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Dynamics of post-critically finite maps in higher dimension

MATTHIEU ASTORG

Université d'Orléans, Collegium Sciences et Techniques, Bâtiment de mathématiques – Rue de Chartres, B.P. 6759 – 45067 Orléans Cedex 2, France
(e-mail: matthieu.astorg@univ-orleans.fr)

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Abstract. We study the dynamics of post-critically finite endomorphisms of $\mathbb{P}^k(\mathbb{C})$. We prove that post-critically finite endomorphisms are always post-critically finite all the way down under a regularity condition on the post-critical set. We study the eigenvalues of periodic points of post-critically finite endomorphisms. Then, under a transversality condition and assuming Kobayashi hyperbolicity of the complement of the post-critical set, we prove that the only possible Fatou components are super-attracting basins; thus, partially extending to any dimension is a result of Fornaess–Sibony and Rong holding in the case $k = 2$.

1. *Introduction*

This paper deals with the dynamics of some endomorphisms of the complex projective space $\mathbb{P}^k$, called post-critically finite (PCF). An endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ is PCF if

$$\mathcal{P}(f, \mathbb{P}^k) := \bigcup_{n \geq 1} f^n(\mathcal{C}(f))$$

is algebraic, where $\mathcal{C}(f, \mathbb{P}^k) := \{z \in \mathbb{P}^k, Df(z) \text{ is not invertible}\}$. The set $\mathcal{C}(f, \mathbb{P}^k)$ is called the critical set of f, and the set $\mathcal{P}(f, \mathbb{P}^k)$ is called the post-critical set of f. We will often consider restrictions of f to an invariant subvariety $L \subset \mathbb{P}^k$: in this case, $\mathcal{C}(f, L)$ and $\mathcal{P}(f, L)$ will refer to the critical and post-critical sets of the restriction map $f : L \to L$. When there is no ambiguity, we will just denote those sets by $\mathcal{P}(f)$ and $\mathcal{C}(f)$.

In the case where $k = 1$, this class of endomorphism has been particularly studied, mainly because of an important classification theorem due to Thurston [**Thu85**] (see also [**DH93**]). The dynamics of those maps is also well understood.

There is no analog of Thurston's theorem in higher dimension ($k > 1$), but PCF maps are still of interest, for example because of a construction due to Koch [**Koc13**]. The dynamics of PCF maps has been investigated by several authors. In [**FS92**], Fornaess

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and Sibony studied the dynamics of two specific examples of PCF endomorphisms of $\mathbb{P}^2$. In [**FS94**], the same authors considered more generally PCF endomorphisms of $\mathbb{P}^k$ and studied the case $k = 2$. They notably proved that if $\mathbb{P}^2 \backslash \mathcal{P}(f)$ is Kobayashi hyperbolic, where f is a PCF endomorphism of $\mathbb{P}^2$, then the only Fatou components of f are basins of super-attracting cycles. As Jonsson notes in [**Jon98**], the case of dimension two is still special, as in this case the post-critical set has dimension one, and it can be proved that the normalization of post-critical components must be either tori or projective lines. In a still higher dimension ($k \geq 3$), no such classification is readily available.

In [**Ued98**], Ueda defined the notion of an endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ that is PCF of order m: informally, a PCF endomorphism is PCF of order 1. Then every critical component is eventually mapped to a periodic post-critical component, and one can consider the first return maps induced by restriction of iterates of f to those periodic components. If all of them are themselves PCF, then f is PCF of order 2. One can then define inductively what it means to be post-critically finite of order m, with $m \leq k$ (see Definition 4 for a precise definition). PCF endomorphisms of $\mathbb{P}^k$ of order k are referred to as *PCF all the way down* (the authors of [**GHK16**] use the term *strongly PCF*).

Those notions lead to defining post-critical finiteness for self-branched covers $f : X \to X$, where X is a regular subvariety of $\mathbb{P}^k$, or more generally a complex manifold. Such maps may have an empty post-critical set (if they are unbranched self-covers); in this case, most authors do not consider them to be PCF. We shall call them *weakly PCF*, and we say that f is weakly PCF all the way down if every first return map on periodic post-critical components is weakly PCF. Again, see Definition 4 for full details.

In [**Jon98**], Jonsson remarks that PCF endomorphisms of $\mathbb{P}^2$ are always PCF all the way down. In [**GHK16**], the authors prove that the same is true for a very specific subclass of PCF endomorphisms of $\mathbb{P}^k$, and ask if the same holds for any PCF endomorphism of $\mathbb{P}^k$. They also give examples of rational maps $f : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ that are PCF but not PCF all the way down. A key point is that those examples are not endomorphisms of $\mathbb{P}^k$ as they have a non-empty indeterminacy set. We give a partial answer to this question, using a purely complex-analytical approach.

THEOREM A. *Let X be a complex manifold and let $f : X \to X$ be a PCF branched covering. Let $(L_i)_{i \in I}$ be the periodic irreducible components of $\mathcal{P}(f, X)$ and assume that for any $J \subset I$, $\bigcap_{j \in J} L_j$ is smooth. Then f is weakly PCF all the way down.*

In fact, we prove a slightly more precise statement, see Theorem 1. It should be noted that our assumptions imply the smoothness of the post-critical components at every level. If $f : X \to X$ is an endomorphism of a singular complex curve X, then post-critical finiteness of f is defined using the normalization $\tilde{X}$ (see [**Jon98**]). However, this is specific to dimension one, and if X is a higher dimensional singular analytic space, then it is not quite clear how one should define post-critical finiteness, and our methods do not apply. Rong gave in [**Ron08**] an example of PCF endomorphisms of $\mathbb{P}^2$ with a singular periodic post-critical curve. It is, to the best of our knowledge, the only such example in the literature.

The local dynamics of a periodic point p of period m is governed by the eigenvalues of $Df^m(p)$; abusing terminology, we call them the eigenvalues of p. If $\lambda \in \mathbb{C}$ is such an

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eigenvalue, we say that λ is as follows:

(1) attracting if $|\lambda| < 1$;
(2) repelling if $|\lambda| > 1$;
(3) super-attracting if $\lambda = 0$;
(4) neutral if $|\lambda| = 1$: more precisely, λ is
   (a) parabolic if λ is a root of unity;
   (b) irrationally neutral otherwise.

If p is a periodic point such that all of its eigenvalues are attracting (respectively repelling), we say that p itself is attracting (respectively repelling).

In [**Jon98**], Jonsson studied PCF endomorphisms of $\mathbb{P}^2$ such that every critical component is strictly preperiodic, and such that the first return maps of the periodic components of $\mathcal{P}(f)$ also have only strictly preperiodic critical points: such PCF endomorphisms are called *2-critically finite*. He proved that for such maps, every periodic point is repelling, and that repelling periodic points are dense in $\mathbb{P}^2$. This result, together with the classical fact that PCF maps on $\mathbb{P}^1$ can only have repelling or super-attracting periodic points, raises the question of knowing which eigenvalues a periodic point of a PCF endomorphism on $\mathbb{P}^k$ may have. We prove the following the theorem.

THEOREM B. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a PCF endomorphism such that the periodic irreducible components of $\mathcal{P}(f, \mathbb{P}^k)$ are weakly transverse (see Definition 1). Let p be a periodic point of period l for f. Then the following hold:*

(1) *p does not have any parabolic or non-zero attracting eigenvalue;*
(2) *either p has at least one repelling eigenvalue, or else all eigenvalues are super-attracting (i.e. $Df^l(p)$ is nilpotent);*
(3) *if p does not have any super-attracting eigenvalue, then p is Lyapunov unstable and at least one of its eigenvalues is repelling.*

See Definition 6 for a definition of the notion of Lyapunov unstability. We could not exclude the possibility of a periodic point having some irrationnally neutral eigenvalues. To the best of our knowledge, no example of PCF endomorphism of $\mathbb{P}^k$ with a periodic point having such an eigenvalue is known (it is known that it cannot occur in dimension one). It would be interesting to know whether such examples exist or not.

Lastly, we study the Fatou dynamics of PCF endomorphisms of $\mathbb{P}^k$. For an endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$, the Fatou set is the largest open subset of $\mathbb{P}^k$ on which the iterates $\{f^n, n \in \mathbb{N}\}$ form a normal family. We denote it by $\mathcal{F}(f)$. Its complement is the Julia set $\mathcal{J}(f) = \mathcal{J}_1(f)$. Alternatively, one can construct a naturel $(1, 1)$ positive closed current T_f called the Green current of f, whose support is exactly $\mathcal{J}(f)$, and define a stratification of Julia sets $\mathcal{J}_m(f)$ (with $1 \leq m \leq k$) as the support of $\bigwedge_{i=1}^{m} T_f$. As m increases, $\mathcal{J}_m(f)$ decreases and the dynamics on $\mathcal{J}_m(f)$ become more chaotic in some sense. In [**DT05**], de Thélin analyzed the structure of T_f on $\mathcal{J}_1(f) \backslash \mathcal{J}_2(f)$ for PCF endomorphisms of $\mathbb{P}^2$: he proved that in that case T_f is laminar, something that is not true for general endomorphisms of $\mathbb{P}^2$. Our next result focuses on the dynamics in the Fatou set, outside of $\mathcal{J}_1(f)$.

