 \ldots \partial u_k}, \tag{7}$$

where, as before, $\partial^k C_{t_1,\ldots,t_k,t_{k+1}}(u_1, \ldots, u_k, u_{k+1})/\partial u_1 \ldots \partial u_k$ and $\partial^k C_{t_1,\ldots,t_k}(u_1, \ldots, u_k)/\partial u_1 \ldots \partial u_k$ denote partial derivatives of the copulas $C_{t_1,\ldots,t_k,t_{k+1}}(u_1, \ldots, u_k, u_{k+1})$ and $C_{t_1,\ldots,t_k}(u_1, \ldots, u_k)$.

Theorem 1 and Corollary 1 provide copula-based characterizations of higher order Markov processes that are an alternative to the conventional characterization using their transition probabilities (and the initial distribution). One can specify a Markov process of order k by prescribing all one-dimensional marginal distributions and a family of $(k+1)$-dimensional copulas. Then one can generate the copulas of higher order and, thus, the finite-dimensional c.d.f.s using (5). As discussed in the introduction, the advantage of the approach based on copulas is that it allows one to separate in the analysis the properties of time series determined by one-dimensional distributions from their dependence characteristics. Relations (7), on the other hand, allow one to recover one characterization of higher order Markov processes given the other.

Corollary 1, together with the inversion method for constructing copulas described in Appendix A, provides a device for obtaining new Markov processes of an arbitrary order that exhibit dependence properties similar to those of a given Markov process of the same order but have different marginals. Namely, let $\{X_t\}_{t=1}^{\infty}$ be a stationary Markov process of order $k \geq 1$ with $(k+1)$-dimensional c.d.f. $\tilde{F}(x_1, \ldots, x_{k+1})$ and the one-dimensional marginal c.d.f. F. Then the $(k+1)$-dimensional copula generating the process $\{X_t\}_{t=1}^{\infty}$ is, via formula (A.1) in Appendix A, $C(u_1, \ldots, u_{k+1}) = \tilde{F}(F^{-1}(u_1), \ldots, F^{-1}(u_{k+1}))$. Given an arbitrary one-dimensional c.d.f. G, the stationary kth-order Markov process that has a dependence structure similar to that of $\{X_t\}_{t=1}^{\infty}$ but a different one-dimensional marginal c.d.f. G can be constructed via (6) by generating its copulas of an arbitrary order and substituting the new one-dimensional c.d.f. to obtain its finite-dimensional c.d.f.s. For instance, taking $\tilde{F}$ to be the $(k+1)$-dimensional normal c.d.f. with a linear correlation matrix R: $\tilde{F} = \Phi_R^{k+1}(x_1, \ldots, x_{k+1})$ as in (A.5) with $n = k+1$, one obtains a stationary kth-order Markov process based on the normal copula $C_R^{k+1}(u_1, \ldots, u_{k+1})$ in (A.6) with $n = k+1$. In the case $k = 1$ and

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

$T = \mathbf{R}$, the construction provides a first-order Markov process in Example 4.3 in Darsow et al. (1992) that is referred to as a Brownian motion with non-Gaussian marginal distributions therein. More generally, in the case $k \geq 1$, one obtains a process with an arbitrary one-dimensional marginal c.d.f. whose dependence structure is similar to that of a Gaussian autoregressive process of order k (see the discussion at the beginning of Section 4). In addition, using examples of (possibly higher order) Markov processes that satisfy additional dependence assumptions available in the literature, such as examples of k-independent kth-order Markov processes or m-dependent Markov processes of the first order constructed in the works by Lévy (1949), Rosenblatt and Slepian (1962), Aaronson, Gilat, and Keane (1992), and Matúš (1998) (see the discussion in the next section), one can use the preceding inversion procedure to construct Markov processes that exhibit similar dependence properties but have one-dimensional marginals different from those in the examples.

In what follows, we refer to the processes $\{X_t\}_{t=1}^{\infty}$ constructed via (6) as stationary kth-order Markov processes based on the ($(k+1)$-dimensional) copula C or as stationary C-based kth-order Markov processes for short.

Remark 1. Let $f : \mathbf{R} \to \mathbf{R}$ be a strictly increasing function and let $\{X_t\}_{t=1}^{\infty}$ be a stationary C-based kth-order Markov process. Denote $Y_t = f(X_t)$, $t \geq 1$. Because copulas are invariant under strictly increasing transformations of r.v.s (see Proposition 2 in Appendix A) and the Markov property is preserved by strictly monotone (hence one-to-one) functions of Markov processes, $\{Y_t\}_{t=1}^{\infty}$ is a stationary kth-order Markov process based on the same copula C.[4] More generally, it is easy to derive the expressions relating the copulas of the kth-order Markov process $\{X_t\}_{t=1}^{\infty}$ to those of the kth-order Markov process determined by $Y_t = f_t(X_t)$, $t \geq 1$, where $f_t : \mathbf{R} \to \mathbf{R}$ are strictly monotone functions. For instance, in the case of the first-order Markov processes X_t with bivariate copulas $C_{t_1 t_2}$, the copulas $\tilde{C}_{t_1 t_2}$ of the process $Y_t = f_t(X_t)$ are related to $C_{t_1 t_2}$ as follows (see Nelsen, 1999, Thm. 2.4.4):

$\tilde{C}_{t_1,t_2}(u,v) = u - C_{t_1,t_2}(u, 1-v)$, if f_{t_1} is strictly increasing and f_{t_2} is strictly decreasing;

$\tilde{C}_{t_1,t_2}(u,v) = v - C_{t_1,t_2}(1-u, v)$, if f_{t_1} is strictly decreasing and f_{t_2} is strictly increasing;

$\tilde{C}_{t_1,t_2}(u,v) = u + v - 1 + C_{t_1,t_2}(1-u, 1-v)$, if both f_{t_1} and f_{t_2} are strictly decreasing.

3. APPLICATIONS TO COMBINING HIGHER ORDER MARKOVNESS WITH OTHER DEPENDENCE PROPERTIES

From copula-based characterizations of Markov processes of an arbitrary order in Section 2 it follows, in particular, that any copulas and, thus, any U-statistics-based representations for copula functions in de la Peña et al. (2006) (see Appendix A in this paper) can be used to construct higher order Markov processes. The results in this and the next section show that additional dependence

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

properties of such time series impose further restrictions on the U-statistics employed in the representations. These restrictions allow one to obtain, for instance, characterizations of Markov processes of an arbitrary order that satisfy additional assumptions of r-independence or m-dependence defined as follows.

DEFINITION 2. *Let $r \geq 2$ and let $T \subseteq \mathbf{R}$ be an index set that contains at least $r+1$ elements. A process $\{X_t\}_{t\in T}$ is called r-independent if any r r.v.s among X_t, $t \in T$, are jointly independent.*

DEFINITION 3. *Let $m \geq 1$ and let $T \subseteq \mathbf{R}$ be an index set that contains at least $m+2$ elements. A process $\{X_t\}_{t\in T}$ is called m-dependent if, for all $1 \leq a \leq l-1$ and any indexes $j_s \in T$, $s = 1, \ldots, l$, such that $1 \leq j_1 < \cdots < j_a < \cdots < j_l$ and $j_{a+1} - j_a \geq m+1$, the vectors $(X_{j_1}, X_{j_2}, \ldots, X_{j_{a-1}}, X_{j_a})$ and $(X_{j_{a+1}}, X_{j_{a+2}}, \ldots, X_{j_{l-1}}, X_{j_l})$ are independent.*

A number of studies have focused on problems of combining Markovian structures with other types of dependence. Lévy (1949) constructed a second-order Markov process consisting of pairwise independent uniformly distributed r.v.s (a second-order pairwise independent Markov process). Motivated by applications in the study of the mechanism of human vision, Rosenblatt and Slepian (1962) constructed stationary Nth-order Markov processes consisting of discrete r.v.s such that every N variables of the process are independent whereas $N+1$ adjacent variables of the process are not independent (stationary Nth-order N-independent Markov process). Rosenblatt and Slepian (1962) also obtained a result that is natural to refer to as an impossibility or a reduction property for Markov processes. This result shows that all Nth order N-independent Markov processes with two-valued X_t's are trivial in that they are processes of independent r.v.s. Higher order Markov r-independent processes are important in testing empirically the sensitivity of statistical procedures developed on the independence assumption to weak dependence in the data generating process (see Rosenblatt and Slepian, 1962). In addition, such processes are of interest because they provide examples of processes that are not Markovian of first order but whose first-order transition probabilities $P(s,x,t,A) = P(X_t \in A | X_s = x)$ nevertheless satisfy the Chapman–Kolmogorov stochastic equation

$$P(s,x,t,A) = \int_{-\infty}^{\infty} P(u,\xi,t,A) P(s,x,u,d\xi) \tag{8}$$

for all Borel sets A, all $s < t$ in T, $u \in (s,t) \cap T$, and for almost all $x \in \mathbf{R}$.[5]

Markov processes with 1-dependence appeared for the first time in Aaronson et al. (1992) and were considered, for example, by Burton, Goulet, and Meester (1993) and Matúš (1996), where the focus was on 1-dependent Markov shifts and on the structure of block factors. Matúš (1998) studied m-dependent Markov sequences consisting of discrete r.v.s and showed, in particular, that generally no stationary sequence of r.v.s that is Markov of order n but not of order $n-1$ and m-dependent but not $(m-1)$-dependent exists if the state space of the sequence has

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

small cardinality (another type of an impossibility/reduction result for Markov processes). Matúš (1998) also showed that to ensure the existence of Markov processes of order $n = 1$ that are m-dependent but not $(m-1)$-dependent the number of attainable states must be at least $m+2$ and this bound is tight.

