Can. Math. Commun. Vol. 1 (e4), 2026, pp. 1–48
http://dx.doi.org/10.4153/S2976859426100034
© The Author(s), 2026. Published by Cambridge University Press on behalf of Canadian Mathematical Society. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

Splicing braid varieties

Eugene Gorsky, Soyeon Kim, Tonie Scroggin, and José Simental

Abstract. For a positive braid $\beta \in \mathrm{Br}_k^+$, we consider the braid variety $X(\beta)$. We define a family of open sets $\mathcal{U}_{r,w}$ in $X(\beta)$, where $w \in S_k$ is a permutation and r is a positive integer no greater than the length of β. For fixed r, the sets $\mathcal{U}_{r,w}$ form an open cover of $X(\beta)$. We conjecture that $\mathcal{U}_{r,w}$ is given by the nonvanishing of some cluster variables in a single cluster for the cluster structure on $\mathbb{C}[X(\beta)]$ constructed in Casals et al. (2025, *J. Amer. Math. Soc.* 38, 369–479), Galashin et al. (2026, *Invent. Math.* 243, 1079–1127), and Galashin et al. (2022, *Braid variety cluster structures, I: 3D plabic graphs*) and that $\mathcal{U}_{r,w}$ admits a cluster structure given by freezing these variables. Moreover, we show that $\mathcal{U}_{r,w}$ is always isomorphic to the product of two braid varieties, and we conjecture that this isomorphism is quasi-cluster. In some important special cases, we are able to prove our conjectures.

1 Introduction

In this article, we study splicing maps between braid varieties, generalizing the constructions of [17, 18].

1.1 Splicing braid varieties

We consider the positive braid monoid Br_k^+. For an element $\beta \in \mathrm{Br}_k^+$, the *braid variety* $X(\beta)$ is a smooth affine algebraic variety defined in terms of configurations of complete flags in the k-dimensional space $\mathbb{C}^k$, see Section 4 for a precise definition. In recent years, braid varieties have gained considerable attention due to their connections to character varieties, Legendrian link invariants, cluster algebras, and link homology. Additionally, many classical varieties appearing in geometric representation theory, such as open Richardson varieties or double Bruhat cells, are isomorphic to braid varieties.

From now on, we will assume that β contains a reduced expression for the longest element $w_0 \in S_k$ as a (not necessarily consecutive) subword (by [2, Lemma 3.4], this is without loss of generality). We consider a decomposition $\beta = \beta^1\beta^2$ of β as a product of positive braids. A *splicing map* is an open embedding

$$X\left(\widetilde{\beta}^1\right) \times X\left(\widetilde{\beta}^2\right) \to X(\beta),$$

Received by the editors August 10, 2025; revised January 29, 2026; accepted February 7, 2026.
E.G. and S.K. were partially supported by the NSF grant DMS-2302305. T.S. was partially supported by the NSF grant DMS-2302305 and the UC President's Pre-Professoriate Fellowship (PPPF). J.S. was partially supported by CONAHCyT project CF-2023-G-106 and UNAM's PAPIIT Grant IA101526.
AMS subject classification: **13F60**, **14M15**.
Keywords: Braid varieties, quasi-cluster isomorphisms, Richardson varieties.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

where $\widetilde{\beta}^1$ and $\widetilde{\beta}^2$ are braids explicitly determined by β^1 and β^2 and some auxiliary data from the splicing map. Motivated by their applications to the study of the Khovanov–Rozansky homology of torus links, splicing maps are studied by the first and third authors in [18, 25] in the special case of the top positroid variety in the Grassmannian $\mathrm{Gr}(k,n)$. This construction was later generalized by the authors in [17] to the case of *skew shaped positroids*, that is, those positroids that are open in the Grassmannian Richardson variety that contains them. In a different guise, another example of a splicing map is given by [6, Definition 3.1], where these maps are used to construct a convolution product on the compactly supported cohomology of Grassmannian Richardson varieties. Our first construction is a common generalization of these.

Assume that $\beta = \sigma_{i_1} \dots \sigma_{i_r}$, where $\sigma_1, \dots, \sigma_{k-1}$ are the usual simple generators of the positive braid monoid Br_k^+. For an element $u \in S_k$, we denote by $\underline{u} \in \mathrm{Br}_k^+$ its braid lift of minimal length. For each $r_1 = 1, \dots, r$, we let $\beta^1 = \beta^1(r_1) = \sigma_{i_1} \dots \sigma_{i_{r_1}}$ and $\beta^2 = \beta^2(r_1) = \sigma_{i_{r_1+1}} \dots \sigma_{i_r}$, so that $\beta = \beta^1\beta^2$. For each element $w \in S_k$, we define a principal open set $\mathcal{U}_{r_1,w} \subseteq X(\beta)$ (see (5.1)) and prove the following.