We call Fatou's component a connected component of the Fatou set. For more background on Fatou–Julia theory, see for example [**DS10**].

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In [**FS94**], Fornaess and Sibony proved that if $f : \mathbb{P}^2 \to \mathbb{P}^2$ is PCF and $\mathbb{P}^2 \backslash \mathcal{P}(f)$ is Kobayashi hyperbolic, then the only Fatou components of f are basins of super-attracting cycles. Using methods from [**Ued98**], Rong was able in [**Ron08**] to remove the assumption of Kobayashi hyperbolicity, still in the case of dimension two.

We give another partial generalization of this result, which is available in any dimension but requires Kobayashi hyperbolicity and a transversality condition on the irreducible components of the post-critical set.

THEOREM C. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a PCF endomorphism, and let $L_i, i \in I$ be the irreducible components of its post-critical set $\mathcal{P}(f, \mathbb{P}^k)$. Assume the following:*

(1) *$(L_i)_{i\in I}$ is weakly transverse;*

(2) *$\mathbb{P}^k \backslash \mathcal{P}(f, \mathbb{P}^k)$ is Kobayashi hyperbolic.*

Then the Fatou set of f is either empty or a finite union of basins of super-attracting cycles, whose points are zero-dimensional intersections of components L_i.

Note that this in particular implies that there are only finitely many periodic Fatou components, something that it always true for endomorphisms of $\mathbb{P}^1$ but not necessarily so for endomorphisms of $\mathbb{P}^k$ with $k \geq 2$, and that every Fatou component is preperiodic (the first examples of endomorphisms of $\mathbb{P}^k$ having a non-preperiodic Fatou component have been constructed in [**ABD⁺16**]).

Finally, Koch has given in [**Koc13**] a systematic method for constructing examples of PCF endomorphisms of $\mathbb{P}^k$. This is interesting, as constructing non-trivial PCF endomorphisms is not easy. Indeed, most examples in the literature can be recovered from this construction. The construction starts with a purely topological object called a *topological polynomial*, and gives a holomorphic PCF endomorphism of $\mathbb{P}^k$ whose dynamics is related to the action of that topological polynomial on a Teichmüller space. In §7, we recall Koch's results and we show that Theorems A, B, and C apply to her examples.

Outline. In §2, we introduce a weak notion of transversality and we prove some lemmas that will be required later. In §3, we define properly the notions of weakly PCF, PCF all the way down, and weakly PCF all the way down, and we prove Theorem A. In §4, we prove a slightly generalized version of a theorem of Ueda on the absence of rotation domains for PCF maps, that will be required for the proof of Theorems B and C. In §5, we prove Theorem B, and in §6, we prove Theorem C. Finally, in §7, we discuss how our results apply to the examples from [**Koc13**].

Notation. In the rest of the article, if L is an analytic subset of a complex manifold X, we will denote the singular part of L by $\mathrm{Sing}(L)$ and its regular part by $\mathrm{Reg}(L)$.

2. *Weak transversality and Kobayashi metric*

Definition 1. Let X be a complex manifold, and let $L_i, i \in I$ be a finite set of irreducible hypersurfaces of X. We say that the $(L_i)_{i\in I}$ are *weakly transverse* if the following holds: for any $x_0 \in \bigcup_{i\in I} L_i$, if $J = \{i \in I, x_0 \in L_i\}$ and if locally near x_0 the sets L_i can be

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written $L_i = \{P_i = 0\}$ (where the P_i are holomorphic functions defined locally near x_0), then $x \mapsto (P_j(x))_{j\in J}$ has constant rank near x_0.

Note that in the particular case where $x \mapsto (P_j(x))_{j\in J}$ is surjective, the implicit function theorem implies that $\bigcup_{i\in I} L_i$ has simple normal crossings. This is a weaker requirement: for example any finite collection of hyperplanes in a projective space will be weakly transverse in the sense of the above definition.

A straightforward consequence of the Constant rank theorem is that if $(L_i)_{i\in I}$ are weakly transverse, then for any $J \subset I$, $\bigcap_{j\in J} L_j$ is a smooth submanifold (though not necessarily of codimension card J), of tangent space at x_0 given by $\bigcap_{j\in J} T_{x_0} L_j$.

Let us begin by proving two lemmas that will be used in later sections.

LEMMA 1. *Let X be a complex manifold, and let $(L_i)_{i\in I}$ be weakly transverse. Let $x_0 \in \bigcup_{i\in I} L_i$ and $J = \{i \in I, x_0 \in L_i\}$. Let $v \in \bigcap_{j\in J} T_{x_0} L_j$. There exists a holomorphic vector field ξ defined in a neighborhood U of x_0, with $\xi(x_0) = v$, and such that ξ is uniformly bounded in the Kobayashi pseudometric of $X \setminus \bigcup_{i\in I} L_i$ on $U \setminus \bigcup_{i\in I} L_i$.*

Proof. For $j \in J$, let P_j be a holomorphic function defined in the neighborhood of x_0 such that locally near x_0, $L_j = \{P_j = 0\}$. Let $\Phi : x \mapsto (P_j(x))_{j\in J}$. Then by weak transversality, Φ has constant rank, so in local coordinates it may be written $\Phi(z_1, \ldots, z_k) = (z_1, \ldots, z_m, 0, \ldots, 0)$ (here $k = \dim X$ and $m = \operatorname{codim} \bigcap_{j\in J} L_j$ so $k < m$). Since $v \in \ker D\Phi(x_0)$, v is of the form $v = \sum_{j=m+1}^{\operatorname{card} J} \lambda_i \partial/\partial z_i$, for some $\lambda_i \in \mathbb{C}$. Now define $\xi := \sum_{j=m+1}^{\operatorname{card} J} \lambda_i \partial/\partial z_i$. Clearly, ξ is a germ of holomorphic vector field at x_0, and moreover $\xi(x_0) = v$. Now let us prove that it is locally uniformly bounded in the Kobayashi pseudo-metric of $X - \bigcup_{j\in J} L_j$.

Let V be a small neighborhood of x_0 in which ξ is well defined. Let U be a relatively compact open subset of V. There exists $r > 0$ such that the complex flow $(\phi_t)_t$ of ξ is well defined for initial conditions $x \in U$ for times $|t| < r$. It follows from the construction of ξ that for all $x \in V$, $D\Phi(\xi) = 0$. Therefore, if $x \in U - \bigcup_{i\in I} L_i$, then for all $|t| < r$, $\phi(t, x) \notin \bigcup_{i\in I} L_i$. Moreover, from the fact that $(d/dt)_{|t=0}\phi(t, x) = \xi(x)$ and that $t \mapsto \phi(t, x)$ is holomorphic and defined on the disk of radius r, the definition of the Kobayashi pseudo-metric implies that

$$\rho_K(x; \xi(x)) \leq \frac{1}{r}. \qquad \square$$

LEMMA 2. *Let X be a complex manifold, and $(L_i)_{i\in I}$ be a family of weakly hypersurfaces. Let $J \subset I$, and $i_1, i_2 \in I \setminus J$ with $i_1 \neq i_2$. Let*

$$L := \bigcap_{j\in J} L_j, \quad M_1 := \bigcap_{j\in J\cup\{i_1\}} L_j, \quad M_2 := \bigcap_{j\in J\cup\{i_2\}} L_j.$$

Then if $M_1 \neq M_2$ and $p \in M_1 \cap M_2$, we have $T_p L \subset T_p M_1 + T_p M_2$.

Proof. Suppose $p \in M_1 \cap M_2$ and we do not have $T_p L \subset T_p M_1 + T_p M_2$. We can assume that both of the M_i have codimension 1 in L, otherwise one of them must be equal to L and we are done. This means that $T_p M_1 = T_p M_2$. Let U be a small neighborhood of p,

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and for any $i \in I$, let $P_i : U \to \mathbb{C}$ be a holomorphic function such that $L_i \cap U = \{P_i = 0\}$. Let

$$\begin{aligned} \phi : U &\to \mathbb{C}^{J \cup \{i_1, i_2\}} \\ x &\mapsto (P_j(x))_{j \in J \cup \{i_1, i_2\}}. \end{aligned}$$

Let m be the codimension of L in $\mathbb{P}^k$. Since $T_p M_1 = T_p M_2$, the rank of $D\phi(p)$ is $m+1$. By definition of weak transversality, up to reducing U, the rank of $D\phi$ is $m+1$ on U. But this implies that codim $M_1 \cap M_2 = m+1$, which is impossible since M_1 and M_2 both have (pure) codimension $m+1$ and $M_1 \neq M_2$. □

3. *Post-critical finiteness all the way down*

Recall the notion of (analytic) branched cover.