The following result gives a characterization of stationary k-independent kth-order Markov processes. Subsequently, $[x]$ stands for the integer part of $x \in \mathbf{R}$.

THEOREM 2. *Let C be a $(k+1)$-dimensional copula. A sequence of r.v.s $\{X_t\}_{t=1}^{\infty}$ is a stationary k-independent C-based kth-order Markov process if and only if the density of C has the form*

$$\frac{\partial^{k+1} C(u_1, \ldots, u_{k+1})}{\partial u_1 \ldots \partial u_{k+1}} = 1 + g(u_1, \ldots, u_{k+1}), \tag{9}$$

where $g : [0,1]^{k+1} \to \mathbf{R}$ is a function satisfying the conditions

$$\int_0^1 \ldots \int_0^1 |g(u_1, \ldots, u_{k+1})| du_1 \ldots du_{k+1} < \infty, \tag{10}$$

$$\int_0^1 \ldots \int_0^1 \prod_{j=1}^{s} g(u_j, \ldots, u_{k+j}) du_{i_1} \ldots du_{i_s} = \int_0^1 \ldots \int_0^1$$

$$g(u_1, \ldots, u_{k+1}) g(u_2, \ldots, u_{k+2}) \ldots g(u_s, \ldots, u_{k+s}) du_{i_1} \ldots du_{i_s} = 0 \tag{11}$$

for all $s \le i_1 < \cdots < i_s \le k+1$, $s = 1, 2, \ldots, [(k+1)/2]$ and

$$g(u_1, \ldots, u_{k+1}) \ge -1. \tag{12}$$

Remark 2. Integration in condition (11) is with respect to all combinations of s variables among the arguments $u_s, u_{s+1}, \ldots, u_{k+1}$ that are common to all functions $g(u_1, \ldots, u_{k+1})$, $g(u_2, \ldots, u_{k+2})$, $\ldots$, $g(u_s, \ldots, u_{k+s})$ appearing in the integrand. These conditions ensure that all k-dimensional marginals of the copula of $X_1, \ldots, X_{k+s}$, $s \ge 1$, are product copulas (A.4) with $n = k$ and thus the k-independence property is satisfied for the stationary kth-order Markov process under consideration.

The following theorem provides a characterization of Markov processes satisfying m-dependence properties.

THEOREM 3. *Let C be a bivariate copula. A sequence of r.v.s $\{X_t\}_{t=1}^{\infty}$ is a stationary m-dependent C-based first-order Markov process if and only if the density of C satisfies*

$$\frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2} = 1 + g(u_1, u_2), \tag{13}$$

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

where $g : [0,1]^2 \to \mathbf{R}$ *is a function satisfying the conditions*

$$\int_0^1 \int_0^1 |g(u_1,u_2)|\,du_1 du_2 < \infty, \tag{14}$$

$$\int_0^1 g(u_1,u_2)\,du_i = 0, \qquad i = 1,2, \tag{15}$$

$$g(u_1,u_2) \geq -1 \tag{16}$$

and such that

$$\int_0^1 \dots \int_0^1 \prod_{i=1}^{m+1} g(u_i,u_{i+1})\,du_2\,du_3 \dots du_{m+1}$$
$$= \int_0^1 \dots \int_0^1 g(u_1,u_2)g(u_2,u_3)\dots g(u_{m+1},u_{m+2})\,du_2\,du_3\dots du_{m+1} = 0. \tag{17}$$

Remark 3. Similar to (11), integration in condition (17) is with respect to the variables $u_2, u_3, \dots, u_{m+1}$ that appear more than once among the arguments of the functions $g(u_1,u_2)$, $g(u_2,u_3),\dots,g(u_{m+1},u_{m+2})$. This condition ensures that the r.v.s X_1 and X_{m+2} are independent and, more generally, independence holds between the vectors $(X_1,\dots,X_n)$ and $(X_{m+n+1},\dots,X_{m+n+j})$, $n, j \geq 1$.

Remark 4. Similar to the proof of Theorems 2 and 3, one can obtain necessary and sufficient conditions for a Markov process of an arbitrary order to be m-dependent and also exhibit certain additional dependence properties. For instance, let C be a $(k+1)$-dimensional copula and suppose that $\{X_t\}_{t=1}^{\infty}$ is a stationary C-based kth-order Markov process such that any k r.v.s among $X_1,\dots,X_{k+1}$ are independent (clearly, this assumption is weaker than k-independence of the process $\{X_t\}_{t=1}^{\infty}$ in Theorem 3). Using Propositions 3 and 4 and Corollary 1 as in the proof of Theorems 2 and 3, one can show that the process $\{X_t\}_{t=1}^{\infty}$ is m-dependent for some $m \geq k+1$ if and only if the density of C has form (9) with a function $g : [0,1]^{k+1} \to \mathbf{R}$ that satisfies conditions (10) and (12) and is such that

$$\int_0^1 \dots \int_0^1 \prod_{j=1}^{m-k+2} g(u_j,\dots,u_{k+j})\,du_2\dots du_{m+1}$$
$$= \int_0^1 \dots \int_0^1 g(u_1,\dots,u_{k+1})g(u_2,\dots,u_{k+2})\dots$$
$$\times\, g(u_{m-k+2},\dots,u_{m+2})\,du_2\dots du_{m+1} = 0.$$

In a number of applications, for instance, in finance, Markov and martingale properties hold simultaneously. The martingale property, in contrast to

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

the Markov (first- and higher order) properties is not determined by finite-dimensional copulas only and can be affected by changes in one-dimensional marginal distributions. Indeed, using (2) with $m = 2$ and $k = 1$ (or Darsow et al., 1992, Thm. 3.1), it is not difficult to see that, for a bivariate copula C, a stationary C-based first-order Markov process $\{X_t\}_{t=1}^{\infty}$ with the univariate c.d.f. $F(x)$ is a martingale difference sequence with respect to the natural filtration $\Im_t = \sigma(X_1, \ldots, X_t)$ if and only if $\int_{-\infty}^{\infty} x \frac{\partial C(F(x), F(y))}{\partial u \partial v} dF(x) = 0$. The martingale property is determined by copulas alone for the class of martingale differences that satisfy conditional symmetry assumptions.

DEFINITION 4. *A sequence $\{X_t\}_{t=1}^{\infty}$ on a probability space $(\Omega, \Im, P)$ is a conditionally symmetric martingale difference with respect to an increasing sequence of σ-algebras $\Im_0 = (\Omega, \emptyset) \subseteq \Im_1 \subseteq \Im_2 \subseteq \cdots \subseteq \Im_n \subseteq \Im$ if, for all $t \geq 1$, the r.v. X_t is $\Im_t$-measurable and conditionally symmetric given $\Im_{t-1}$, that is, $P(X_t > x|\Im_{t-1}) = P(X_t < -x|\Im_{t-1})$, $x \geq 0$.*

THEOREM 4. *Let C be a bivariate copula. A stationary C-based first-order Markov process $\{X_t\}_{t=1}^{\infty}$ consisting of symmetric r.v.s is a conditionally symmetric martingale difference with respect to the natural filtration $\Im_0 = (\Omega, \emptyset)$, $\Im_t = \sigma(X_1, \ldots, X_t)$, $t \geq 1$, if and only if*

$$\frac{\partial C(u_1,\ 1/2-u)}{\partial u_1} + \frac{\partial C(u_1,\ 1/2+u)}{\partial u_1} = 1 \tag{18}$$

for all $u_1 \in [0, 1]$, $u \in [0, \frac{1}{2})$, or, equivalently, if the density of C satisfies

$$\frac{\partial C(u_1,\ 1/2-u)}{\partial u_1 \partial u_2} = \frac{\partial C(u_1,\ 1/2+u)}{\partial u_1 \partial u_2} \tag{19}$$

for all $u_1 \in [0, 1]$, $u \in \left[0, \frac{1}{2}\right)$.

4. REDUCTION AND IMPOSSIBILITY THEOREMS FOR MARKOV PROCESSES OF AN ARBITRARY ORDER

For a Gaussian process, pairwise independence coincides with joint independence. Therefore, it is not surprising that a stationary higher order Markov process based on a normal copula exhibits r-independence if and only if it is an independently and identically distributed (i.i.d.) sequence. Formally, let $C(u_1, \ldots, u_{k+1}) = C_R(u_1, \ldots, u_{k+1})$ be the normal copula with linear correlation matrix R defined in (A.6) with $n = k + 1$ and let $\{X_t\}_{t=1}^{\infty}$ be a stationary C-based kth-order Markov process. Let $Y_t = \Phi^{-1}[F(X_t)]$, $t \geq 1$, where $\Phi(x)$ is the standard normal univariate c.d.f. and $F(x)$ is the c.d.f. of X_t. Because the random vector $(Y_1, \ldots, Y_{k+1})$ has the multivariate normal distribution $\mathcal{N}(0, R)$ with correlation matrix R,

$$\left(Y_1, \ldots, Y_{k+1}\right) \sim \mathcal{N}(0, R), \tag{20}$$

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

we conclude that the process $\{X_t\}_{t=1}^{\infty}$ is r-independent for some $r \geq 2$ if and only if $R = I$, that is, if and only if $\{X_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. r.v.s.

Now let $k = 1$ and let $C(u_1, u_2) = C_\rho(u_1, u_2)$ be the bivariate normal copula with correlation coefficient ρ (see (A.7)). Then (as in, e.g., Chen and Fan, 2004, Exmp. 1) we conclude that $\{Y_t\}_{t=1}^{\infty}$ is a Gaussian process and, thus, for $t \geq m+2$,

$$Y_t = \rho Y_{t-1} + \epsilon_t = \rho^{m+1} Y_{t-m-1} + \sum_{k=0}^{m} \rho^k \epsilon_{t-k}, \tag{21}$$

where ϵ_t has a normal distribution: $\epsilon_t \sim \mathcal{N}(0, 1-\rho^2)$. From (21) we conclude that Y_t and Y_{t-m-1} (and, thus, X_t and X_{t-m-1}) are independent if and only if $\rho = 0$. Thus, $\{X_t\}_{t=1}^{\infty}$ is a stationary m-dependent C_ρ-based Markov process (of the first order) if and only if $\rho = 0$, that is, if and only if $\{X_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. r.v.s.