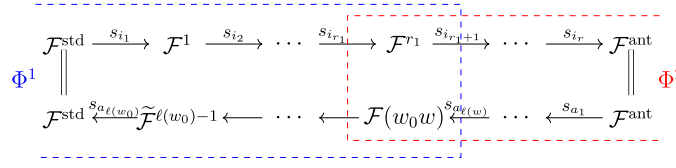
Theorem 1.1 *For each $r_1 = 1, \dots, r$ and $w \in S_k$, we have an isomorphism of algebraic varieties*

$$\Psi_{r_1,w} : X\left(\underline{(w^{-1}w_0)}\beta^1\right) \times X\left(\beta^2\underline{w}\right) \xrightarrow{\cong} \mathcal{U}_{r_1,w}, \tag{1.1}$$

where w_0 is the longest element of S_k.

Remark 1.2 For simplicity, we choose to state and prove all the results in this article only in type A. However, it is not hard to see from the proof (see Section 5 below) that Theorem 1.1 holds in arbitrary type.

More precisely, the braid variety $X(\beta)$ is defined via chains of flags with specified relative positions, and the open set $\mathcal{U}_{r_1,w}$ is given by the condition that the r_1-th flag is transverse to the coordinate flag $\mathcal{F}(w_0w)$. Schematically, the inverse $\Phi_{r_1,w}$ of the map $\Psi_{r_1,w}$ is given as follows:



(1.2)

In (1.2), the flags on the bottom row are all coordinate flags, and their successive relative positions spell a reduced word for w_0. If $\mathcal{F}^{r_1}$ is transverse to $\mathcal{F}(w_0w)$, then the blue part of the diagram belongs, up to an overall shift, to $X\left(\underline{(w^{-1}w_0)}\beta^1\right)$. Similarly, the red part of (1.2) belongs to $X\left(\beta^2\underline{w}\right)$.

Note that it may be that $\mathcal{U}_{r_1,w} = \varnothing$, in which case the product on the left of (1.1) is also empty. This phenomenon can be easily understood using the notion of the Demazure product (see Section 2). The open set $\mathcal{U}_{r_1,w}$ is not empty precisely when both Demazure products $\delta\left(\underline{(w^{-1}w_0)}\beta^1\right)$ and $\delta\left(\beta^2\underline{w}\right)$ are equal to w_0.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

One advantage of the maps (1.1) is that, for fixed $r_1 = 1, \dots, r$, we have

$$\bigcup_{w \in S_k} \mathcal{U}_{r_1,w} = X(\beta),$$

that is, the sets $\mathcal{U}_{r_1,w}$ form a cover of $X(\beta)$ by open subsets which are themselves isomorphic to products of braid varieties for simpler braids.

A disadvantage is that the properties of the map (1.1) remain mysterious at the moment, especially those regarding the relationship between (1.1) and the cluster structure on $X(\beta)$ constructed in [2, 12, 13]. The following surprising inequality is an easy consequence of Theorem 1.1 (see Remark 5.8).

Corollary 1.3 *Let f, f_1, f_2 denote the numbers of frozen variables for $X(\beta), X(\underline{(w^{-1}w_0)}\beta^1)$ and $X\left(\beta^2\underline{w}\right)$, respectively. Then we have an inequality*

$$f_1 + f_2 \geq f. \tag{1.3}$$

More precisely, we formulate the following conjecture relating the cluster structures.

Conjecture 1.4 *For each $r_1 = 1, \dots, r$ and $w \in S_k$ such that $\mathcal{U}_{r_1,w} \neq \varnothing$, there exists a seed $\Sigma = (Q, \mathbf{x})$ in $\mathbb{C}[X(\beta)]$ with cluster variables $x_{a_1}, \dots, x_{a_s} \in \mathbf{x}$ such that:*

(a) *The open set $\mathcal{U}_{r_1,w}$ is the common non-vanishing locus of $x_{a_1}, \dots, x_{a_s}$.*
(b) *The variety $\mathcal{U}_{r_1,w}$ is the cluster variety associated with the seed obtained from Σ upon freezing the cluster variables $x_{a_1}, \dots, x_{a_s}$.*
(c) *The map (1.1) is a quasi-cluster isomorphism.*

In particular, Conjecture 1.4(b) ensures that $\mathcal{U}_{r_1,w}$ is a cluster chart in $X(\beta)$ in the sense of Muller [23]. The open set $\mathcal{U}_{r_1,w}$ is defined in $X(\beta)$ by the non-vanishing of minors $\Delta_{ww_0[i],[i]}(M_{r_1})$, $i = 1, \dots, k$, where M_{r_1} is a certain matrix related to the flag $\mathcal{F}^{r_1}$ in (1.2). Conjecture 1.4(b) then implies that these minors are cluster monomials in the seed Σ, and their irreducible factors (up to monomials in frozen variables) are precisely $x_{a_1}, \dots, x_{a_s}$. The number s of cluster variables that one needs to freeze in Conjecture 1.4 equals

$$s = f_1 + f_2 - f,$$

which is nonnegative by Corollary 1.3. See Sections 5.2 and 5.3 for more details and examples. Also, see Lemma 5.7 for more details and dependencies between (a) and (c).