Definition 2. Let X, Y be smooth complex manifolds of the same dimension, and $f : X \to Y$ an analytic map. We say that f is a branched cover if f is open, surjective, and proper.

The set

$$\mathcal{C}(f) := \{x \in X,\ Df(x) \text{ is not invertible}\}$$

is either empty or a closed analytic hypersurface of X. We call it the critical set of f, and its image is the critical value set or the branch locus of f, denoted by $\mathrm{CV}(f)$. It is easy to check that $f : X \backslash f^{-1}(\mathrm{CV}(f)) \to Y \backslash \mathrm{CV}(f)$ is a covering map.

Definition 3. Let X be a complex manifold, and let $f : X \to X$ be a branched cover. Let $\mathcal{P}(f, X)$ denote $\bigcup_{n \geq 0} f^n(\mathrm{CV}(f))$: $\mathcal{P}(f)$ is called the post-critical set of f. We say that f is PCF if $\mathcal{P}(f, X)$ is non-empty and analytic with only finitely many irreducible components.

If f is unbranched, then $\mathcal{P}(f, X) = \varnothing$. When there is no ambiguity, we will just write $\mathcal{P}(f)$.

Note that any endomorphism of $\mathbb{P}^k$ is a branched cover, see [**Ued98**]. Let us now define the notions of post-critical finiteness all the way down.

Definition 4. Let X be an irreducible analytic set, and $f : X \to X$ be a PCF endomorphism. Let $f_0 := f$, $\Omega_0 = X$, and denote by P_0 the post-critical set of f. Let us make the following definition inductively on m: if the restriction of f_m to each irreducible component of Ω_m is either unbranched or PCF, then the following are true.

- Ω_{m+1} is the union of the f_m-periodic irreducible components of P_m, and k_m is the least common multiple of the periods.
- f_{m+1} is the restriction of $f_m^{k_m}$ to Ω_{m+1}.
- If every irreducible component of P_m is smooth, P_{m+1} is the union of the post-critical sets of f_{m+1} restricted to each irreducible component of Ω_{m+1}.

Note that if the restriction of f_m to each irreducible component of Ω_m is PCF or unbranched, then Ω_{m+1} is either empty or an analytic set of pure codimension dim $\Omega_m - 1$.

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Definition 5. Using the same notation as the definition above.

- If for some $k \in \mathbb{N}$ the restriction of f_m to each irreducible component of Ω_m is PCF for every $m \leq k$, then we say that f is PCF of order $k+1$.
- If f is PCF of order $\dim X$, we say that f is PCF all the way down.
- If for all $k \leq \dim X - 1$ the restriction of f_m to each irreducible component of Ω_m is either unbranched or PCF, then we say that f is *weakly PCF all the way down*.

Note that if f is PCF all the way down then it is weakly PCF all the way down, and that if f is weakly PCF all the way down then for all $m \leq \dim X$, Ω_m is either empty or an analytic subset of X of pure codimension m.

THEOREM 1. *Let X be a complex manifold and let $f : X \to X$ be a PCF branched covering. Let $(L_i)_{i\in I}$ be the periodic irreducible components of $\mathcal{P}(f, X)$ and assume that for any $J \subset I$, $\bigcap_{j\in J} L_j$ is smooth. Then the following can be said.*

(1) *f is weakly PCF all the way down and the irreducible components of Ω_m are of the form $L_{i_1} \cap \cdots \cap L_{i_m}$ for some $(i_1, \ldots, i_m) \in I^m$.*
(2) *In fact, for any $(i_1, \ldots, i_m) \in I_m$, if we set $L := L_{i_1} \cap \cdots \cap L_{i_m}$, then $f_m : L \to L$ is either unbranched or PCF.*

Before we prove Theorem 1, we will need the following lemma.

LEMMA 3. *Let X be a complex manifold, and let $f : X \to X$ be a branched cover. Let C be a smooth irreducible hypersurface of X, and assume that $f(C) = C$. Then the critical set of the restriction map $f_{|C} : C \to C$ is included in $\bigcup_i C \cap C_i$, where the C_i are the irreducible components of the critical set of f other than C.*

Proof. This is clear if C is not a component of the critical set of f, since any critical point for the restriction of f to C is also a critical point for the ambient map $f : X \to X$. Therefore, let us assume that $C \subset \mathcal{C}(f)$. Let x be a critical point of the restriction of f to C.

All our arguments in this lemma are local. Therefore, if $\dim X = k$, we may assume without loss of generality that C is locally written $\{z_k = 0\}$ in local holomorphic coordinates $z = (z_1, \ldots, z_k)$ near x and $f(x)$, and that $x = (0, \ldots, 0) \in \mathbb{C}^k$. In those coordinates, we may write $f(z_1, \ldots, z_k) = (f_1(z), \ldots, f_k(z))$: then $f_k(z_1, \ldots, z_{k-1}, 0) = 0$ for every $(z_1, \ldots, z_{k-1}) \in \Delta^k$. Therefore, for every $i \leq k-1$, $\partial_i f_k = 0$ on C. So locally near x, for every $z \in C$, $Df(z)$ can be represented by the matrix:

$$Df(z) = \left(\begin{array}{c|c} A_k & * \\ \hline 0 & \partial_k f_k \end{array}\right)$$

and A_k is $Df_{|C}(z)$. Moreover, since C is in the critical set of f, we have

$$\operatorname{Jac} f = \det A_k \times \partial_k f_k = 0$$

on C. Also, $\det A_k$ cannot vanish identically on C because $f_{|C} : C \to C$ is finite to one (since the ambient map $f : X \to X$ a branched cover). So $\partial_k f_k$ vanishes identically on C. Therefore, there is some maximal $m \in \mathbb{N}^*$ and some germ of holomorphic function g such that $f_k(z_1, \ldots, z_k) = z_k^m g(z_1, \ldots, z_k)$, and z_k does not divide g. Then note that z_k^m

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divides $\partial_i f_k$ for all $i < k$, but does not divide $\partial_k f_k$, while z_k^{m-1} does divide $\partial_k f_k$ (indeed, for all $i < k$, $\partial_i f_k = z_k^m \partial_i g$ and $\partial_k f_k = m z_k^{m-1} g + z_k^m \partial_k g$).

For every $i \leq k$, let A_i be the minor of size $k-1$ corresponding to the (k, i) entry of the matrix representation of Df in the local coordinates $(z_1, \dots, z_k)$. Then we have, by expanding the last row:

$$\operatorname{Jac} f = \sum_{i \leq k} \partial_i f_k \det A_i.$$

Now if we let

$$r = mg \det A_k + z_k \sum_{i \leq k} \partial_i g \det A_i,$$

notice that $\operatorname{Jac} f(z) = z_k^{m-1} r(z)$. Moreover, since $\det A_k$ is not identically zero on C and neither is g, the function r is not identically zero on C. Now remember that by assumption, $x = 0$ is in the critical locus of $f_{|C}$: therefore, $\det A_k(0, \dots, 0) = 0$. This means that $r(0, \dots, 0) = 0$, that is to say, x belong to the analytic set $r^{-1}(0)$. Since r is not identically zero on C, this means that x belong to some irreducible component of the zero locus of $\operatorname{Jac} f$ that is not C, which is the desired result. □

We will also need the following lemma due to Ueda.

LEMMA 4. [**Ued98**, Lemma 3.5] *Let X, Y be complex manifolds, and $f : X \to Y$ a holomorphic branched cover with branch locus $D \subset Y$. If $x \in \operatorname{Sing}(f^{-1}(D))$, then $f(x) \in \operatorname{Sing}(D)$.*

Now we can prove Theorem 1.

Proof of Theorem 1. Let us prove by induction on $0 \leq m \leq \dim X - 1$ that for any $(i_1, \dots, i_m) \in I^m$, the restriction of f_m to $L := L_{i_1} \cap \cdots \cap L_{i_m}$ (with the convention that $L = X$ if $m = 0$) is either unbranched or PCF and that the irreducible components of $\mathcal{P}(f_m, L)$ are of the form $L_{i_1} \cap \cdots \cap L_{i_m} \cap L_{i_{m+1}}$.

If $m = 0$, then by definition the irreducible components of Ω_1 are some of the L_i, and by assumption $f : X \to X$ is PCF, so there is nothing to prove.

Now let $0 \leq m \leq \dim X - 1$ be such that for any $(i_1, \dots, i_m) \in I^m$, the restriction of f_m to $L := L_{i_1} \cap \cdots \cap L_{i_m}$ (with the convention that $L = X$ if $m = 0$) is either unbranched or PCF and that the irreducible components of Ω_{m+1} are of the form $L_{i_1} \cap \cdots \cap L_{i_{m+1}}$.