Theorems 2 and 3 imply several further reduction and impossibility results for Markov processes satisfying m-dependence and r-independence conditions that are similar in spirit to those in the case of normal copulas. The results show that a number of copula-based time series that simultaneously exhibit Markovness and m-dependence or r-independence properties are, in fact, sequences of independent r.v.s.

Theorem 5, which follows, shows that a construction of nontrivial Markov processes of higher order that exhibit r-independence properties is impossible on the basis of copulas whose densities C in Theorem 2 have functions g with a separable product form.

THEOREM 5. *Let $k \geq 2$ and let C be a $(k+1)$-dimensional copula that has density (9), where $g(u_1, u_2, \ldots, u_{k+1}) = \alpha f(u_1) f(u_2) \ldots f(u_{k+1})$ for some $\alpha \in \mathbf{R}$ and some continuous function $f : [0,1] \to \mathbf{R}$. A sequence of r.v.s $\{X_t\}_{t=1}^{\infty}$ is a stationary k-independent C-based kth-order Markov process if and only if $\{X_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. r.v.s.*

An example of copulas C in the separable product form of Theorem 5 is given by the special case of $(k+1)$-dimensional Eyraud–Farlie–Gumbel–Morgenstern copulas (A.10):

$$C(u_1, u_2, \ldots, u_{k+1})$$

$$= \prod_{i=1}^{k+1} u_i (1 + \alpha(1-u_1)(1-u_2)\ldots(1-u_{k+1})), \qquad -1 \leq \alpha \leq 1. \tag{22}$$

These copulas have densities (9) with

$$g(u_1, u_2, \ldots, u_{k+1}) = \alpha(1-2u_1)(1-2u_2)\ldots(1-2u_{k+1}). \tag{23}$$

COROLLARY 2. *Let $k \geq 2$ and let C be a $(k+1)$-dimensional Eyraud–Farlie–Gumbel–Morgenstern copula (22) with density (9) where g is given by*

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

(23). A sequence of r.v.s $\{X_t\}_{t=1}^{\infty}$ is a stationary k-independent C-based kth-order Markov process if and only if it is a sequence of i.i.d. r.v.s.

Corollary 3 is a generalization of Corollary 2 to the special case of $(k+1)$-dimensional power copulas (A.11) given by

$$C(u_1,u_2,\ldots,u_{k+1}) = \prod_{i=1}^{k+1} u_i \left(1+\alpha\left(u_1^l-u_1^{l+1}\right)\left(u_2^l-u_2^{l+1}\right)\ldots\left(u_{k+1}^l-u_{k+1}^{l+1}\right)\right),$$

$$-1 \le \alpha \le 1, \qquad \textbf{(24)}$$

where $l \in \{0,1,2,\ldots\}$ (copulas (24) reduce to those in (22) for $l=0$). These copulas have density (9) in which

$$g(u_1,u_2,\ldots,u_{k+1}) = \alpha\left((l+1)u_1^l-(l+2)u_1^{l+1}\right)\left((l+1)u_2^l-(l+2)u_2^{l+1}\right)\ldots$$

$$\times\left((l+1)u_{k+1}^l-(l+2)u_{k+1}^{l+1}\right). \qquad \textbf{(25)}$$

COROLLARY 3. *Let $k \ge 2$ and let C be a $(k+1)$-dimensional power copula (24) with density (9), where g is given by (25). A sequence of r.v.s $\{X_t\}_{t=1}^{\infty}$ is a stationary k-independent C-based kth-order Markov process if and only if $\{X_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. r.v.s.*

Theorem 6, which follows, is an analogue of Theorem 5 that provides impossibility/reduction results for m-dependent Markov processes. Theorem 6 shows that construction of nontrivial examples (i.e., those more general than sequences of i.i.d. r.v.s) of stationary Markov processes exhibiting m-dependence is impossible on the basis of bivariate copulas that have, similar to Theorem 5, the function g in a separable product form.

THEOREM 6. *Suppose that C is a bivariate copula that has the density $\partial^2 C(u_1,u_2)/\partial u_1 \partial u_2 = 1+\alpha f(u_1) f(u_2)$ for some $\alpha \in \mathbf{R}$ and some continuous function $f : [0,1] \to \mathbf{R}$. A sequence of r.v.s $\{X_t\}_{t=1}^{\infty}$ is a stationary m-dependent C-based Markov process (of the first order) if and only if $\{X_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. r.v.s.*

The following corollary is a specialization of Theorem 6 to the special case of bivariate Eyraud–Farlie–Gumbel–Morgenstern copulas (22) with $k=1$,

$$C(u_1,u_2) = u_1u_2\Big(1+\alpha(1-u_1)(1-u_2)\Big), \qquad -1 \le \alpha \le 1, \qquad \textbf{(26)}$$

that have density (13) with

$$g(u_1,u_2) = \alpha(1-2u_1)(1-2u_2). \qquad \textbf{(27)}$$

COROLLARY 4. *Let C be a bivariate copula Eyraud–Farlie–Gumbel–Morgenstern copula (26) with density (13), where g is given by (27). A sequence of r.v.s $\{X_t\}_{t=1}^{\infty}$ is a stationary m-dependent C-based Markov process (of the first order) if and only if $\{X_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. r.v.s.*

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

The results in this section that demonstrate that Markov processes with m-dependence and r-independent Markov processes of higher order cannot be constructed from Eyraud–Farlie–Gumbel–Morgenstern copulas and other separable copulas complement and substantially generalize the results of Cambanis (1991). Cambanis (1991) showed that the most common dependence structures such as constant, exponential, and m-dependence cannot be exhibited by stationary processes $\{X_t\}$ whose finite-dimensional copulas are the following multivariate analogues of bivariate Eyraud–Farlie–Gumbel–Morgenstern copulas (26):

$$C_{j_1,\ldots,j_n}(u_{j_1},\ldots,u_{j_n}) = \prod_{s=1}^{n} u_{j_k}\left(1+\sum_{1\le l<m\le n}\alpha_{lm}(1-u_{j_l})(1-u_{j_m})\right).$$

The results also complement the previously mentioned results by Rosenblatt and Slepian (1962) on nonexistence of nontrivial Nth-order N-independent Markov processes consisting of two-valued r.v.s because, as follows from Sharakhmetov and Ibragimov (2002), the finite-dimensional copulas of sequences of r.v.s concentrated on two points have multivariate Eyraud–Farlie–Gumbel–Morgenstern structure (A.10).

Remark 5. Interestingly, in contrast to normal copulas, t-copulas cannot be used to construct (possibly higher order) Markov processes that exhibit r-independence or m-dependence even if their correlation matrices are identity matrices corresponding to the case of uncorrelatedness. For instance, let $C(u_1,\ldots,u_{k+1}) = C^t_{\nu,R}(u_1,\ldots,u_{k+1})$ be a t-copula with correlation matrix R defined in (A.8) with $n = k+1$ and let $\{X_t\}_{t=1}^{\infty}$ be a stationary C-based kth-order Markov process. Then relation (20) holds for $Y_t = \left(\sqrt{S}/\sqrt{\nu}\right)\cdot t_\nu^{-1}[F(X_t)]$, where $S \sim \chi^2_\nu$ is a r.v. with chi-square distribution with ν degrees of freedom that is independent of $\{X_t\}_{t=1}^{\infty}$, $t_\nu(x)$ is the c.d.f. of the univariate Student t-distribution with ν degrees of freedom, and $F(x)$ is the c.d.f. of X_t. As before, we obtain that $R = I$ and $\{Y_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. standard normal r.v.s if $\{X_t\}_{t=1}^{\infty}$ is r-independent for some $r \ge 2$. Because the components of the random vector

$$\left(F^{-1}\left[t_\nu\left(\sqrt{\nu}Y_{t-1}/\sqrt{S}\right)\right], F^{-1}\left[t_\nu\left(\sqrt{\nu}Y_t/\sqrt{S}\right)\right]\right) \tag{28}$$

are dependent if Y_{t-1} and Y_t are i.i.d. standard normal r.v.s independent of $S \sim \chi^2_\nu$, we thus conclude that there does not exist a stationary r-independent $C^t_{\nu,R}$-based kth-order Markov process for any correlation matrix R.

Let $k = 1$ and let $C(u_1,u_2) = C^t_{\nu,\rho}(u_1,u_2)$ be a bivariate t-copula in (A.9). Then the process $Y_t = \left(\sqrt{S}/\sqrt{\nu}\cdot\right)t_\nu^{-1}[F(X_t)]$ satisfies (21). We thus conclude that if $\{X_t\}_{t=1}^{\infty}$ is a stationary m-dependent $C^t_{\nu,\rho}$-based Markov process (of the first order), then $\rho = 0$ and $\{Y_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. standard normal r.v.s. As before, this, together with dependence of the components of random vector (28), implies that there does not exist a stationary m-dependent $C^t_{\nu,\rho}$-based Markov process of the first order for any value of ρ.

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

5. FOURIER COPULAS

The results on limitations of Eyraud–Farlie–Gumbel–Morgenstern, separable, normal, and t-copulas presented in the previous section emphasize the substantial technical difficulty in constructing copula-based time series with flexible dependence structures. A class of copulas based on expansions by Fourier polynomials we introduce in this section allows one to overcome this difficulty.