1.2 Splicing open Richardson varieties

Open Richardson varieties are smooth, affine subvarieties of the flag variety, given by specifying relative positions with respect to the standard flag and the antistandard flag (see Section 5.3 for details). For $v \leq w \in S_k$, the open Richardson variety is

$$R(v, w) := \{\mathcal{F} \in \mathcal{F}\ell(k) \mid \mathcal{F}^{\mathrm{std}} \xrightarrow{w} \mathcal{F} \xrightarrow{v^{-1}w_0} \mathcal{F}^{\mathrm{ant}}\},$$

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

see Section 2 for details on relative position and unexplained notation. It is known, see, for example, [2, 12, 13] and Section 5.3 below, that open Richardson varieties are special cases of braid varieties. Specializing Theorem 1.1 to this setting, we obtain the following result.

Theorem 1.5 *For $u \le v \le w \in S_k$, define the set*

$$\mathcal{U}_{u,v,w} := \left\{ \mathcal{F} \in R(u,w) \mid \mathcal{F} \xrightarrow{w_0} \mathcal{F}(vw_0) \right\}.$$

Then, $\mathcal{U}_{u,v,w}$ is principal open in $R(u,w)$ and

$$\mathcal{U}_{u,v,w} \cong R(u,v) \times R(v,w). \tag{1.4}$$

Note that Theorem 1.5 implies that, if $f_{v,w}$ denotes the number of frozen variables in $R(v,w)$, then we have the inequality $f_{u,v} + f_{v,w} \ge f_{u,w}$. The problem of finding a combinatorial rule to compute the quantity $f_{v,w}$ is an interesting one. In Section 5.3, we apply Theorem 1.5 to this problem. In particular, we find that $\max\{f_{v,w} \mid v \le w \in S_k\} - k + 1$ grows at least linearly in k (see Remark 5.18).

We can identify $R(u,w)$ with a locally closed subset of the affine space $\mathbb{C}^{\ell(w)}$, by identifying an explicit matrix form of all flags $\mathcal{F}$ such that $\mathcal{F}^{\mathrm{std}} \xrightarrow{w} \mathcal{F}$. It is a hard problem to determine which minors are cluster monomials. If true, Conjecture 1.4 applied to the Richardson setting would imply that, for every $u \le v \le w$, the minors $\Delta_{v[i],[i]}$ are cluster monomials in $R(u,w)$ for all $i = 1, \dots, k$. Note that it was recently shown in [22, Theorem C] that for open positroid varieties in the Grassmannian $\mathrm{Gr}(r,k)$ *all* nonzero Plücker coordinates are cluster monomials. Viewed in the Richardson setting, the Plücker coordinates correspond to minors of the form $\Delta_{I,[r]}$, where $I \subseteq [k]$ is an r-element set.

One can also iterate Theorem 1.5 to obtain the following.

Corollary 1.6 *Suppose that $u = v_0 < v_1 < v_2 < \cdots < v_\ell = w$ is a maximal chain from u to w in the Bruhat poset. Let $\sigma \in S_{\ell-1}$ be a permutation specifying an order in which to apply the splicing isomorphisms (1.4). Then we have an open embedding*

$$\iota_\sigma : (\mathbb{C}^\times)^\ell \simeq R(v_0,v_1) \times R(v_1,v_2) \times \cdots \times R(v_{\ell-1},v_\ell) \hookrightarrow R(u,w). \tag{1.5}$$

Here, we used the fact that one-dimensional Richardson varieties $R(v_{i-1},v_i)$ are isomorphic to $\mathbb{C}^\times$. If true, Conjecture 1.4 would imply that all tori (1.5) are cluster tori in $R(u,w)$.

Remark 1.7 We note that (the image of) the embedding (1.5) depends on the permutation σ and not just on the maximal Bruhat chain. As an example, consider the chain

$$v_0 = e < v_1 = s_1 < v_2 = s_1s_2 < v_3 = s_1s_2s_1.$$

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

The Richardson variety $R(v_0, v_3)$ can be identified with $\{(x, y, z) \in \mathbb{C}^3 \mid y \neq 0, xz - y \neq 0\}$ via the map

$$(x, y, z) \mapsto \begin{pmatrix} y & -z & 1 \\ x & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Consider the case when the permutation σ is the identity, so first we are splicing at v_1 and consider the open set $\mathcal{U}_{v_0,v_1,v_3} \cong R(v_0, v_1) \times R(v_1, v_3)$. By Proposition 5.12 and Lemma 2.7(a), this open set is $\{z \neq 0\}$. Note that $R(v_1, v_3)$ is already a torus, so in this case, we have that the image of ι_σ is $\{z \neq 0\}$.