Let $N := L_{i_1} \cap \cdots \cap L_{i_m}$ of codimension m (if it has a codimension $n < m$, then it can be written as $L_{j_1} \cap \cdots L_{j_n}$ and the desired properties follow from the induction hypothesis). By definition of $f_{m+1} = f_m^{k_{m+1}}$, the map f_{m+1} fixes all of its post-critical components, so that $\mathcal{P}(f_{m+1}, N) = \operatorname{CV}(f_{m+1}, N)$ and f_{m+1} restricts to an endomorphism of any periodic component of $\mathcal{P}(f_m, N)$. Let $L = L_{i_1} \cap \cdots L_{i_{m+1}}$ be a periodic component of $\mathcal{P}(f_m, N)$, and let C be an irreducible component of $\mathcal{C}(f_{m+1}, L)$ (if $\mathcal{C}(f_{m+1}, L) = \varnothing$, then $f_{m+1} : L \to L$ is unbranched and there is nothing to prove).

By Lemma 3, there is a component C' of $\mathcal{C}(f_{m+1}, N)$ with $C' \neq L$ such that $C \subset C' \cap L$. Therefore, $C \subset \operatorname{Sing}(f_{m+1}^{-1}(\operatorname{CV}(f_{m+1}, N)))$. In fact, for any $n \in \mathbb{N}$, $C \subset \operatorname{Sing}(f_{m+1}^{-n}(\operatorname{CV}(f_{m+1}, N)))$. By Lemma 4 applied to f_{m+1}^n for any arbitrary $n \in \mathbb{N}$,

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we have

$$f_{m+1}^n(C) \subset \mathrm{Sing}(\mathrm{CV}(f_{m+1}^n, N)) \subset \mathrm{Sing}(\mathcal{P}(f_m, N)).$$

Hence, the increasing sequence of analytic sets of pure codimension $m+2$ given by

$$C_N := \bigcup_{n \le N} f_1^n(C)$$

is contained in the analytic set of pure codimension $m+2$ given by $\mathrm{Sing}(\mathcal{P}(f_m, N))$. Moreover, by assumption, each intersection of components of $\mathcal{P}(f, X)$ is smooth so the induction hypothesis implies that the singular part of $\mathcal{P}(f_m, N)$ consists in points belonging to at least two different irreducible components of $\mathcal{P}(f_m, N)$. Those are of the form $L_{i_1} \cap L_{i_{m+2}}$, $(i_1, i_{m+2}) \in I$. In particular, $\mathrm{Sing}(\mathcal{P}(f_m, N))$ has only finitely many irreducible components, so the sequence C_N is eventually stationary. Since this holds for any irreducible component C of $\mathcal{C}(f_{m+1}, L)$, we have proved that $f_{m+1} : L \to L$ is PCF and that the irreducible components of $\mathcal{P}(f_{m+1}, L)$ are of the form $L_{i_1} \cap L_{i_{m+2}}$. This finishes the proof. □

An endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ may be represented in homogeneous coordinates by $k+1$ homogeneous polynomials of a same degree $d \ge 1$. The integer d is called the algebraic degree of f (not to be confused with the topological degree, which is d^k). The following classical lemma is due to Dinh and Sibony.

LEMMA 5. [**DS10**, Lemma 1.48] *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be an endomorphism of algebraic degree d, and let $L \subset \mathbb{P}^k$ be an algebraic set of pure dimension m, such that $L = f(L)$. Then for n large enough, f^n fixes all irreducible components of L, and the restriction of f^n to L has topological degree d^{nm}.*

COROLLARY 1. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a PCF endomorphism such that $\mathcal{P}(f, \mathbb{P}^k)$ is a union of hyperplanes. Then f is PCF all the way down.*

Proof. In this case we do not need to use Lemma 5. Any intersection of hyperplanes is a projective subspace, and hence smooth. In view of Theorem 1, it is enough to prove that if $L \subset \mathbb{P}^k$ is a projective subspace with $f^p(L) = L$ for some $p \in \mathbb{N}^*$, then the restriction map $f^p : L \to L$ cannot be unbranched. Let d be the algebraic degree of f on $\mathbb{P}^k$: $d \ge 2$ since f is PCF, and therefore, has a non-empty critical set. After a change of coordinates, we may assume that L can be written as $\{z_k = \cdots = z_m = 0\}$ in homogeneous coordinates. In those homogeneous coordinates, f is of the form

$$(z_0 : \ldots : z_k) \mapsto (P_0(z_0, \ldots, z_k), \ldots, P_k(z_0, \ldots, z_k)),$$

where the P_i are homogeneous polynomials of degree d^p. The restriction of f^p to L is then of the form

$$(z_0 : \ldots : z_{m-1}) \mapsto (P_0(z_0, \ldots, z_{m_1}, 0, \ldots, 0), \ldots, P_{m-1}(z_0, \ldots, z_{m-1}, 0, \ldots, 0))$$

and since f^p has no indeterminacy points, $f^p : L \to L$ has algebraic degree d^p so its Jacobian (in homogeneous coordinates) is a non-constant polynomial, so the critical locus of $f^p : L \to L$ is not empty. □

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4. *Absence of rotation domains*

Let Z be an analytic set, and $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism. Let $h : Z \to \mathbb{P}^k$ be an analytic map. Following Ueda [**Ued98**], we say that a family of maps $g_n : Z \to \mathbb{P}^k$ is a family of lifts of h by iterates of f if for all $n \in \mathbb{N}$, there is $m_n \in \mathbb{N}$ such that $f^{m_n} \circ g_n = h$.

We will rely quite heavily on the following useful theorem, due to Ueda [**Ued98**, Theorem 2.1].

THEOREM 2. *Let Z be an analytic set, and $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism. Let $h : Z \to \mathbb{P}^k$ be an analytic map. Let $(g_n)_{n\in\mathbb{N}}$ be a family of lifts of h by iterates of f. Then $(g_n)_{n\in\mathbb{N}}$ is a normal family.*

The following result is a straightforward adaptation of [**Ued98**, Theorem 4.15]. We include a proof for the convenience of the reader.

THEOREM 3. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be an endomorphism of algebraic degree $d \geq 2$. Let L be a closed regular subvariety of $\mathbb{P}^k$ with $L = f(L)$, and assume that the restriction $f : L \to L$ has a rotation domain V. Then ∂V (the boundary of V in L) is included in the post-critical set of the restriction map $f : L \to L$.*

Remark 1. The assumption that $f : L \to L$ extends to an endomorphism of $\mathbb{P}^k$ is necessary: indeed, there are automorphisms of projective varieties (which, therefore, have an empty post-critical set) with rotation domains that are not all of L. The crucial point here is the normality of the inverse branches afforded by Theorem 2.

Proof of Theorem 3. Let $x \in \partial V$. Assume that x does not belong to the post-critical set $\mathcal{P}(f, L)$ of $f : L \to L$. Let D be a simply connected neighborhood of x in L that does not meet the post-critical set of $f : L \to L$. Let $(n_j)_{j\in\mathbb{N}}$ be a subsequence such that f^{n_j} converges locally uniformly to the identity on S. For all $j \in \mathbb{N}$, $f^{n_j} : V \to V$ is an automorphism of S, so we can consider $h_j := (f^{n_j})^{-1} : V \to V$. Moreover, since D is simply connected and disjoint from the post-critical set of $f : L \to L$, for all $j \in \mathbb{N}$ branches of f^{-n_j} are well defined, so we can extend h_j to D.

By Theorem 2, the family $(h_j)_{j\in\mathbb{N}}$ is normal on D, and from the fact that V is a rotation domain, it is easy to see that $(h_j)_{j\in\mathbb{N}}$ converges to the identity on V, and therefore, also on D by the identity principle. Let W be a connected, relatively compact open subset of D containing x such that for all $j \in \mathbb{N}$ large enough, $W \subset h_j(D)$. Then $f^{n_j}_{|W}$ converges to $\mathrm{Id}_{|W}$, and hence W is included in the rotation domain V. This contradicts the fact that $x \in \partial V$. □

COROLLARY 2. *Assume that $f : \mathbb{P}^k \to \mathbb{P}^k$ is PCF, and call $L_i, i \in I$ the irreducible components of Ω_1. Let $J \subset I$, $L := \bigcap_{j\in J} L_j$ and m the least common multiple of the periods of the L_j, $j \in J$. Then $f^m : L \to L$ has no rotation domain.*

Proof. Assume that $f : L \to L$ does have a rotation domain V. By Theorem 3, the boundary ∂V is contained in the post-critical set of $f : L \to L$. By Theorem 1, $f : L \to L$ is either unbranched or PCF.

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Let us first assume that $f : L \to L$ is unbranched: then $\partial V = \varnothing$, so $L = V$. In particular, $f : L \to L$ is an automorphism of L, which contradicts Lemma 5 (indeed, f must have algebraic degree at least 2 since it is PCF).