It is not difficult to check that the conditions of Theorem 2 are satisfied for the following functions g:

$$g(u_1,\dots,u_{k+1}) = \sum_{j=1}^{N}\left[\alpha_j \sin\left(2\pi\sum_{i=1}^{k+1}\beta_i^j u_i\right) + \gamma_j \cos\left(2\pi\sum_{i=1}^{k+1}\beta_i^j u_i\right)\right], \qquad (29)$$

where $N \geq 1$, and $\alpha_j, \gamma_j \in \mathbf{R}$, and $\beta_i^j \in \mathbf{Z}$, $i = 1,\dots,k+1$, $j = 1,\dots,N$, are arbitrary numbers such that

$$\beta_1^{j_1} + \sum_{l=2}^{s}\delta_{l-1}\beta_l^{j_l} \neq 0,$$

for $j_1,\dots,j_s \in \{1,\dots,N\}$, $\delta_1,\dots,\delta_{s-1} \in \{-1,1\}$, $s = 2,\dots,k+1$, and

$$1 + \sum_{j=1}^{N}\left[\alpha_j\delta_j + \gamma_j\delta_{j+N}\right] \geq 0$$

for $\delta_1,\dots,\delta_{2N} \in \{-1,1\}$. We refer to the copulas C corresponding to the functions g in such a way,

$$C(u_1,\dots,u_{k+1}) = \int_0^{u_1}\dots\int_0^{u_{k+1}}(1+g(u_1,\dots,u_{k+1}))\,du_1\dots du_{k+1},$$

as $(k+1)$-dimensional *Fourier* copula. Each such copula can thus be used to construct a stationary k-independent kth-order Markov process via (6).

Similarly, conditions of Theorem 3 are satisfied with $m = 1$ for the bivariate Fourier copulas corresponding to the functions g defined in (29) with $k = 1$, that is, for the Fourier copulas

$$C(u_1,u_2) = \int_0^{u_1}\int_0^{u_2}(1+g(u_1,u_2))\,du_1du_2, \qquad (30)$$

where

$$g(u_1,u_2) = \sum_{j=1}^{N}\left[\alpha_j \sin\left(2\pi\left(\beta_1^j u_1 + \beta_2^j u_2\right)\right) + \gamma_j \cos\left(2\pi\left(\beta_1^j u_1 + \beta_2^j u_2\right)\right)\right], \qquad (31)$$

$N \geq 1$, $\alpha_j, \gamma_j \in \mathbf{R}$, and $\beta_1^j, \beta_2^j \in \mathbf{Z}$, $j = 1,\dots,N$, are arbitrary numbers such that $\beta_1^{j_1} + \beta_2^{j_2} \neq 0$ for $j_1, j_2 \in \{1,\dots,N\}$ and $\beta_1^{j_1} - \beta_2^{j_2} \neq 0$, $1 + \sum_{j=1}^{N}\left[\alpha_j\delta_j + \gamma_j\delta_{j+N}\right] \geq 0$ for $\delta_1,\dots,\delta_{2N} \in \{-1,1\}$. The processes constructed from copulas

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

(30) via (6) thus give examples of stationary 1-dependent first-order Markov processes.

6. CONCLUSION

The results in this paper provide a justification for estimation of $(k+1)$-dimensional copulas for stationary time series with kth-order Markovian dependence structure. For instance, the results imply that all finite-dimensional copulas and, thus, all copula-based multivariate dependence measures and properties of such time series (such as, e.g., multivariate analogues of Spearman's rho or relative entropy for finite-dimensional distributions and β-mixing properties) are determined by and can be recovered from their $(k+1)$-dimensional copulas. The paper also shows how dependence properties of time series place additional nontrivial restrictions on copulas of the processes under consideration that can be applied in inference on the properties of the processes. These restrictions also allow one to construct time series with prescribed dependence structures that can be used, for instance, in the analysis of the robustness of statistical and econometric procedures to dependence. The results further provide an approach to obtaining approximations of higher order Markov processes and their functionals, including those that arise in contingent claim pricing and other applications in economics, finance, and risk management (see Cherubini et al., 2004; McNeil et al., 2005), using approximations to their copulas of a given order. This can be accomplished, for instance, using expansions of the copulas by orthogonal functions or degenerate U-statistic kernels as in de la Peña et al. (2006) and Proposition 3 in this paper, Bernstein polynomials as in Sancetta and Satchell (2004), Hermite polynomials as in Gram–Charlier copulas discussed in Ibragimov (2005a), or Fourier polynomials similar to the constructions discussed in Section 5.[6] The study of the preceding problems is left for future research.

NOTES

1. The concept of copulas is closely related to the probability integral transformation (see Rosenblatt, 1952; Gouriéroux and Monfort, 1979; Breymann, Dias, and Embrechts, 2003, Sect. 4) and to Fréchet classes of joint distributions (see Joe, 1997, Ch. 3).

2. I thank a referee for the reference to Doukhan et al. (2004).

3. Analogues of the copula-based higher order Markov characterizations in the paper for not necessarily continuous univariate c.d.f.s can be obtained similarly to the corresponding extensions in the case of first-order Markov processes in Darsow et al. (1992), using linear (in each place) interpolation between the points at which the finite-dimensional copulas of the processes are determined uniquely. Extensions of other results in the paper can be obtained similarly to their proofs in this work.

4. In general, Markovness is not preserved by many-to-one transformations. Rosenblatt (1971, Ch. III) provides conditions under which general functions of Markov processes are still Markovian.

5. Examples of non-Markovian processes for which the Chapman–Kolmogorov equation is satisfied were also given, for example, by Feller (1959) and Rosenblatt (1960).

6. See also the recent work by Lowin (2007) for the analysis of properties and economic and financial applications of Fourier copulas introduced in Section 5 and their generalizations.

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

REFERENCES

Aaronson, J., D. Gilat, & M. Keane (1992) On the structure of 1-dependent Markov chains. *Journal of Theoretical Probability* 5, 545–561.

Breymann, W., A. Dias, & P. Embrechts (2003) Dependence structures for multivariate high-frequency data in finance. *Quantitative Finance* 3, 1–14.

Burton, R.M., M. Goulet, & R. Meester (1993) On 1-dependent processes and k-block factors. *Annals of Probability* 21, 2157–2168.

Cambanis, S. (1977) Some properties and generalizations of multivariate Eyraud-Gumbel-Morgenstern distributions. *Journal of Multivariate Analysis* 7, 551–559.

Cambanis, S. (1991) On Eyraud-Farlie-Gumbel-Morgenstern random processes. In G. Dall'aglio, S. Kotz, & G. Salinetti (eds.), *Advances in probability Distributions with Given Marginals*, pp. 207–222. Mathematics and Its Applications 67. Kluwer.

Chen, X. & Y. Fan (2004) Evaluating density forecasts via the copula approach. *Finance Research Letters* 1, 74–84.

Chen, X. & Y. Fan (2006) Estimation of copula-based semiparametric time series models. *Journal of Econometrics* 130, 307–335.

Cherubini, U., E. Luciano, & W. Vecchiato (2004) *Copula Methods in Finance.* Wiley.

Cont, R. (2001) Empirical properties of asset returns: Stylized facts and statistical issues. *Quantitative Finance* 1, 223–236.

Darsow, W.F., B. Nguyen, & E.T. Olsen (1992) Copulas and Markov processes. *Illinois Journal of Mathematics* 36, 600–642.

de la Peña, V.H., R. Ibragimov, & S. Sharakhmetov (2006) Characterizations of joint distributions, copulas, information, dependence and decoupling, with applications to time series. In J. Rojo (ed.), *2nd Erich L. Lehmann Symposium—Optimality*, pp. 183–209. IMS Lecture Notes—Monograph Series 49. Available at http://dx.doi.org/10.1214/074921706000000455.

Doukhan, P., J.-D. Fermanian, & G. Lang (2004) Copulas of a Vector-Valued Stationary Weakly Dependent Process. Working paper, CREST.

Embrechts, P., F. Lindskog, & A. McNeil (2003) Modeling dependence with copulas and applications to risk management. In S. Rachev (ed.), *Handbook of Heavy Tailed Distributions in Finance*, pp. 329–384. Elsevier.

Embrechts, P., A. McNeil, & D. Straumann (2002) Correlation and dependence in risk management: Properties and pitfalls. In M.A.H. Dempster (ed.), *Risk Management: Value at Risk and Beyond*, pp. 176–223. Cambridge University Press.

Feller, W. (1959) Non-Markovian processes with the semi-group property. *Annals of Mathematical Statistics* 30, 1252–1253.

Fermanian, J.-D., D. Radulović, & M. Wegkamp (2004) Weak convergence of empirical copula process. *Bernoulli* 10, 847–860.

Gouriéroux, C. & A. Monfort (1979) On the characterization of a joint probability distribution by conditional distributions. *Journal of Econometrics* 10, 115–118.

Granger, C.W.J., T. Teräsvirta, & A.J. Patton (2006) Common factors in conditional distributions. *Journal of Econometrics* 132, 43–57.

Hu, L. (2006) Dependence patterns across financial markets: A mixed copula approach. *Applied Financial Economics* 16, 717–729.

Ibragimov, R. (2005a) Copula-Based Dependence Characterizations and Modeling for Time Series. Harvard Institute of Economic Research Discussion paper 2094. Available at http://www.economics.harvard.edu/pub/hier/2005/HIER2094.pdf.

Ibragimov, R. (2005b) New majorization theory in economics and martingale convergence results in econometrics. Ph.D. dissertation, Yale University.

Ibragimov, R. (2009) Heavy-tailed densities. In S.N. Durlauf & L.E. Blume (eds.), *The New Palgrave Dictionary Online*. Palgrave Macmillan. Available at http://www.dictionaryofeconomics.com/article?id=pde2008_H000191.