Now assume that $\sigma \in S_2$ is not the identity, so that we first splice at v_2 and consider the open set $\mathcal{U}_{v_0,v_2,v_3}$. Similarly to the previous paragraph, the set $\mathcal{U}_{v_0,v_2,v_3}$ is given by $\{x \neq 0\}$, so in this case, the image of ι_σ is $\{x \neq 0\}$. We remark that $\{x \neq 0\}$ and $\{z \neq 0\}$ are precisely the cluster tori in the cluster structure on $R(v_0, v_3)$ with the following initial seed:

$$\boxed{xz - y} \longrightarrow z \longrightarrow \boxed{y}$$

1.3 Splicing double Bott–Samelson varieties

While we do not have a proof of Conjecture 1.4 in full generality, in some cases, we are able to prove it, namely, those cases where we start with a *double Bott–Samelson variety*. For each positive braid $\beta \in \mathrm{Br}_k^+$, the double Bott–Samelson variety $\mathrm{BS}(\beta)$ is defined as a space of flag configurations dictated by the braid β in a way similar to, but subtly different from, the definition of a braid variety. In fact, we have isomorphisms

$$\varphi_1 : \mathrm{BS}(\beta) \to X(\beta\Delta), \qquad \varphi_2 : \mathrm{BS}(\beta) \to X(\Delta\beta), \tag{1.6}$$

where Δ is a positive braid lift of the longest element $w_0 \in S_k$. Double Bott–Samelson varieties were introduced by Elek and Lu in [7], and a cluster structure on them was studied in [15, 26]. Given a braid decomposition $\beta = \beta^1\beta^2$ as above, we define an open set $\mathcal{U}_{r_1} = \mathcal{U}(\beta^1, \beta^2) \subseteq \mathrm{BS}(\beta)$ and show the following.

Theorem 1.8 *Let $\beta = \beta^1\beta^2 \in \mathrm{Br}_k^+$ be a positive braid. Then, we have an isomorphism:*

$$\Psi_{r_1} : \mathrm{BS}(\beta^1) \times \mathrm{BS}(\beta^2) \xrightarrow{\cong} \mathcal{U}_{r_1}. \tag{1.7}$$

Moreover, the map (1.7) and the open set $\mathcal{U}_{r_1}$ satisfy the properties predicted by Conjecture 1.4.

We remark that, upon some identifications made possible by (1.6), the map (1.7) is a special case of the maps (1.1). Recall that we denote by $\Phi_{r_1,w}$ the inverse to $\Psi_{r_1,w}$. Similarly, denote by Φ_{r_1} the inverse to Ψ_{r_1} from (1.7).

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

Theorem 1.9 *Let $\beta = \beta^1\beta^2$ be a positive braid. The following diagrams commute:*

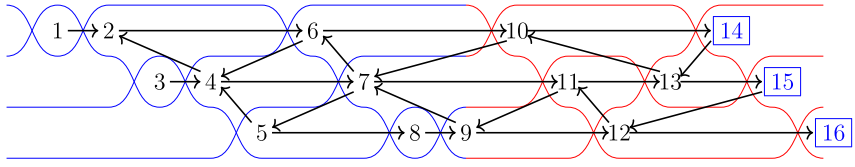
$$\begin{array}{ccc} \mathrm{BS}(\beta) \supseteq \mathcal{U}_{r_1} & \xrightarrow{\Phi_{r_1}} & \mathrm{BS}(\beta^1) \times \mathrm{BS}(\beta^2) \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2 \times \varphi_1} \\ X(\beta\Delta) \supseteq \mathcal{U}_{r_1,e} & \xrightarrow{\Phi_{r_1,e}} & X(\Delta\beta^1) \times X(\beta^2\Delta) \end{array}$$

$$\begin{array}{ccc} \mathrm{BS}(\beta) \supseteq \mathcal{U}_{r_1} & \xrightarrow{\Phi_{r_1}} & \mathrm{BS}(\beta^1) \times \mathrm{BS}(\beta^2) \\ \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_2 \times \varphi_1} \\ X(\Delta\beta) \supseteq \mathcal{U}_{r_1+\ell(w_0),w_0} & \xrightarrow{\Phi_{r_1+\ell(w_0),w_0}} & X(\Delta\beta^1) \times X(\beta^2\Delta). \end{array}$$

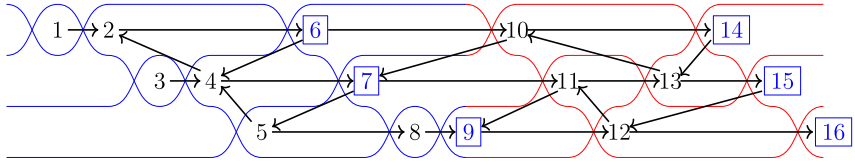
Here φ_1, φ_2 are isomorphisms from (1.6).

We visually describe the splicing map in the following example, but leave the technical details needed to verify that the map is a quasi-cluster isomorphism to Example 6.10.

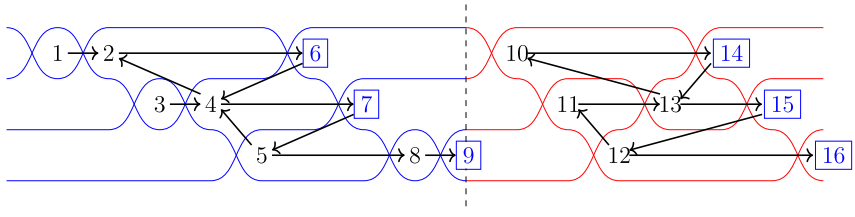
Example 1.10 Consider $\beta = \sigma_3^2\sigma_2^2\sigma_1\sigma_3\sigma_2\sigma_1^2\sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1$, where β^1 and β^2 are as indicated by colors. We have the quiver Q_β:



The open set $\mathcal{U}_{r_1}$ is the cluster variety corresponding to the following quiver, which is obtained by freezing the vertices 6, 7, and 9. Note that these are the vertices on the right of the rightmost appearance of a crossing in β^1.