Let us now assume that $f : L \to L$ is PCF instead of unbranched. Then ∂V is contained in an algebraic hypersurface of L. Since hypersurfaces of L do not disconnect L, $L \backslash \partial V$ is connected, so $V = L \backslash \partial V$ and V contains the complement of the post-critical set of $f : L \to L$. By [**Ued98**, Lemma 4.13], this means that $f : L \to L$ is in fact an automorphism of L and that $V = L$. Again, this contradicts Lemma 5. □

5. *Eigenvalues of periodic points of PCF endomorphisms*

Recall the following definition from topological dynamics.

Definition 6. Let X be a Hausdorff topological space, $f : X \to X$ a continuous map, and $p \in X$ a fixed point. The fixed point p is said to be Lyapunov stable if for every neighborhood V of p, there exists an open set U containing p such that for all $n \in \mathbb{N}$, $f^n(U) \subset V$. We will also say that p is Lyapunov unstable if f has a local inverse g fixing p such that p is Lyapunov stable for g.

Remark 2. In the case where X is a complex manifold and f is holomorphic, a necessary (but not sufficient) condition for Lyapunov stability is for the eigenvalues of $Df(p)$ to have modulus smaller or equal to one. A sufficient (but not necessary) condition is that every eigenvalue of $Df(p)$ has modulus strictly smaller than one.

LEMMA 6. *Let L be a complex manifold, and $f : L \to L$ a PCF ramified cover. Let $p \in \mathrm{Reg}(\mathcal{P}(f, L)) \backslash \mathcal{C}(f, L)$ be a Lyapunov unstable fixed point for $f : \mathcal{P}(f, L) \to \mathcal{P}(f, L)$. Then there is a simply connected open subset $U \subset L$ containing p such that for all $n \in \mathbb{N}$, there is a well-defined holomorphic lift $g_n : U \to L$ of f^n fixing p.*

Proof. Let us begin with the case $n = 1$. Let V be a small enough neighborhood of p in L that f admits a unique local inverse g_1 defined on V, with $g_1(p) = p$. Let U be an open subset given by the Lyapunov stability of p as a fixed point of g_1. Up to restricting U, we can take U to be simply connected.

We will prove the following statement by induction: for all $n \in \mathbb{N}^*$, there is a well-defined holomorphic lift $g_n : U \to L$ of f^n fixing p, such that

$$g_n(U) \cap \mathcal{P}(f, L) \subset g_1^n(U) \cap \mathcal{P}(f, L).$$

Now assume that g_n is constructed, and let us set $V_n := g_n(U)$. Let us prove that g_1 admits analytic continuations along any path in V_n starting at p. Let $\gamma : [0, 1] \to V_n$ be a path with $\gamma(0) = p$. By the induction hypothesis and the fact that p is a Lyapunov unstable point,

$$V_n \cap \mathcal{P}(f, L) \subset g_1^n(U) \cap \mathcal{P}(f, L) \subset V \cap \mathcal{P}(f, L).$$

Using the compacity of $\gamma([0, 1])$, we may find a finite subcover of $\gamma([0, 1])$ by small balls $(U_i)_{i \leq m}$ such that either $U_i \subset V$ or $U_i \cap \mathcal{P}(f, L) = \varnothing$, and then define an analytic continuation of g_1 along γ in the usual way, using compatible local inverses on each U_i.

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Since U is simply connected and g_n is univalent, V_n is also simply connected, so by the Monodromy theorem g_1 extends to a single-valued map $g_{1,n}$ defined on V_n†. We then set

$$g_{n+1} := g_{1,n} \circ g_n \quad \text{on } U.$$

Clearly this defines a lift of f^n fixing p.

Now set $V_{n+1} := g_{n+1}(U)$. To finish the proof by induction, we need to prove that $V_{n+1} \cap \mathcal{P}(f, L) \subset g_1^{n+1}(U) \cap \mathcal{P}(f, L)$. Let $x \in V_{n+1} \cap \mathcal{P}(f, L)$, and let y be the unique element of V_n such that $x = g_{1,n}(y)$. Then $f(x) = y \in \mathcal{P}(f, L)$ by forward invariance of $\mathcal{P}(f, L)$, so by the induction hypothesis, $y \in g_1^n(U) \cap \mathcal{P}(f, L)$. So finally $x = g_{1,n}(y) = g_1(y) \in g_1^{n+1}(U) \cap \mathcal{P}(f, L)$ (the second equality comes from the fact that $g_{1,n} = g_1$ on V), and the proof is finished. □

LEMMA 7. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be an endomorphism, and assume that L is a subvariety such that $L = f(L)$ and $f : L \to L$ is PCF. Assume that $p = f(p) \in \mathrm{Reg}(\mathcal{P}(f, L)) \cap \mathcal{C}(f, L)$. Up to replacing f by an iterate, there is a fixed component P_i of $\mathcal{P}(f, L)$ such that the image of $Df(p)_{|T_pL}$ is included in $T_p P_i$.*

Proof. Up to replacing f by an iterate, we may assume that all irreducible components of $\mathcal{C}(f, L)$ are mapped by f to fixed post-critical components. According to [**GR58**], Satz 10, we can find local coordinates (different in domain and range) in which $f : L \to L$ takes locally near p (in L) the form $(z_1, z_2, \ldots) \mapsto (z_1^m, z_2, \ldots)$. In particular, p belongs to the regular part of $\mathcal{C}(f, L)$, and $Df(p)$ maps $T_p\mathcal{C}(f, L)$ injectively into $T_p\mathcal{P}(f, L)$. Since $Df(p)$ has rank at most $\dim L - 1$ at p, the image of $Df(p)$ must be exactly $T_p\mathcal{P}(f, L)$. □

The following lemma builds on ideas from [**FS94**].

LEMMA 8. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be an endomorphism, and assume that L is a subvariety such that $L = f(L)$ and $f : L \to L$ is weakly PCF. Let $p = f(p) \notin \mathcal{C}(f, L)$, and assume that there is an open neighborhood U of p in L such that for all $n \in \mathbb{N}$ there is a holomorphic lift $g_n : U \to L$ of f^n with $g_n(p) = p$. Then p is Lyapunov unstable, and at least one of its eigenvalues is repelling. Moreover, p is linearizable through an analytic submanifold tangent to the subspace associated to neutral eigenvalues.*

Proof. By Theorem 2, the family $\{g_n, n \in \mathbb{N}\}$ is normal. Moreover, $Dg_n(p) = Df(p)^{-n}$, so $Df(p)$ cannot have any attracting eigenvalue, otherwise $Dg_n(p)$ would blow up, contradicting the normality of the g_n, $n \in \mathbb{N}$.

Therefore, all eigenvalues must be either neutral or repelling; assume for a contradiction that none of them are repelling. Then every eigenvalue of $Df(p)$, and hence of $Dg_1(p)$, have modulus one. Moreover, the sequence $Dg_n(p) = Dg_1^n(p)$ is bounded; by considering a Jordan decomposition, one easily sees that this implies that $Dg_1(p)$ is unitary. By [**FS94**, Lemma 6.8], g_1 and therefore, f are linearizable in an open neighborhood of p in L. This implies that p belongs to a rotation domain of $f : L \to L$, which contradicts Corollary 2.

Let us prove that p is Lyapunov unstable: let V be an open neighborhood of p. Up to reducing V, we may assume that each g_n coincide with g_1^n on V. Since $\{g_1^n, n \in \mathbb{N}\}$ is

† We use the notation $g_{1,n}$ to emphasize the fact that the values of this extension depend a priori on n on $V_n \setminus V$.

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normal on V, there is a neighborhood $U \subset V$ such that for all $n \in \mathbb{N}$, $g_1^n(U) \subset V$. Thus, p is Lyapunov unstable.

Finally, let us prove the linearizing statement. Let

$$T_pL = E^c \oplus E^u$$

be the decomposition of T_pL into invariant subspaces associated to neutral and repelling eigenvalues respectively. We will prove that there is a germ of invariant analytic submanifold $W^c_{\text{loc}}(p)$ tangent to E^c on which f is linearizable. Let $G = \lim_{j\to\infty} g_{n_j}$ be a limit function. Up to extracting further subsequences, we may assume that $n_{j+1} - 2n_j \to +\infty$, and that $g_{n_{j+1}-n_j}$ converges to some other limit function $\rho : U \to L$, while $g_{n_{j+1}-2n_j}$ converges to some limit function $\phi : U \to L$. Observe that we have

$$G \circ \rho = \rho \circ G = G$$

and

$$G \circ \phi = \phi \circ G = \rho.$$

Therefore, we have

$$\rho \circ \rho = \phi \circ G \circ \rho = \phi \circ G = \rho. \tag{1}$$

A holomorphic map ρ satisfying $\rho \circ \rho = \rho$ is called a retraction; in [**Car86**], Cartan gave a very elementary proof that the image of a retraction is smooth. Therefore, if we let $S := \rho(U)$, then S is a smooth analytic subset of U, and it is easy to check that it is invariant under g_1. Again, using the fact that the iterates of $Dg_1(p)$ are bounded and looking at its Jordan decomposition, we can see that all the Jordan blocks associated to neutral eigenvalues must have size at most 1, and so $Dg_1(p)$ restricted to T_pS is unitary. Again, by [**FS94**, Lemma 6.8], this implies that g_1 is linearizable on $W^c_{\text{loc}}(p) := S$, and therefore the same holds for f. □

Let us now prove Theorem B on the eigenvalues of cycles of PCF endomorphisms of $\mathbb{P}^k$.