Joe, H. (1987) Majorization, randomness and dependence for multivariate distributions. *Annals of Probability* 15, 1217–1225.

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

Joe, H. (1989) Relative entropy measures of multivariate dependence. *Journal of the American Statistical Association* 84, 157–164.
Joe, H. (1997) *Multivariate Models and Dependence Concepts*. Monographs on Statistics and Applied Probability 73. Chapman and Hall.
Johnson, N.L. & S. Kotz (1975) On some generalized Farlie-Gumbel-Morgenstern distributions. *Communications in Statistics* 4, 415–424.
Lévy, P. (1949) Exemples de processus pseudo-markoviens. *Comptes Rendus de l'Académie des Sciences* 228, 2004–2006.
Loretan, M. & P.C.B. Phillips (1994). Testing the covariance stationarity of heavy-tailed time series. *Journal of Empirical Finance* **3**, 211–248.
Lowin, J. (2007) The Fourier copula: Theory and applications. Senior thesis, Harvard University.
Matúš, F. (1996) On two-block-factor sequences and one-dependence. *Proceedings of the American Mathematical Society* 124, 1237–1242.
Matúš, F. (1998) Combining m-dependence with Markovness. *Annales de l'Institut Henri Poincaré. Probabilités et Statistiques* 34, 407–423.
McNeil, A.J., R. Frey, & P. Embrechts (2005) *Quantitative Risk Management. Concepts, Techniques and Tools.* Princeton University Press.
Nelsen, R.B. (1996) Nonparametric measures of multivariate association. In L. Rüschendorf, B. Schweizer, & M.D. Taylor (eds.), *Distributions with Fixed Marginals and Related Topics (Seattle, WA, 1993)*, pp. 223–232. IMS Lecture Notes—Monograph Series 28.
Nelsen, R.B. (1999) *An Introduction to Copulas*. Lecture Notes in Statistics 139. Springer-Verlag.
Patton, A. (2004) On the out-of-sample importance of skewness and asymmetric dependence for asset allocation. *Journal of Financial Econometrics* 2, 130–168.
Patton, A. (2006) Modelling asymmetric exchange rate dependence. *International Economic Review* 47, 527–556.
Rosenblatt, M. (1952) Remarks on a multivariate transformation. *Annals of Mathematical Statistics* 23, 470–472.
Rosenblatt, M. (1960) An aggregation problem for Markov chains. In R.E. Machol (ed.), *Information and Decision Processes,* pp. 87–92. McGraw-Hill.
Rosenblatt, M. (1971) *Markov Processes. Structure and Asymptotic Behavior.* Springer-Verlag.
Rosenblatt, M. & D. Slepian (1962) Nth order Markov chains with every N variables independent. *Journal of the Society for Industrial and Applied Mathematics* 10, 537–549.
Sancetta, A. & S. Satchell (2004). The Bernstein copula and its applications to modeling and approximations of multivariate distributions. *Econometric Theory* 20, 535–562.
Sharakhmetov, S. & R. Ibragimov (2002) A characterization of joint distribution of two-valued random variables and its applications. *Journal of Multivariate Analysis* 83, 389–408.
Sklar, A. (1959) Fonctions de répartition à n dimensions et leurs marges. *Publications de l'Institut de Statistique de l'Université de Paris* 8, 229–231.
Slepian, D. (1972) On the symmetrized Kronecker power of a matrix and extensions of Mehler's formula for Hermite polynomials. *SIAM Journal on Mathematical Analysis* 3, 606–616.

APPENDIX A: Copulas and Their Properties—U-Statistics-Based Copula Characterizations

We begin with the definition of copulas and formulation of Sklar's theorem referred to in the introduction (see, e.g., Nelsen, 1999; Embrechts et al., 2002; Embrechts, Lindskog, and McNeil, 2003).

DEFINITION 5. *A function* $C : [0,1]^n \to [0,1]$ *is called an* n*-dimensional* copula *if it satisfies the following conditions:*

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

1. $C(u_1, \ldots, u_n)$ *is increasing in each component* u_i.
2. $C(u_1, \ldots, u_{k-1}, 0, u_{k+1}, \ldots, u_n) = 0$ *for all* $u_i \in [0,1]$, $i \neq k$, $k = 1, \ldots, n$.
3. $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ *for all* $u_i \in [0,1]$, $i = 1, \ldots, n$.
4. *For all* $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n) \in [0,1]^n$ *with* $a_i \leq b_i$,

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C(x_{1i_1}, \ldots, x_{ni_n}) \geq 0,$$

where $x_{j1} = a_j$ *and* $x_{j2} = b_j$ *for all* $j \in \{1, \ldots, n\}$. *Equivalently,* C *is an* n*-dimensional copula if it is a joint c.d.f of* n *r.v.s* $U_1, \ldots, U_n$ *each of which is uniformly distributed on* $[0,1]$.

DEFINITION 6. *A copula* $C : [0,1]^n \to [0,1]$ *is called* absolutely continuous *if, when considered as a joint c.d.f., it has a joint density given by* $\partial C^n(u_1, \ldots, u_n)/\partial u_1 \ldots \partial u_n$.

PROPOSITION 1 (Sklar, 1959). *If* $X_1, \ldots, X_n$ *are r.v.s defined on a common probability space, with the one-dimensional c.d.f.s* $F_{X_k}(x_k) = P(X_k \leq x_k)$ *and the joint c.d.f.* $F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = P(X_1 \leq x_1, \ldots, X_n \leq x_n)$, *then there exists an* n*-dimensional copula* $C_{X_1,\ldots,X_n}(u_1, \ldots, u_n)$ *such that* $F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = C_{X_1,\ldots,X_n}(F_{X_1}(x_1), \ldots, F_{X_n}(x_n))$ *for all* $x_k \in \mathbf{R}$, $k = 1, \ldots, n$. *If the univariate marginal c.d.f.s* $F_{X_1}, \ldots, F_{X_n}$ *are all continuous, then the copula is unique and can be obtained via the* inversion *method:*

$$C_{X_1,\ldots,X_n}(u_1, \ldots, u_n) = F_{X_1,\ldots,X_n}\left(F_{X_1}^{-1}(u_1), \ldots, F_{X_n}^{-1}(u_n)\right), \tag{A.1}$$

where $F_{X_k}^{-1}(u_k) = \inf\{x : F_{X_k}(x) \geq u_k\}$. *Otherwise, the copula is uniquely determined at points* $(u_1, \ldots, u_n)$, *where* u_k *is in the range of* F_k, $k = 1, \ldots, n$.

Copulas are invariant under strictly increasing transformations of r.v.s with continuous univariate c.d.f.s.

PROPOSITION 2 (Nelson, 1999, Thm. 2.4.4; McNeil, 2005, Prop. 5.6). *Let* X_k, $1 \leq k \leq n$, *be r.v.s with continuous univariate marginal c.d.f.s* F_{X_k} *and a copula* C. *If* $f_k : \mathbf{R} \to \mathbf{R}$, $1 \leq k \leq n$, *are strictly increasing functions, then the r.v.s* $Y_k = f_k(X_k)$ *have the same copula* C.

In a more general case when f_k are strictly monotone (either strictly increasing or decreasing) functions there exist simple relations expressing the copula of $Y_k = f(X_k)$, $k = 1, \ldots, n$, in terms of the copula of X_k, $k = 1, \ldots, n$ (see Nelsen, 1999, Thm. 2.4.4; and Remark 1 in the present paper for the case $n = 2$).

The following proposition provides U-statistics-based characterizations of copulas.

PROPOSITION 3 (de la Peña et al., 2006). *A function* $C : [0,1]^n \to [0,1]$ *is an absolutely continuous* n*-dimensional copula if and only if there exist functions* $g_{i_1,\ldots,i_c} : [0,1]^c \to \mathbf{R}$, $1 \leq i_1 < \cdots < i_c \leq n$, $c = 2, \ldots, n$, *satisfying the conditions*

A1 (integrability):

$$\int_0^1 \cdots \int_0^1 |g_{i_1,\ldots,i_c}(t_{i_1}, \ldots, t_{i_c})| \, dt_{i_1} \ldots dt_{i_c} < \infty,$$

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

A2 (degeneracy):

$$\mathrm{E}(g_{i_1,\ldots,i_c}(u_{i_1},\ldots,u_{i_{k-1}},u_{i_k},u_{i_{k+1}},\ldots,u_{i_c})|u_{i_1},\ldots,u_{i_{k-1}},u_{i_{k+1}},\ldots,u_{i_c})$$
$$= \int_0^1 g_{i_1,\ldots,i_c}(u_{i_1},\ldots,u_{i_{k-1}},t_{i_k},u_{i_{k+1}},\ldots,u_{i_c})\,dt_{i_k} = 0,$$
$$1 \le i_1 < \cdots < i_c \le n, \qquad k = 1,2,\ldots,c, \qquad c = 2,\ldots,n,$$

A3 (positive definiteness):

$$\sum_{c=2}^{n} \sum_{1\le i_1<\cdots<i_c\le n} g_{i_1,\ldots,i_c}(u_{i_1},\ldots,u_{i_c}) \ge -1,$$

and such that

$$C(u_1,\ldots,u_n)$$
$$= \int_0^{u_1} \cdots \int_0^{u_n} \left(1 + \sum_{c=2}^{n} \sum_{1\le i_1<\cdots<i_c\le n} g_{i_1,\ldots,i_c}(t_{i_1},\ldots,t_{i_c})\right) \prod_{i=1}^{n} dt_i, \tag{A.2}$$

or, equivalently, such that the density of C satisfies

$$\frac{\partial^n C(u_1,\ldots,u_n)}{\partial u_1 \ldots \partial u_n} = 1 + \sum_{c=2}^{n} \sum_{1\le i_1<\cdots<i_c\le n} g_{i_1,\ldots,i_c}(u_{i_1},\ldots,u_{i_c}). \tag{A.3}$$