On the other hand, the product $\mathrm{BS}(\beta^1) \times \mathrm{BS}(\beta^2)$ is the cluster variety corresponding to the disjoint union of the quivers Q_{β^1} and Q_{β^2}:



https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

1.4 Connections to other work

The splicing maps for double Bott–Samelson varieties are predicted by results in link homology. In particular, the work of Trinh [27] relates the equivariant Borel–Moore homology of $\mathrm{BS}(\beta)$ (together with the weight filtration) to the Khovanov–Rozansky homology $\mathrm{HHH}^{a=0}(\beta)$. The corresponding multiplication maps in link homology

$$\mathrm{HHH}^{a=0}(\beta^1) \otimes \mathrm{HHH}^{a=0}(\beta^2) \to \mathrm{HHH}^{a=0}(\beta^1\beta^2)$$

are well known and very useful (see, e.g., [16] and the references therein).

More generally, the equivariant Borel–Moore homology (again with the weight filtration) of the variety $X(\beta)$ is related to the Khovanov–Rozansky homology $\mathrm{HHH}^{a=0}(\beta\Delta^{-1})$. In the setting of Theorem 1.1, note that the braid

$$\left(\Delta^{-1}\underline{w^{-1}w_0}\beta^1\right)\cdot\left(\beta^2\underline{w}\Delta^{-1}\right)$$

is conjugate to $\beta^1\beta^2\Delta^{-1}$, so they represent the same link. Thus, we have a map in link homology

$$\mathrm{HHH}^{a=0}\left(\Delta^{-1}\underline{w^{-1}w_0}\beta^1\right) \otimes \mathrm{HHH}^{a=0}\left(\beta^2\underline{w}\Delta^{-1}\right) \to \mathrm{HHH}^{a=0}\left(\beta^1\beta^2\Delta^{-1}\right),$$

which suggests the splicing map $X\left(\underline{w^{-1}w_0}\beta^1\right) \times X\left(\beta^2\underline{w}\right) \to X\left(\beta^1\beta^2\right)$ as in (1.1).

Another motivation for splicing maps comes from [6] that studies homology of (parabolic) open Richardson varieties. Let u, v be two permutations in S_k such that $u \le v$ in Bruhat order. As mentioned above, the open Richardson variety $R(u,v)$ is isomorphic to a certain braid variety:

$$R(u,v) \simeq X\left(\underline{v}\cdot\underline{u^{-1}w_0}\right).$$

It is known that compactly supported cohomology of $R(u,v)$ is closely related to the Ext group between the Verma modules in the category $\mathcal{O}$ for $\mathfrak{gl}_k$:

$$H_c^*(R(u,v)) \simeq \mathrm{Ext}^*_{\mathcal{O}}(\Delta_u, \Delta_v). \tag{1.8}$$

The homological grading and the weight filtration on the left-hand side correspond to two gradings on the right-hand side (see [6, Theorem 12.5], [11], and the references therein).

Given $u \le v \le w$ in S_k, [6, Section 3] constructs a rational map

$$R(u,v) \times R(v,w) \to R(u,w), \tag{1.9}$$

which corresponds under (1.8) to compositions of extensions

$$\mathrm{Ext}^*_{\mathcal{O}}(\Delta_u, \Delta_v) \otimes \mathrm{Ext}^*_{\mathcal{O}}(\Delta_v, \Delta_w) \to \mathrm{Ext}^*_{\mathcal{O}}(\Delta_u, \Delta_w),$$

see [6, Corollary 3.3]. This can be compared with Theorem 1.5, however we do not know whether our maps coincide with those of [6] and plan to investigate this in the future.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

1.5 Organization of the article

Sections 2–4 are mostly preparatory: in Section 2, we give the necessary background on relative positions of flags and its relation to matrix minors. In Section 3, we recall the definition of cluster algebras and quasi-cluster isomorphisms. We define braid and double Bott–Samelson varieties in Section 4. In particular, in Section 4.3, we give details on the cluster structure on braid and double Bott–Samelson varieties obtained in [2, 12, 13, 26].

The technical heart of the article is Section 5. In Section 5.1, we define the open sets $\mathcal{U}_{r_1,w}$, see (5.1), and prove Theorem 1.1 as Theorem 5.2. In Section 5.2, we elaborate on the conjectural properties of the splicing map (1.1), giving a more precise version of Conjecture 1.4 as Conjecture 5.6. We elaborate on the relations between the different parts of this conjecture, and show a weaker version of the conjecture for the case $w = w_0$, and illustrate this with examples. Finally, in Section 5.3, we specialize the map (1.1) to the case of open Richardson varieties and prove Theorem 1.5.