THEOREM B. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a PCF endomorphism such that the irreducible components of $\mathcal{P}(f, \mathbb{P}^k)$ are weakly transverse. Let p be a periodic point of period l for f. Then the following are true:*

(1) *p has no parabolic or non-zero attracting eigenvalue;*
(2) *either p has at least one repelling eigenvalue, or else all eigenvalues are zero (i.e. $Df^l(p)$ is nilpotent);*
(3) *if p does not have zero as an eigenvalue, then p is Lyapunov unstable and at least one of its eigenvalues is repelling.*

Proof. Up to replacing f by an iterate, we lose no generality in considering only fixed points and in assuming that all irreducible components of the Ω_m, $0 \le m \le k$, are fixed. Let $p = f(p)$ be a fixed point of f.

If $p \notin \mathcal{P}(f, \mathbb{P}^k)$, we can choose a simply connected neighborhood U of p in L, and for all $n \in \mathbb{N}$ we have a well-defined inverse branch $g_n : L \to L$ fixing p. Then by Lemma 8, we are done. Therefore, we will now assume that $p \in \mathcal{P}(f, \mathbb{P}^k)$.

Let m_0 be the greatest $m \le k-1$ such that $p \in \Omega_m$. We shall prove the following by descending induction on the codimension m, for $0 \le m \le m_0$.

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For any irreducible component L of Ω_m containing p:

(1) the non-zero eigenvalues of p associated to vectors tangent to L are not attracting or parabolic;
(2) if none of those eigenvalues are repelling, then they are all 0;
(3) if $p \notin \mathcal{C}(f, L)$, then p is a Lyapunov unstable fixed point of f in L.

If $m_0 = k - 1$ and L is an irreducible component of Ω_{k-1} containing p, then $f : L \to L$ is weakly PCF according to Theorem 1, and L is a compact Riemann surface. Therefore, L must be either a torus or a projective line (indeed, in all other cases, f would have to restrict to a degree one self-map of L, contradicting Lemma 5). If L is a torus, then all cycles are repelling, and if L is a projective line, it is classical that $f : L \to L$ may only have super-attracting or repelling cycles.

If $m_0 < k - 1$ and L is an irreducible component of Ω_{m_0} containing p, then by Theorem 1, $f : L \to L$ is an unbranched cover, and $\mathcal{P}(f, L) = \varnothing$. In particular, for any simply connected neighborhood U of p we can take inverse branches g_n of f^n fixing p. Then, by Lemma 8 p is Lyapunov unstable, has at least one repelling eigenvalue and no attracting or parabolic eigenvalue. Thus, the statement is proved for $m = m_0$.

Now let $m \geq 1$ and assume that $p \in \Omega_m$ and that the induction hypothesis is satisfied for m. Let L be an irreducible component of Ω_{m-1} containing p. Note that by assumption, $p \in \mathcal{P}(f, L)$ since $p \in \Omega_m$. We will distinguish three mutually exclusive cases:

(1) $p \in \mathrm{Reg}(\mathcal{P}(f, L)) \cap \mathcal{C}(f, L)$;
(2) $p \in \mathrm{Reg}(\mathcal{P}(f, L)) \backslash \mathcal{C}(f, L)$;
(3) $p \in \mathrm{Sing}(\mathcal{P}(f, L))$.

Case (1) follows from Lemma 7 and the induction hypothesis.

Assume we are in case (2). Let P be the irreducible post-critical component of $\mathcal{P}(f, L)$ containing p. By item (3) of the induction hypothesis, p is a Lyapunov unstable fixed point of the restriction $f : P \to P$. By Lemmas 6 and 8, p is also a Lyapunov unstable fixed point for $f : L \to L$, and it has at least one repelling eigenvalue, so the three items of the induction are proved.

Finally, let us assume that we are in case (3). Since we assumed that any intersection of components of Ω_1 is smooth, Theorem 1 implies that p belongs to at least two different components P_1 and P_2 of Ω_m. Since the irreducible components of Ω_1 are weakly transverse, Lemma 2 implies that $T_p L \subset T_p P_1 + T_p P_2$. Also note that each of the $T_p P_i$ are invariant under $Df(p)$. Therefore, any eigenvalue of $Df(p)$ on $T_p L$ must be an eigenvalue on either $T_p P_1$ or $T_p P_2$. This together with the induction hypothesis imply that all three desired items of the induction are true.

Thus, the induction is finished and the theorem is proved, since it corresponds to the case $m = 0$. □

6. *Super-attracting basins are the only Fatou components*

We are now able to prove Theorem C.

THEOREM C. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a PCF endomorphism, and let $L_i, i \in I$ be the irreducible components of its post-critical set $\mathcal{P}(f)$. Assume the following:*

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(1) $(L_i)_{i\in I}$ *is weakly transverse;*
(2) $\mathbb{P}^k\backslash\mathcal{P}(f)$ *is Kobayashi hyperbolic.*

Then the Fatou set of f *is empty or a finite union of basins of super-attracting cycles, whose points are zero-dimensional intersections of components* L_i.

Proof of Theorem C. Let us start by introducing some notation. Denote by $(L_i)_{i\in I}$ the family of the irreducible components of $\mathcal{P}(f)$.

For every $y \in \mathcal{P}(f)$, denote by M_y the intersection of all irreducible components of $\mathcal{P}(f)$ containing y:

$$M_y = \bigcap_{i\in I_y} L_i,$$

where I_y is the set of $i \in I$ such that $y \in L_i$.

For any $i \in I_y$, there exists $(k_i, m_i) \in \mathbb{N} \times \mathbb{N}^*$ such that $f^{m_i+k_i}(L_i) = f^{m_i}(L_i)$. Let $m_y := \operatorname{lcm}(m_i, i \in I_y)$ and

$$N_y := \bigcap_{i\in I_y} f^{k_i}(L_i).$$

Then we have $f^{m_y}(N_y) \subset N_y$.

For any closed hypersurface V, denote by $\|\cdot\|_{\mathbb{P}^k-V}$ the Kobayashi pseudo-metric of $\mathbb{P}^k - V$.

Step 1. Let $x \in \mathbb{P}^k - f^{-1}(\mathcal{P}(f))$. Then for every $v \in T_x\mathbb{P}^k$, we have

$$\|Df(x)\cdot v\|_{\mathbb{P}^k-\mathcal{P}(f)} \geq \|v\|_{\mathbb{P}^k-\mathcal{P}(f)}.$$

Proof. Since the restriction $f : \mathbb{P}^k - f^{-1}(\mathcal{P}(f)) \to \mathbb{P}^k - \mathcal{P}(f)$ is a covering map, we have

$$\|Df(x)\cdot v\|_{\mathbb{P}^k-\mathcal{P}(f)} = \|v\|_{\mathbb{P}^k-f^{-1}(\mathcal{P}(f))}.$$

Moreover, since there is an inclusion $i : \mathbb{P}^k - f^{-1}(\mathcal{P}(f)) \to \mathbb{P}^k - \mathcal{P}(f)$, we have

$$\|v\|_{\mathbb{P}^k-f^{-1}(\mathcal{P}(f))} \geq \|v\|_{\mathbb{P}^k-\mathcal{P}(f)}.$$ □

Step 2. Let $z \in \mathcal{F}(f)$ be a point whose orbit never lands in $\mathcal{P}(f)$, and let f^{n_j} be a subsequence converging to some map h on a neighborhood of z. Let $y := h(z)$. Then $y \in \mathcal{P}(f)$, and $Dh(z) : T_z\mathbb{P}^k \to T_yM_y$ is surjective.