The random variables $X_1,\ldots,X_n$ with copula $C(u_1,\ldots,u_n)$ are jointly independent if and only if C is the product copula:

$$C(u_1,\ldots,u_n) = u_1 \ldots u_n. \tag{A.4}$$

Well-studied examples of copulas are given by, for example, Clayton, Gumbel, and Frank copulas (see Joe, 1997; Nelsen, 1999). Taking in (A.1) F to be the n-dimensional normal c.d.f. with linear correlation matrix R,

$$F(x) = \Phi_R^n(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \phi_{n,R}(x)\,dx, \tag{A.5}$$

with $\phi_{n,R}(x) = 1/\left((2\pi)^{n/2}|R|^{1/2}\right)\exp\left(-\frac{1}{2}xR^{-1}x\right)$, one obtains the well-known normal, or Gaussian, copula $C_R^n(u_1,\ldots,u_n)$:

$$C_R^n(u_1,\ldots,u_n) = \Phi_R^n(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_n)), \tag{A.6}$$

where, as in Section 4, $\Phi(x)$ denotes the standard normal univariate c.d.f. In the bivariate case, the normal copula can be written as

$$C_\rho(u_1,u_2) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{u_1^2 - 2\rho u_1 u_2 + u_2^2}{2(1-\rho^2)}\right) du_1 du_2, \tag{A.7}$$

where ρ is the linear correlation coefficient of the corresponding bivariate normal distribution. (Using approximations to multinormal c.d.f.s via multidimensional Hermite polynomials [see Slepian, 1972], one can obtain a class of approximations to normal copulas (A.6) by so-called Gram-Charlier copulas discussed in Ibragimov [2005a].)

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

Let $\nu > 0$ and let F be the n-dimensional Student-t c.d.f. $t^n_{\nu,R}$ with ν degrees of freedom, the linear correlation matrix R, and the location parameter $0 \in \mathbf{R}^n$. That is, $F = t^n_{\nu,R}$ is the joint c.d.f. of the random vector $\sqrt{\nu}Y/\sqrt{S}$, where $Y \sim \mathcal{N}_n(0, R)$ has the n-dimensional normal distribution with the correlation matrix R and $S \sim \chi^2(\nu)$ is a chi-square r.v. with ν degrees of freedom that is independent of Y. Formula (A.1) then gives n-dimensional t-copulas with correlation matrix R,

$$C^t_{\nu,R}(u_1, \ldots, u_n) = t^n_{\nu,R}\left(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_n)\right), \tag{A.8}$$

where $t_\nu(x)$ denotes the c.d.f. of the univariate Student t-distribution with ν degrees of freedom.

In the bivariate case, t-copulas take the form

$$C^t_{\nu,\rho}(u_1, u_2) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \times \left(1 + \frac{u_1^2 - 2\rho u_1 u_2 + u_2^2}{\nu(1-\rho^2)}\right)^{-(\nu+2)/2} du_1 du_2, \tag{A.9}$$

where $\rho \in (-1, 1)$.

Let $\alpha_{i_1,\ldots,i_c} \in \mathbf{R}$ be constants such that $\sum_{c=2}^n \sum_{1 \le i_1 < \cdots < i_c \le n} \alpha_{i_1,\ldots,i_c} \delta_{i_1} \ldots \delta_{i_c} \ge -1$ for all $\delta_i \in \{-1, 1\}$, $i = 1, \ldots, n$. Taking in Proposition 3 $g_{i_1,\ldots,i_c}(t_{i_1}, \ldots, t_{i_c}) = \alpha_{i_1,\ldots,i_c}$ $(1-2t_{i_1})(1-2t_{i_2})\ldots(1-2t_{i_c})$, $1 \le i_1 < \cdots < i_c \le n$, $c = 2, \ldots, n$, we obtain the following generalized multivariate Eyraud–Farlie–Gumbel–Morgenstern copulas (see Johnson and Kotz, 1975; Cambanis, 1977):

$$C(u_1, \ldots, u_n) = \prod_{k=1}^n u_k \left(1 + \sum_{c=2}^n \sum_{1 \le i_1 < \cdots < i_c \le n} \alpha_{i_1,\ldots,i_c}(1-u_{i_1})\ldots(1-u_{i_c})\right). \tag{A.10}$$

In the bivariate case these copulas have form (26). As shown in Sharakhmetov and Ibragimov (2002), the generalized Eyraud–Farlie–Gumbel–Morgenstern copulas and corresponding c.d.f.s completely characterize joint distributions of two-valued r.v.s.

Now let $\alpha_{i_1,\ldots,i_c} \in \mathbf{R}$ be such that $\sum_{c=2}^n \sum_{1 \le i_1 < \cdots < i_c \le n} |\alpha_{i_1,\ldots,i_c}| \le 1$ (it is easy to see that this condition is satisfied if $\alpha_{i_1,\ldots,i_c} = \lambda_{i_1} \ldots \lambda_{i_c}$, where $\sum_{i=1}^n |\lambda_i| \le 1$). Taking in Proposition 3

$$g_{i_1,\ldots,i_c}(t_{i_1}, \ldots, t_{i_c}) = \alpha_{i_1,\ldots,i_c}\left((l+1)t_{i_1}^l - (l+2)t_{i_1}^{l+1}\right)\ldots\left((l+1)t_{i_c}^l - (l+2)t_{i_c}^{l+1}\right),$$

where $l \in \{0, 1, 2, \ldots\}$, we get the following extensions of Eyraud–Farlie–Gumbel–Morgenstern copulas (A.10) that are natural to call power copulas:

$$C(u_1, \ldots, u_n) = \prod_{i=1}^n u_i \left(1 + \sum_{c=2}^n \sum_{1 \le i_1 < \cdots < i_c \le n} \alpha_{i_1,\ldots,i_c}(u_{i_1}^l - u_{i_1}^{l+1})\ldots(u_{i_c}^l - u_{i_c}^{l+1})\right). \tag{A.11}$$

Proposition 3 implies the following characterizations of r.v.s satisfying m-dependence or r-independence.

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

PROPOSITION 4 (de la Peña et al., 2006, Thm. 13). *Let* $2 \le r < n$. *Random variables* $X_1, \ldots, X_n$ *are* r*-independent if and only if the functions* $g_{i_1,\ldots,i_c}$ *in representation (A.2) satisfy the conditions* $g_{i_1,\ldots,i_c}(u_{i_1}, \ldots, u_{i_c}) = 0$, $1 \le i_1 < \cdots < i_c \le n$, $c = 2, \ldots, r$.

PROPOSITION 5 (de la Peña et al., 2006, Thm. 11). *Let* $1 \le m \le n-2$. *Random variables* $X_1, \ldots, X_n$ *are* m*-dependent if and only if the functions* g *in representation (A.2) satisfy the conditions* $g_{i_1,\ldots,i_k,i_{k+1},\ldots,i_c}(u_{i_1}, \ldots, u_{i_k}, u_{i_{k+1}}, \ldots, u_{i_c}) = g_{i_1,\ldots,i_k}(u_{i_1}, \ldots, u_{i_k})$ $g_{i_{k+1},\ldots,i_c}(u_{i_{k+1}}, \ldots, u_{i_c})$ *for all* $1 \le i_1 < \cdots < i_k < i_{k+1} \ldots < i_c \le n$, $i_{k+1} - i_k \ge m+1$, $k = 1, \ldots, c-1$, $c = 2, \ldots, n$.

Darsow et al. (1992) obtained the following necessary and sufficient conditions for a time series process based on bivariate copulas to be first-order Markov. For copulas $A, B : [0,1]^2 \to [0,1]$, set

$$(A * B)(x, y) = \int_0^1 \frac{\partial A(x,t)}{\partial t} \cdot \frac{\partial B(t,y)}{\partial t}\, dt.$$

Further, for copulas $A : [0,1]^m \to [0,1]$ and $B : [0,1]^n \to [0,1]$, define their $\star$-product $A \star B : [0,1]^{m+n-1} \to [0,1]$ via

$$A \star B(x_1, \ldots, x_{m+n-1}) = \int_0^{x_m} \frac{\partial A(x_1, \ldots, x_{m-1}, \xi)}{\partial \xi} \cdot \frac{\partial B(\xi, x_{m+1}, \ldots, x_{m+n-1})}{\partial \xi} d\xi.$$

As shown in Darsow et al. (1992), the operators $*$ and $\star$ on copulas are distributive over convex combinations, associative, and continuous in each place but not jointly continuous.

Darsow et al. (1992) proved that the transition probabilities $P(s, x, t, A) = P(X_t \in A | X_s = x)$ of a real-valued stochastic process $\{X_t\}_{t \in T}$, $T \subseteq \mathbf{R}$, satisfy Chapman–Kolmogorov equations (8) if and only if the copulas corresponding to bivariate distributions of X_t satisfy

$$C_{st} = C_{su} * C_{ut} \tag{A.12}$$

for all $s, u, t \in T$ such that $s < u < t$. Darsow et al. (1992) also showed that a real-valued stochastic process $\{X_t\}_{t \in T}$ is a first-order Markov process if and only if the copulas corresponding to the finite-dimensional distributions of $\{X_t\}$ satisfy the conditions

$$C_{t_1,\ldots,t_n} = C_{t_1 t_2} \star C_{t_2 t_3} \star \cdots \star C_{t_{n-1} t_n}$$

for all $t_1, \ldots, t_n \in T$ such that $t_k < t_{k+1}$, $k = 1, \ldots, n-1$.