In Section 6, we deal with the case of double Bott–Samelson varieties. We show Theorem 1.8 in two parts: first, in Theorem 6.1, we construct the map (1.7) and show that it is an isomorphism; second, we study the cluster-theoretic properties of (1.7) in Section 6.4, showing the second part of Theorem 1.8 as Theorem 6.9. Finally, Theorem 1.9 is proved as Lemmas 6.4 and 6.5.

1.6 Notations

For a tuple of matrices $(A_1, \dots, A_k)$ and a matrix M, we write $M(A_1, \dots, A_k) = (MA_1, \dots, MA_k)$. Similarly, given a tuple of flags $(\mathcal{F}^1, \dots, \mathcal{F}^k)$ and a matrix M, we write $M(\mathcal{F}^1, \dots, \mathcal{F}^k) = (M\mathcal{F}^1, \dots, M\mathcal{F}^k)$.

2 Background

We recall some standard facts we will use throughout the article, in the way setting up notation and conventions.

2.1 Braids and Demazure product

We will work with the positive braid monoid on k strands Br_k^+:

$$\mathrm{Br}_k^+ = \left\langle \sigma_1, \dots, \sigma_{k-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, i = 1, \dots, k-2 \right\rangle.$$

We have a surjective homomorphism $\pi : \mathrm{Br}_k^+ \to S_k$ given by $\pi(\sigma_i) = s_i$, where $s_1, \dots, s_{k-1}$ are the simple transpositions in the symmetric group S_k. We also have the Demazure product $\delta : \mathrm{Br}_k^+ \to S_k$, defined inductively by

$$\delta(e) = e,\ \delta(\beta\sigma_i) = \begin{cases} \delta(\beta)s_i & \text{if } \delta(\beta)s_i > \delta(\beta) \\ \delta(\beta) & \text{else.} \end{cases} \tag{2.1}$$

Note that, as opposed to π, the Demazure product δ is not a morphism of monoids. If $w \in S_k$, we denote by $\underline{w} \in \mathrm{Br}_k^+$ the unique lift of minimal length (either under the projection π or the Demazure product δ) of w to Br_k^+. We will denote by $w_0 \in S_k$

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

the longest element, and its lift $\underline{w_0}$ will be denoted by Δ. If $v, w \in S_k$, we define their Demazure product by

$$v \star w := \delta(\underline{v} \cdot \underline{w}). \tag{2.2}$$

We finish this section with the following result.

Lemma 2.1 *Let $v, w \in S_k$. The following are equivalent:*

(a) $v \star w = w_0$.
(b) $v \geq w_0 w^{-1}$ *in Bruhat order.*
(c) $w \geq v^{-1} w_0$ *in Bruhat oder.*

Proof We show that (a) implies (b). First, we remark that the Demazure product $\delta(\beta)$ is the longest reduced word contained in β. Definition (2.1) is a greedy method to compute it – another equally effective greedy method is to read the word β in the opposite direction. That being said, we have $v \star w = s_{i_1} \dots s_{i_k} w$, where $\sigma_{i_1} \dots \sigma_{i_k}$ is a reduced subexpression of $\underline{v}$. If $v \star w = w_0$, then $s_{i_1} \dots s_{i_k} = w_0 w^{-1}$, so $v \geq w_0 w^{-1}$ as needed. Now, we show that (b) implies (a). If $\underline{w_0 w^{-1}}$ appears as a reduced subexpression in $\underline{v}$, then a reduced subexpression for w_0 appears as a subexpression in $\underline{v} \dots \underline{w}$, and we obtain $v \star w = w_0$. This proves that (a) and (b) are equivalent. The equivalence between (a) and (c) is proved similarly. ∎

2.2 Flags and relative position

We will denote by $\mathcal{F}\ell_k$ the variety of complete flags in the k-dimensional complex vector space $\mathbb{C}^k$. For a matrix $M \in \mathrm{GL}(k)$ with columns $m_1, \dots, m_k \in \mathbb{C}^k$, we define the flag

$$\mathcal{F}(M) = \left(\{0\} \subseteq \langle m_1 \rangle \subseteq \langle m_1, m_2 \rangle \subseteq \cdots \subseteq \langle m_1, \dots, m_{k-1} \rangle \subseteq \langle m_1, \dots, m_k \rangle = \mathbb{C}^k\right).$$

The assignment $M \mapsto \mathcal{F}(M)$ gives rise to the usual identification $\mathcal{F}\ell_k = \mathrm{GL}(k)/\mathsf{B}(k)$, where $\mathsf{B}(k) \subseteq \mathrm{GL}(k)$ is the subgroup of upper triangular matrices. Note that the group $\mathrm{GL}(k)$ acts on $\mathcal{F}\ell_k$ by multiplication on the left, or, equivalently,

$$g.(\{0\} \subseteq F_1 \subseteq \cdots \subseteq F_{k-1} \subseteq \mathbb{C}^k) = (\{0\} \subseteq g(F_1) \subseteq \cdots \subseteq g(F_{k-1}) \subseteq \mathbb{C}^k),$$

so that $g.\mathcal{F}(M) = \mathcal{F}(gM)$.