Proof. The fact that y must be in $\mathcal{P}(f)$ is well known; we refer the reader to [**Ron08**] for instance. Let $(P_i)_{i\in I_y}$ denote a collection of irreducible polynomials whose zero loci correspond to the irreducible components L_i of $\mathcal{P}(f)$ containing y, and let $v \in \bigcap_{i\in I_y} \ker DP_i(y)$. Let us prove that there exists $w \in T_z\mathbb{P}^k$ such that $Dh(z)\cdot w = v$. Let ξ be a holomorphic vector field defined on a neighborhood U of y, such that $\xi(y) = v$ given by Lemma 1. For any $j \in \mathbb{N}$ large enough to ensure that $f^{n_j}(z) \in U$, let $w_j = (Df^{n_j})^{-1}(f^{n_j}(z))\cdot\xi(f^{n_j}(z))$. Since f expands the Kobayashi metric of $\mathbb{P}^k - \mathcal{P}(f)$ (Step 1), we have

$$\|w_j\|_{\mathbb{P}^k-\mathcal{P}(f)} \leq \|\xi(f^{n_j}(z))\|_{\mathbb{P}^k-\mathcal{P}(f)}.$$

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Moreover, by Lemma 1, there exists a constant $C > 0$ independent from j such that

$$\|\xi(f^{n_j}(z))\|_{\mathbb{P}^k - \mathcal{P}(f)} \leq C.$$

Therefore, up to extracting one more time, the sequence w_j converges to some $w \in T_z\mathbb{P}^k$. By continuity of ξ, we have that $Dh(z) \cdot w = v$. □

Step 3. Let h be a Fatou limit function, $z \in \mathcal{F}(f)$ and let $y := h(z)$. Assume that $Dh(z) : T_z\mathbb{P}^k \to T_y M_y$ is surjective. Then we have that the sequence $(Df^k(y)_{|T_y M_y})_{k\in\mathbb{N}}$ is uniformly bounded (in the Fubini–Study metric of $\mathbb{P}^k$).

Proof. Let $v \in T_y L$: we shall prove that the sequence $Df^k(y) \cdot v$ is bounded. According to Step 1, there is $w \in T_z\mathbb{P}^k$ such that $Dh(z) \cdot w = v$. Let $j, k \in \mathbb{N}^*$. We have

$$Df^{n_j+k}(z) \cdot w = Df^k(f^{n_j}(z)) \circ Df^{n_j}(z) \cdot w.$$

Since $Df^{n_j}(z)$ converges to $Dh(z)$, for all $k \in \mathbb{N}^*$ there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$,

$$|\|Df^{n_j+k}(z) \cdot w\| - \|Df^k(y) \cdot v\|| \leq \frac{1}{k}.$$

Finally, since z belongs to the Fatou set of f, the sequence $\|Df^n(z) \cdot w\|$ is bounded, and therefore, so is the sequence $\|Df^k(y) \cdot v\|$. □

Step 4. Let U be a Fatou component of f, and let $h : U \to \mathbb{P}^k$ be a Fatou limit function. There is a unique maximal $J_h \subset I$ such that $h(U) \subset \bigcap_{j\in J_h} L_j := M_h$. Moreover, there is an open and dense subset $W \subset U$ such that $h : W \to M_h$ is a submersion (in particular, open).

Proof. For every component L_i of $\mathcal{P}(f)$, let P_i denote an irreducible homogenous polynomial on $\mathbb{C}^{k+1}$ such that $M_i = \{P_i = 0\}$ in homogenous coordinates. Since $h(U) \subset \mathcal{P}(f)$, we have

$$\prod_{i\in I} P_i \circ h = 0.$$

Let $\phi_i = P_i \circ h$, and let $J = \{i \in I, \phi_i \text{ does not vanish identically on } U\}$.

Then $\bigcup_{i\in J} \phi^{-1}(0)$ is a proper analytic subset of U (possibly empty), so $V := U - \bigcup_{i\in J} \phi^{-1}(0)$ is open and dense in U.

Let $z \in V$ and $y := h(z)$. By definition of V, if y belongs to some component L of $\mathcal{P}(f)$, then $h(U) \subset L$; therefore, $h(U) \subset M_y$. Since this is true for any $z \in V$, we have that for every $z, z' \in V$, $M_{h(z)} = M_{h(z')}$. We can, therefore, set $M_h = M_{h(z)}$ for any arbitrary $z \in V$. Moreover, by Step 1, $Dh(z) : T_z\mathbb{P}^k \to T_y M_y$ is surjective if the orbit of z does not meet $\mathcal{P}(f)$, or in other words, if $z \in V - \bigcup_{n\geq 0} f^{-n}(\mathcal{P}(f))$. But for all $n \in \mathbb{N}$, $f^{-n}(\mathcal{P}(f))$ is a proper closed algebraic hypersurface, so by Baire category $V - \bigcup_{n\geq 0} f^{-n}(\mathcal{P}(f))$ not empty. This implies that outside of an analytic subset of A of U, we have rank $Df =$ dim M_h; we then set $W := U - A$. □

Step 5. With the same notation as the previous step: for any $n \in \mathbb{N}$, let M_n be the irreducible component of $M_{f^n\circ h}$ containing $f^n \circ h(U)$. There exists $(k, m) \in \mathbb{N} \times \mathbb{N}^*$ such that $M_{m+k} = M_k$, and $f^k \circ h(W)$ contains an open subset of the Fatou set of the restriction of f^m to $N_h := M_k$.

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Proof. Since f maps analytic sets to analytic sets of the same dimension, and since $h(U)$ contains an open subset of M_0 by Step 4, all the M_n have the same dimension. Since there are only finitely many analytic sets of the form $\bigcap_{j\in J} L_j$, each with finitely many irreducible components, there exists $(k, m) \in \mathbb{N} \times \mathbb{N}^*$ such that $M_{k+m} = M_k$.

Applying Step 3 to any $z \in W' := (W - \mathcal{C}(f^k))$, we have that the derivatives $Df^n_{|T_{f^k\circ h(z)}M_k}$ are bounded uniformly in n. This implies that the family $\{f^{mn} : f^k \circ h(W') \to N_h\}$ is normal, which is exactly saying that $f^k \circ h(W')$ is in the Fatou set of N_h. □

Step 6. Let U be a Fatou component of f. There exists a subsequence $(m_j)_{j\in\mathbb{N}}$ such that f^{m_j} converges on U to a cycle for f.

Proof. We will prove that there is a constant Fatou limit function on U. To do this, we will prove that for every Fatou limit function h on U such that the M_h given by Step 4 has dimension $\delta \geq 1$, there is a Fatou limit function $\tilde{h}$ on U such that $\dim M_{\tilde{h}} \leq \delta - 1$.

According to the previous step, $f^k \circ h(W)$ contains an open subset W' of the Fatou set of $g := f^m_{|N_h} : N_h \to N_h$. Let $(n_j)_{j\in\mathbb{N}}$ be a subsequence such that g^{n_j} converges to a Fatou limit function h_2 (for g) on W'. We are going to prove that for any $z \in W'$, the rank of $Dh_2(z)$ must be less than or equal to $\delta - 1 = \dim N_h - 1$. Suppose for a contradiction that it is not the case: then by the local inverse theorem, we may find an open set $R \subset W'$ with $R \subset h_2(W')$. Up to extracting further, we may assume without loss of generality that $n_{j+1} - n_j$ tends to infinity. From the equality $g^{n_{j+1}-n_j} \circ g^{n_j} = g^{n_{j+1}}$, we deduce that $g^{n_{j+1}-n_j}$ converges to Id_R on R; therefore, R is included in a rotation domain for g. But this is impossible by Corollary 2.

Therefore, there exists a Fatou limit function h_2 for $g = f^m_{|N_h}$ defined on $W' \subset f^k \circ h(W)$, such that the rank of Dh_2 is strictly less than δ everywhere on W'.

Moreover, $\tilde{h} := h_2 \circ h$ is a Fatou limit function of f (on W), as can be seen by considering iterates of the form $f^{k+mn_j+p_i}$ with $i \in \mathbb{N}$ large and then suitable $j \in \mathbb{N}$, where $(p_i)_{i\in\mathbb{N}}$ is a subsequence such that f^{p_i} tends to h. Moreover, the differential of $\tilde{h}$ has a rank at most $\delta - 1$ everywhere on W. Finally, since $W \subset U$ is open, $\tilde{h}$ extends to a Fatou limit function on U, which again has rank at most $\delta - 1$ on U.

Therefore, by descending induction on δ, we can find a Fatou limit function on U with rank zero, or in other words, a constant Fatou limit function $h = y$, with $y \in \mathcal{P}(f)$. By Step 4, we have $M_y = M_h = \{y\}$, so $\{y\}$ is a zero-dimensional intersection of hyperplanes in $\mathcal{P}(f)$, and therefore y has a finite orbit. So up to replacing h by $f^p \circ h$ for some appropriate value of $p \in \mathbb{N}$ (we may do so since $f^p \circ h$ is also a Fatou limit function), we may assume that y is periodic. □

Step 7. For every z in the Fatou set of f, we have $\lim_{n\to\infty} Df^n(z) = 0$.

Proof. Let U be a Fatou component of f. By the previous step, there is a subsequence f^{n_j} such that f^{n_j} converges on U to a finite set; therefore, Df^{n_j} converges locally uniformly to zero on U.