APPENDIX B: Proofs

As before, for a r.v. X_t in the process $\{X_t\}_{t \in T}$, we denote by F_t its c.d.f. As usual, for a Borel set $A \in \Im$, the notation $X^{-1}(A)$ will stand for the event $\{\omega \in \Omega : X(\omega) \in A\}$.

Proof of Theorem 1. Clearly, it suffices to consider the case $T = \mathbf{N}$. Let $n \ge k+1$. Let us show that the Markovian (order k) property (4) holds for $t_1 = 1, \ldots, t_n = n$ and $t = n+1$ if and only if

$$\begin{aligned} P\big(X_1 < x_1, \ldots, X_{n-k} < x_{n-k}, X_{n+1} < x_{n+1} \big| X_{n-k+1}, \ldots, X_n\big) \\ = P\big(X_1 < x_1, \ldots, X_{n-k} < x_{n-k} \big| X_{n-k+1}, \ldots, X_n\big) \\ \times P\big(X_{n+1} < x_{n+1} \big| X_{n-k+1}, \ldots, X_n\big). \end{aligned} \tag{B.1}$$

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

Indeed, suppose that (4) holds for $t_1 = 1, \dots, t_n = n$ and $t = n+1$. Then we have

$$
\begin{aligned}
&P\left(X_1 < x_1, \dots, X_{n-k} < x_{n-k}, X_{n+1} < x_{n+1} \middle| X_{n-k+1}, \dots, X_n\right) \\
&\quad = \mathrm{E}\left(I_{X_1 < x_1} \dots I_{X_{n-k} < x_{n-k}} I_{X_{n+1} < x_{n+1}} \middle| X_{n-k+1}, \dots, X_n\right) \\
&\quad = \mathrm{E}\left\{ E\left(I_{X_1 < x_1} \dots I_{X_{n-k} < x_{n-k}} I_{X_{n+1} < x_{n+1}} \middle| X_1, \dots, X_n\right) \middle| X_{n-k+1}, \dots, X_n \right\} \\
&\quad = \mathrm{E}\left\{ I_{X_1 < x_1} \dots I_{X_{n-k} < x_{n-k}} E\left(I_{X_{n+1} < x_{n+1}} \middle| X_1, \dots, X_n\right) \middle| X_{n-k+1}, \dots, X_n \right\} \\
&\quad = \mathrm{E}\left\{ I_{X_1 < x_1} \dots I_{X_{n-k} < x_{n-k}} E\left(I_{X_{n+1} < x_{n+1}} \middle| X_{n-k+1}, \dots, X_n\right) \middle| X_{n-k+1}, \dots, X_n \right\} \\
&\quad = \mathrm{E}\left(I_{X_1 < x_1} \dots I_{X_{n-k} < x_{n-k}} \middle| X_{n-k+1}, \dots, X_n\right) P\left(I_{X_{n+1} < x_{n+1}} \middle| X_{n-k+1}, \dots, X_n\right) \\
&\quad = P\left(X_1 < x_1, \dots, X_{n-k} < x_{n-k} \middle| X_{n-k+1}, \dots, X_n\right) \\
&\qquad \times P\left(X_{n+1} < x_{n+1} \middle| X_{n-k+1}, \dots, X_n\right),
\end{aligned}
$$

that is, (B.1) holds. Conversely, if (B.1) holds, then from the preceding chain of equalities read in the opposite order it follows that

$$
\begin{aligned}
&\mathrm{E}\left\{ I_{X_1 < x_1} \dots I_{X_{n-k} < x_{n-k}} \mathrm{E}\left(I_{X_{n+1} < x_{n+1}} \middle| X_1, \dots, X_n\right) \middle| X_{n-k+1}, \dots, X_n \right\} \\
&\quad = \mathrm{E}\left\{ I_{X_1 < x_1} \dots I_{X_{n-k} < x_{n-k}} \mathrm{E}\left(I_{X_{n+1} < x_{n+1}} \middle| X_{n-k+1}, \dots, X_n\right) \middle| X_{n-k+1}, \dots, X_n \right\},
\end{aligned}
$$

that is, for all Borel subsets $B_{n-k+1}, \dots, B_n$ of $\mathbf{R}$,

$$
\begin{aligned}
&\mathrm{E}\left\{ I_{X_1 < x_1} \dots I_{X_{n-k} < x_{n-k}} I_{X_{n-k+1} \in B_{n-k+1}} \dots I_{X_n \in B_n} \mathrm{E}\left(I_{X_{n+1} < x_{n+1}} \middle| X_1, \dots, X_n\right) \right\} \\
&\quad = \mathrm{E}\left\{ I_{X_1 < x_1} \dots I_{X_{n-k} < x_{n-k}} I_{X_{n-k+1} \in B_{n-k+1}} \dots I_{X_n \in B_n} \right. \\
&\qquad \left. \times \mathrm{E}\left(I_{X_{n+1} < x_{n+1}} \middle| X_{n-k+1}, \dots, X_n\right) \right\}.
\end{aligned}
$$

This relation means that (4) holds for $t_1 = 1, \dots, t_n = n$ and $t = n+1$.

Suppose now that $\{X_t\}_{t=1}^{\infty}$ is a Markov process of order k. Integrating (B.1) over $X_{n-k+1}^{-1}((-\infty, x_{n-k+1})) \times \dots \times X_n^{-1}((-\infty, x_n))$, we get

$$
\begin{aligned}
&C_{1,2,\dots,n+1}\left(F_1(x_1), \dots, F_{n+1}(x_{n+1})\right) = F_{1,2,\dots,n+1}\left(x_1, \dots, x_{n+1}\right) \\
&\quad = \int_{-\infty}^{x_{n-k+1}} \dots \int_{-\infty}^{x_n} C_{1,2,\dots,n|n-k+1,\dots,n}\left(F_1(x_1), \dots, F_{n-k}(x_{n-k}), \right. \\
&\qquad \left. \times F_{n-k+1}(\eta_{n-k+1}), \dots, F_n(\eta_n)\right) \\
&\qquad \times C_{n-k+1,\dots,n,n+1|n-k+1,\dots,n}\left(F_{n-k+1}(\eta_{n-k+1}), \dots, F_n(\eta_n), F_{n+1}(x_{n+1})\right) \\
&\qquad \times dF_{n-k+1,\dots,n}\left(\eta_{n-k+1}, \dots, \eta_n\right)
\end{aligned}
$$

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

$$= \int_0^{F_{n-k+1}(x_{n-k+1})} \cdots \int_0^{F_n(x_n)} C_{1,2,\ldots,n|n-k+1,\ldots,n}\left(F_1(x_1), \ldots, F_{n-k}(x_{n-k}), \right.$$
$$\left. \times \xi_{n-k+1}, \ldots, \xi_n\right)$$
$$\times C_{n-k+1,\ldots,n,n+1|n-k+1,\ldots,n}\left(\xi_{n-k+1}, \ldots, \xi_n, F_{n+1}(x_{n+1})\right)$$
$$\times C_{n-k+1,\ldots,n}\left(d\xi_{n-k+1}, \ldots, d\xi_n\right)$$
$$= C_{1,2,\ldots,n} \star^k C_{n-k+1,\ldots,n+1}\left(F_1(x_1), \ldots, F_{n+1}(x_{n+1})\right).$$

By induction, this implies that (5) holds.

Conversely, suppose that (5) holds. Then we have

$$\mathrm{E} I_{X_1<x_1} \ldots I_{X_{n-k}<x_{n-k}} I_{X_{n-k+1}<x_{n-k+1}} \cdots I_{X_n<x_n} I_{X_{n+1}<x_{n+1}}$$
$$= C_{1,2,\ldots,n+1}\big(F_1(x_1), \ldots, F_{n-k}(x_{n-k}), F_{n-k+1}(x_{n-k+1}), \ldots,$$
$$\times\ F_n(x_n), F_{n+1}(x_{n+1})\big)$$
$$= \int_0^{F_{n-k+1}(x_{n-k+1})} \cdots \int_0^{F_n(x_n)} C_{1,2,\ldots,n|n-k+1,\ldots,n}(F_1(x_1), \ldots,$$
$$\times F_{n-k}(x_{n-k}), \xi_{n-k+1}, \ldots, \xi_n)$$
$$\times C_{n-k+1,\ldots,n,n+1|n-k+1,\ldots,n}(\xi_{n-k+1}, \ldots, \xi_n, F_{n+1}(x_{n+1}))$$
$$\times C_{n-k+1,\ldots,n}(d\xi_{n-k+1}, \ldots, d\xi_n)$$
$$= \mathrm{E}\big(\mathrm{E}(I_{X_1<x_1} \ldots I_{X_{n-k}<x_{n-k}} | X_{n-k+1}, \ldots, X_n) I_{X_{n-k+1}<x_{n-k+1}} \ldots I_{X_n<x_n}$$
$$\times \mathrm{E}(X_{n+1} | X_{n-k+1}, \ldots, X_n)\big).$$

This implies that (B.1) holds. ■

Proof of Theorem 2. Let C be a $(k+1)$-dimensional copula and let $\{X_t\}_{t=1}^{\infty}$ be a stationary C-based kth-order Markov process. Using Propositions 3 and 4, we obtain that if the process $\{X_t\}_{t=1}^{\infty}$ is k-independent, then the density of the copula C has form (A.3) with $n = k+1$ and the functions g such that $g_{i_1,\ldots,i_c}(u_{i_1}, \ldots, u_{i_c}) = 0$, $1 \le i_1 < \cdots < i_c \le n$, $c = 2, \ldots, k$, that is, (9) holds with $g(u_1, \ldots, u_{k+1}) = g_{1,\ldots,k+1}(u_1, \ldots, u_{k+1})$. In addition, by the same propositions, the preceding function g satisfies conditions (10) and (12) and is such that

$$\int_0^1 g(u_1, \ldots, u_{k+1})\, du_j = 0, \qquad j = 1, 2, \ldots, k+1. \tag{B.2}$$