We will denote by $e_1, \dots, e_k \in \mathbb{C}^k$ the usual standard basis. Given an element $w \in S_k$, we define the coordinate flag

$$\mathcal{F}(w) := (\{0\} \subseteq \langle e_{w(1)} \rangle \subseteq \langle e_{w(1)}, e_{w(2)} \rangle \subseteq \cdots \subseteq \langle e_{w(1)}, \dots, e_{w(k-1)} \rangle \subseteq \mathbb{C}^k),$$

in particular, we have the standard and antistandard flags:

$$\mathcal{F}^{\mathrm{std}} := \mathcal{F}(e), \qquad \mathcal{F}^{\mathrm{ant}} := \mathcal{F}(w_0).$$

We say that two flags $\mathcal{F}$ and $\mathcal{F}'$ are in (relative) position $w \in S_k$, and write $\mathcal{F} \xrightarrow{w} \mathcal{F}'$ if there exists $g \in \mathrm{GL}(k)$ such that $g\mathcal{F} = \mathcal{F}^{\mathrm{std}}$ and $g\mathcal{F}' = \mathcal{F}(w)$. Such g is uniquely defined up to left multiplication by an element of $\mathsf{B}(k) \cap w\mathsf{B}(k)w^{-1}$.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

Note that given two flags $\mathcal{F}, \mathcal{F}' \in \mathcal{F}\ell_k$, there exists a unique $w \in S_k$ such that $\mathcal{F} \xrightarrow{w} \mathcal{F}'$, and such w is determined by

$$\dim(F_i \cap F'_j) = \#\left(\{1, \dots, i\} \cap \{w(1), \dots, w(j)\}\right).$$

In particular,

$$\mathcal{F} \xrightarrow{s_i} \mathcal{F}' \text{ if and only if } F_i \neq F'_i \text{ and } F_j = F'_j \text{ for } j \neq i.$$

Relative position of flags is closely related to the Bruhat decomposition of the group $\mathrm{GL}(k)$:

$$\mathrm{GL}(k) = \bigsqcup_{w \in S_k} \mathsf{B}(k) w \mathsf{B}(k).$$

Indeed, note that if $M, M' \in \mathrm{GL}(k)$, then $\mathcal{F}(M) \xrightarrow{w} \mathcal{F}(M')$ if and only if there exist upper triangular matrices $U_1, U_2 \in \mathsf{B}(k)$ and $g \in \mathrm{GL}(k)$ such that $gM = U_1, gM' = wU_2$, and this is in turn equivalent to $M^{-1}M' \in \mathsf{B}(k) w \mathsf{B}(k)$. The Bruhat decomposition satisfies the following multiplicative property. If $w \in S_k$ and $i = 1, \dots, k-1$, we have

$$(\mathsf{B}w\mathsf{B})(\mathsf{B}s_i\mathsf{B}) = \begin{cases} \mathsf{B}ws_i\mathsf{B} & \text{if } \ell(ws_i) > \ell(w), \\ (\mathsf{B}ws_i\mathsf{B}) \sqcup (\mathsf{B}w\mathsf{B}) & \text{else.} \end{cases} \tag{2.3}$$

This implies the following standard lemma that we will use repeatedly. For a proof, see, for example, [2, Lemma 3.2].

Lemma 2.2 *Let $\mathcal{F}, \mathcal{F}' \in \mathcal{F}\ell_k$ and $w \in S_k$. The following are equivalent:*

(1) $\mathcal{F} \xrightarrow{w} \mathcal{F}'$.

(2) *There exist a reduced decomposition $w = s_{i_1} \dots s_{i_r}$ and flags $\mathcal{F}^1, \dots, \mathcal{F}^{r-1}$ such that*

$$\mathcal{F} \xrightarrow{s_{i_1}} \mathcal{F}^1 \xrightarrow{s_{i_2}} \dots \xrightarrow{s_{i_{r-1}}} \mathcal{F}^{r-1} \xrightarrow{s_{i_r}} \mathcal{F}'.$$

(3) *For any reduced decomposition $w = s_{i_1} \dots s_{i_r}$ there exist flags $\mathcal{F}^1, \dots, \mathcal{F}^{r-1}$ such that*

$$\mathcal{F} \xrightarrow{s_{i_1}} \mathcal{F}^1 \xrightarrow{s_{i_2}} \dots \xrightarrow{s_{i_{r-1}}} \mathcal{F}^{r-1} \xrightarrow{s_{i_r}} \mathcal{F}'.$$

Moreover, given a reduced decomposition of w, the flags $\mathcal{F}^1, \dots, \mathcal{F}^{r-1}$ in (3) are unique.

The set Γ_w of pairs of flags $(\mathcal{F}, \mathcal{F}')$ such that $\mathcal{F} \xrightarrow{w} \mathcal{F}'$ is a locally closed subvariety of $\mathcal{F}\ell_k \times \mathcal{F}\ell_k$. The intermediate flags $\mathcal{F}^1, \dots, \mathcal{F}^r$ depend on $(\mathcal{F}, \mathcal{F}')$ algebraically, that is, define $(r-1)$ regular maps from Γ_w to $\mathcal{F}\ell_k$.