Now observe that for any $n \in \mathbb{N}$,

$$\|Df^n\| \leq \|Df^{n_j}\| \|Df^{n-n_j}\|,$$

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where $j = \max\{i,\, n_i \leq n\}$, and that $\|Df^{n-n_j}\|$ is uniformly bounded on every compact of Ω (by normality). Here we used the Fubini–Study metric on $\mathbb{P}^k$ (or any other continuous Hermitian metric on $\mathbb{P}^k$). □

Step 8. For every Fatou component U of f, there is a periodic point y such that f^n converges to that cycle, locally uniformly on U.

Proof. Applying Step 2 and the previous step, we obtain that for every $z \in U$ and every Fatou limit function h, M_y has dimension zero, with $y := h(z)$. Thus, by Step 4, for every Fatou limit function h, there is a $z \in U$ such that if $y := h(z)$, we have that $h(U) = M_y = \{y\}$, and y is a periodic point for f in view of Step 6. Since $M_y = \{y\}$, y can be written as the intersection of at least k hyperplanes that are irreducible components of $\mathcal{P}(f)$. There are only finitely many such points, and they are all fixed by f. Denote by F the set of those points. Let $K \subset U$ be a compact with non-empty interior, and let $\epsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that for all $z \in K$, for all $n \geq n_0$, $f^n(z) \in \bigcup_{p\in F} D(p, \epsilon)$. If ϵ is small enough, the disks $D(p, \epsilon)$, $p \in F$, are pairwise disjoint and for all $p \neq q$, $f(D(p, \epsilon)) \cap D(q, \epsilon) = \varnothing$ (because all the $p \in F$ are fixed points). Therefore, if ϵ is small enough, and $n \geq n_0$, there is a unique $p \in F$ such that $f^n(K) \subset D(p, \epsilon)$, and moreover for all $q \neq p$, we have $f^{n+1}(K) \cap D(q, \epsilon) = 0$. Therefore, by induction, we must have $f^n(K) \subset D(p, \epsilon)$ for all $n \geq n_0$. This proves that the sequence of iterates $f^n_{|K}$ restricted to K converge to p. □

Step 9. The periodic cycles from the previous step must be super-attracting.

Proof. Let $p \in \mathbb{P}^k$ be such a periodic point. Replacing f by one of its iterates if necessary, we assume that p is a fixed point. Since there is an open set $U \subset \mathbb{P}^k$ of points converging locally uniformly to p under iteration, $Df(f^n(z))$ converges locally uniformly to $Df(p)$, and $Df^n(z)$ converges to 0. Therefore, p cannot have any repelling eigenvalue. By Theorem B, this means that $Df(p)$ is nilpotent, and hence p is super-attracting. □

To sum things up, we have proved that for every Fatou component U, the iterates f^n converge locally uniformly on U to some super-attracting periodic point y, that is a maximal codimension intersection of irreducible components of $\mathcal{P}(f, \mathbb{P}^k)$. This proves Theorem C. □

7. *The case of moduli space maps*

In this final section, we recall Koch's results from [**Koc13**] and explain why Theorems A, B, and C apply to her construction. It starts with a topological object on the 2-sphere S^2.

Definition 7. A topological polynomial is an orientation-preserving branched cover of the 2-sphere $h : S^2 \to S^2$, such that there exists $p \in S^2$ with $h^{-1}(p) = \{p\}$, and such that the post-critical set P_h is finite.

PCF polynomials are topological polynomials in the sense of the previous definition, by identifying the Riemann sphere with S^2 and taking $p = \infty$.

Given a topological polynomial h and a complex structure $\mathbf{s}$ on (S^2, P_h), one call pull back the complex structure by h to obtain a (possibly different) complex structure $h^*\mathbf{s}$.

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This descends to a holomorphic map $\sigma_h : \mathrm{Teich}(S^2,\ P_h) \to \mathrm{Teich}(S^2,\ P_h)$ (see [**Koc13**] for more details). This map is of particular interest as it will have a fixed point in $\mathrm{Teich}(S^2,\ P_h)$ if and only if h is equivalent in some sense to an 'usual' polynomial on the Riemann sphere, by a celebrated theorem of Thurston.

The *moduli space* $\mathcal{M}_{P_h}$ is the set of all injections $P_h \hookrightarrow \mathbb{P}^1$ modulo composition by Möbius transformation. If $N = \mathrm{card}\ P_h$, it is easy to see that it identifies to an open subset of $\mathbb{C}^{N-3}$ if $N > 3$, and a single point otherwise. In the following, we will always assume that $N > 3$, and let $k := N - 3$. The Teichmüller space $\mathrm{Teich}(S^2,\ P_h)$ is the universal cover of the moduli space: there is a natural holomorphic universal covering $\pi : \mathrm{Teich}(S^2,\ P_h) \to \mathcal{M}_{P_h}$.

There are several interesting compactifications of the moduli space; in the case of a topological polynomial, there is a distinguished hypersurface corresponding to the 'infinity' point p. Therefore, it is natural to consider the compactification $\mathcal{M}_{P_h} \hookrightarrow \mathbb{P}^k$, where the distinguished hypersurface would be in the hyperplane at infinity.

In [**Koc13**], Koch studies the existence of a holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ making the following diagram commute:

$$\begin{array}{ccc} \mathrm{Teich}(S^2,\ P_h) & \xrightarrow{\sigma_h} & \mathrm{Teich}(S^2,\ P_h) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ \mathbb{P}^k & \xleftarrow[f]{} & \mathbb{P}^k \end{array}$$

When it exists, such an endomorphism is called a *moduli space map*. It is always PCF, and its post-critical set consists in a union of hyperplanes. Moreover, $\mathbb{P}^k \backslash \mathcal{P}(f) \simeq \mathcal{M}_{P_h}$ is Kobayashi hyperbolic [**Koc13**, Corollary 6.3]. Koch proved that if h is a topological polynomial that either is unicritical or has a periodic critical point, then there exists a corresponding moduli space map [**Koc13**, Theorems 5.17 and 5.18].

Since any intersection of projective hyperplanes is a projective space, such intersections are always smooth. Moreover, they intersect each other weakly transversally in the sense of Definition 1.

Therefore, Theorems A, B, and C apply to moduli space maps.

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References

[**ABD$^+$16**] M. Astorg, X. Buff, R. Dujardin, H. Peters and J. Raissy. A two-dimensional polynomial mapping with a wandering Fatou component. *Ann. of Math. (2)* **184**(1) (2016), 263–313.

[**Car86**] H. Cartan. Sur les rétractions d'une variété. *C.R. Acad. Sci. Paris Sér. I Math.* **303** (1986), 715–716.

[**DH93**] A. Douady and J. J. Hubbard. A proof of Thurston's topological characterization of rational functions. *Acta Math.* **171**(2) (1993), 263–297.

[**DS10**] T.-C. Dinh and N. Sibony. Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings. *Holomorphic Dynamical Systems (Lecture Notes in Mathematics, 1998)*. Springer, Berlin, 2010, pp. 165–294.

https://doi.org/10.1017/etds.2018.32 Published online by Cambridge University Press

[**DT05**] H. De Thélin. Un phénomène de concentration de genre. *Math. Ann.* **332**(3) (2005), 483–498.
[**FS92**] J. Fornæss and N. Sibony. Critically finite rational maps on $\mathbb{P}^2$. *The Madison Symposium on Complex Analysis: Proceedings of the Symposium on Complex Analysis, 2–7 June 1991, University of Wisconsin–Madison (Contemporary Mathematics, 137)*. American Mathematical Society, Providence, RI, 1992, p. 245.
[**FS94**] J. E. Fornæss and N. Sibony. Complex dynamics in higher dimension I. Complex Analytic Methods in Dynamical Systems (Rio de Janeiro, 1992). *Astérisque* **5**(222) (1994), 201–231.
[**GHK16**] T. Gauthier, B. Hutz and S. Kaschner. Symmetrization of rational maps: arithmetic properties and families of Lattès maps of $\mathbb{P}^k$. *Preprint*, 2016, arXiv:1603.04887.
[**GR58**] H. Grauert, R. Remmert and K. Räume. *Math. Ann.* **136**(3) (1958), 245–318.
[**Jon98**] M. Jonsson. Some properties of 2-critically finite holomorphic maps of $\mathbb{P}^2$. *Ergod. Th. & Dynam. Sys.* **18**(01) (1998), 171–187.
[**Koc13**] S. Koch. Teichmüller theory and critically finite endomorphisms. *Adv. Math.* **248** (2013), 573–617.
[**Ron08**] F. Rong. The Fatou set for critically finite maps. *Proc. Amer. Math. Soc.* **136**(10) (2008), 3621–3625.
[**Thu85**] W. P. Thurston. *On the Combinatorics of Iterated Rational Maps*. Princeton University, 1985.
[**Ued98**] T. Ueda. Critical orbits of holomorphic maps on projective spaces. *J. Geom. Anal.* **8**(2) (1998), 319–334.

https://doi.org/10.1017/etds.2018.32 Published online by Cambridge University Press