Further, from Corollary 1 it follows that the density of the copula $C_{1,2,\ldots,k+1,k+2}$ of the r.v.s $X_1, X_2, \ldots, X_{k+1}, X_{k+2}$ is given by

$$\frac{\partial^{k+2} C_{1,2,\ldots,k+1,k+2}(u_1, u_2, \ldots, u_{k+1}, u_{k+2})}{\partial u_1 \partial u_2 \ldots \partial u_{k+1} \partial u_{k+2}}$$
$$= (1 + g(u_1, \ldots, u_{k+1}))(1 + g(u_2, \ldots, u_{k+2}))$$
$$= 1 + g(u_1, \ldots, u_{k+1}) + g(u_2, \ldots, u_{k+2}) + g(u_1, \ldots, u_{k+1}) g(u_2, \ldots, u_{k+2}). \tag{B.3}$$

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

Using (B.2) and (B.3) we get that, for $2 \le i_1 < i_2 \le k+1$, the density of the copula of the r.v.s X_j, $j \in \{1,2,\dots,k+2\} \setminus \{i_1, i_2\}$ is given by

$$1 + \int_0^1 \int_0^1 g(u_1,\dots,u_{k+1}) g(u_2,\dots,u_{k+2})\, du_{i_1} du_{i_2}.$$

This and k-independence of $\{X_t\}$ imply that

$$\int_0^1 \int_0^1 g(u_1,\dots,u_{k+1}) g(u_2,\dots,u_{k+2})\, du_{i_1} du_{i_2} = 0, \qquad 2 \le i_1 < i_2 \le k+1. \tag{B.4}$$

In complete similarity, by considering the k-dimensional marginal copulas of the r.v.s $X_1, X_2, \dots, X_{k+2}, X_{k+3}$ and using (B.2)–(B.4), we obtain

$$\int_0^1 \int_0^1 \int_0^1 g(u_1,\dots,u_{k+1}) g(u_2,\dots,u_{k+2}) g(u_3,\dots,u_{k+3})\, du_{i_1} du_{i_2} du_{i_3} = 0,$$

$3 \le i_1 < i_2 < i_3 \le k+1$. Continuing in the same fashion, we get that the property that $\{X_t\}_{t=1}^{\infty}$ is a stationary k-independent C-based kth-order Markov process implies that (11) holds for all $s \le u_{i_1} < \cdots < u_{i_s} \le k+1$, $s = 1, 2, \dots, [(k+1)/2]$.

Suppose now that relation (11) holds for all $s \le u_{i_1} < \cdots < u_{i_s} \le k+1$, $s = 1, 2, \dots, [(k+1)/2]$. One then easily gets that (11) also holds for all $1 \le u_{i_1} < \cdots < u_{i_s} \le k+s$, $s \ge 1$, and the product $g(u_1,\dots,u_{k+1}) g(u_2,\dots,u_{k+2}) \dots g(u_s,\dots,u_{k+s})$ that appears in the density $\frac{\partial^{k+s} C_{1,\dots,k+s}(u_1,\dots,u_{k+s})}{\partial u_1 \dots \partial u_{k+s}} = \prod_{j=1}^{s} (1 + g(u_j,\dots,u_{k+j}))$ of the copula of $X_1, X_2, \dots, X_{k+s}$. It is not difficult to see, similar to the preceding analysis, that this implies that the copula $C_{1,2,\dots,k+s}$ has k-dimensional marginal copulas in product form (A.4) with $n = k$. Thus, the r.v.s $X_1, X_2, \dots, X_{k+s}$ are k-independent for all $s \ge 1$. ■

Proof of Theorem 3. Let C be a bivariate copula and let $\{X_t\}_{t=1}^{\infty}$ be a stationary C-based first-order Markov process. By Proposition 3, the density of the copula C is given by (13) with the function $g(u_1, u_2)$ satisfying conditions (14)–(16). In addition, from Corollary 1 it follows that the density of the copula $C_{1,2,\dots,m+1,m+2}$ of the r.v.s $X_1, X_2, \dots, X_{m+1}, X_{m+2}$ has the form

$$\begin{aligned}
&\frac{\partial^{m+2} C_{1,2,\dots,m+1,m+2}(u_1, u_2, \dots, u_{m+1}, u_{m+2})}{\partial u_1 \partial u_2 \dots \partial u_{m+1} \partial u_{m+2}} \\
&\quad = (1 + g(u_1, u_2))(1 + g(u_2, u_3)) \dots (1 + g(u_{m+1}, u_{m+2})) \\
&\quad = \prod_{s=1}^{m+1} (1 + g(u_s, u_{s+1})).
\end{aligned}$$

Using relation (15) we thus get that the copula $C_{1,m+2}$ of the r.v.s X_1 and X_{m+2} is given by

$$\begin{aligned}
C_{1,m+2}(u_1, u_{m+2}) &= \int_0^1 \dots \int_0^1 \prod_{s=1}^{m+1} (1 + g(u_s, u_{s+1}))\, du_2 \dots du_{m+1} \\
&= 1 + \int_0^1 \dots \int_0^1 \prod_{s=1}^{m+1} g(u_s, u_{s+1})\, du_2 \dots du_{m+1}.
\end{aligned}$$

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

Thus, the copula $C_{1,m+2}$ is the product copula: $C_{1,m+2}(u_1, u_{m+2}) = u_1 u_{m+2}$ if and only if condition (17) is satisfied. It is not difficult to see that if (17) holds, then one also has

$$\int_0^1 \cdots \int_0^1 \prod_{s=1}^{m+n+j} g(u_s, u_{s+1})\, du_{n+1} \ldots du_{m+n} = 0$$

for all $n \geq 1$, $j \geq 0$. Similar to the preceding analysis, this relation, together with (15), implies that the random vectors $(X_1, \ldots, X_n)$ and $(X_{m+n+1}, \ldots, X_{m+n+j+1})$ in the stationary C-based Markov process $\{X_t\}_{t=1}^{\infty}$ are independent for all $n \geq 1$, $j \geq 0$. ■

Proof of Theorem 4. By the Markov property, $P(X_t > x|\Im_{t-1}) = P(X_t < -x|\Im_{t-1})$, $x \geq 0$, if and only if $P(X_t > x|X_{t-1}) = P(X_t < -x|X_{t-1})$, $x \geq 0$. The latter inequality, in turn, is equivalent to $P(V_n > 1/2 + u|V_{n-1}) = P(V_n < 1/2 - u|V_{n-1})$, $u \in [0, 1/2)$, where $V_t = F(X_t)$ and, by stationarity, to $P(V_2 > 1/2 + u|V_1) = P(V_2 < 1/2 - u|V_1)$, $u \in [0, 1/2)$. We have therefore, that $\{X_t\}$ is a conditionally symmetric martingale difference if and only if $\frac{\partial C(V_1, 1/2-u)}{\partial u_1} = 1 - \frac{\partial C(V_1, 1/2+u)}{\partial u_1}$ or, equivalently, if and only if (18) and (19) hold. ■

Proof of Theorem 5. Using relation (11) in Theorem 2 with $s = 2$ and $i_1 = 2$, $i_2 = 3$, we get that if, under the conditions of Theorem 5, $\{X_t\}_{t=1}^{\infty}$ is a stationary k-independent C-based kth-order Markov process, then

$$\int_0^1 \int_0^1 g(u_1, u_2, \ldots, u_{k+1}) g(u_2, u_3, \ldots, u_{k+2})\, du_2\, du_3$$

$$= \int_0^1 \int_0^1 \alpha^2 f(u_1) f^2(u_2) f^2(u_3) \ldots f^2(u_{k+1}) f(u_{k+2})\, du_2\, du_3 = 0,$$

that is,

$$\alpha^2 \left[\int_0^1 f^2(u_2)\, du_2\right] \left[\int_0^1 f^2(u_3)\, du_3\right] f(u_1) f^2(u_3) \ldots f^2(u_{k+1}) f(u_{k+2}) = 0.$$

This evidently implies that $g(u_1, u_2, \ldots, u_{k+1}) = \alpha f(u_1) \ldots f(u_{k+1}) = 0$ and, thus, $\{X_t\}$ is a sequence of i.i.d. r.v.s. ■

Proof of Corollary 2. The corollary is a consequence of Theorem 5 applied to the function $f(u) = 1 - 2u$. ■

Proof of Corollary 3. The corollary is a consequence of Theorem 5 applied to the function $f(u) = (l+1)u^l - (l+2)u^{l+1}$. ■

Proof of Theorem 6. Using relation (17) in Theorem 3, we obtain that, if, under the conditions of Theorem 6, $\{X_t\}_{t=1}^{\infty}$ is a stationary m-dependent C-based Markov process, then

$$\int_0^1 \cdots \int_0^1 \alpha^{m+1} f(u_1) f^2(u_2) \ldots f^2(u_{m+1}) f(u_{m+2})\, du_2 \ldots du_{m+1} = 0,$$

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press

that is,

$$\alpha^{m+1} f(u_1) f(u_{m+2}) \left[\int_0^1 f^2(u_2)\, du_2 \right]^m = 0.$$

This evidently implies that $\alpha = 0$ or $f(u) = 0$ and, thus, $\{X_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. r.v.s. ■

Proof of Corollary 4. The corollary is a consequence of Theorem 6 applied to the function $f(u) = 1 - 2u$. ■

https://doi.org/10.1017/S0266466609090720 Published online by Cambridge University Press