Corollary 2.3 (a) *Suppose that $w = w_1 w_2$, where $w, w_1, w_2 \in S_k$, and $\ell(w) = \ell(w_1) + \ell(w_2)$. Then $\mathcal{F} \xrightarrow{w} \mathcal{F}'$ if and only if there exists a flag $\mathcal{F}''$ such that*

$$\mathcal{F} \xrightarrow{w_1} \mathcal{F}'' \xrightarrow{w_2} \mathcal{F}'.$$

In this case, $\mathcal{F}''$ is unique.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

(b) *Assume that we have* $\mathcal{F} \xrightarrow{w_1} \mathcal{F}' \xrightarrow{w_2} \mathcal{F}''$. *Then,* $\mathcal{F} \xrightarrow{w} \mathcal{F}''$ *with* $w \leq w_1 \star w_2$, *where* $w_1 \star w_2$ *is defined by (2.2).*

Proof Part (a) is immediate from Lemma 2.2. For part (b), assume $\mathcal{F} = \mathcal{F}(M)$, $\mathcal{F}' = \mathcal{F}(M')$, and $\mathcal{F}'' = \mathcal{F}(M'')$. So we have $M^{-1}M' \in \mathsf{B}w_1\mathsf{B}$ and $(M')^{-1}M'' \in \mathsf{B}w_2\mathsf{B}$. Now, $M^{-1}M'' \in (\mathsf{B}w_1\mathsf{B})(\mathsf{B}w_2\mathsf{B})$ and the result follows from (2.3). ∎

2.3 Transverse flags

We will say that two flags $\mathcal{F}$ and $\mathcal{F}'$ are transverse and write $\mathcal{F} \pitchfork \mathcal{F}'$ if $\mathcal{F} \xrightarrow{w_0} \mathcal{F}'$. Note that

$$\mathcal{F} \pitchfork \mathcal{F}' \text{ if and only if } F_i \cap F'_{k-i} = \{0\} \text{ for all } i, \text{ if and only if } F_i + F'_{k-i} = \mathbb{C}^k \text{ for all } i.$$

Lemma 2.4 *Let* M *be a nonsingular matrix and* $w \in S_k$. *Then,* $\mathcal{F}(M) \pitchfork \mathcal{F}(w)$ *if and only if the matrix* M *admits a decomposition of the form*

$$M = ww_0LU,$$

where L *is lower-triangular and* U *is upper-triangular. Such decomposition is unique upon requiring that* L *has* 1*'s on the diagonal.*

Proof We have that $\mathcal{F}(M) \pitchfork \mathcal{F}(w)$ if and only if there exists an element $g \in \mathrm{GL}(k)$ such that $gM \in \mathsf{B}$ and $gw \in w_0\mathsf{B}$. If such an element exists, then for some $U_1, U_2 \in \mathsf{B}$, we get $g = w_0U_1w^{-1}$ and $M = g^{-1}U_2 = (w_0U_1w^{-1})^{-1}U_2 = wU_1^{-1}w_0U_2 = ww_0(w_0U_1^{-1}w_0)U_2$, so such a decomposition of M exists with $L = w_0U_1^{-1}w_0$. Conversely, if $M = ww_0LU$, then setting $g = L^{-1}w_0w^{-1}$ we have that $gM = U \in \mathsf{B}$ and $gw = L^{-1}w_0 \in w_0\mathsf{B}$. Finally, the uniqueness claim follows from the uniqueness of the LU-decomposition. ∎

2.4 Braid matrices

For $i = 1, \ldots, k-1$ and a formal variable z, we define the matrix $B_i(z)$ to be

$$B_i(z) = \begin{pmatrix} 1 & \cdots & & & \ldots & 0 \\ \vdots & \ddots & & & & \vdots \\ 0 & \cdots & z & -1 & \cdots & 0 \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 1 \end{pmatrix},$$

where the non-identity part of $B_i(z)$ is located in the i-th and $(i+1)$-st row and columns.

The matrix $B_i(z)$ is important for us due to the following well-known (and easy) lemma.

Lemma 2.5 *Let* $M \in \mathrm{GL}(n)$ *be a nondegenerate matrix and let* $i = 1, \ldots, n-1$. *Then,* $\mathcal{F} \xrightarrow{s_i} \mathcal{F}(M)$ *if and only if there exists a (necessarily unique)* $z \in \mathbb{C}$ *such that* $\mathcal{F} = \mathcal{F}(MB_i(z))$.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

The following lemma is also easy to check (see [2, Corollary 3.9] and [4, Lemma 2.20]).

Lemma 2.6 *If U is an upper-triangular matrix and $z \in \mathbb{C}$, then there exist unique upper-triangular matrix U' and $z' \in \mathbb{C}$ such that*

$$UB_i(z) = B_i(z')U'.$$

Furthermore, the diagonal entries of U' are permuted from the ones of U by s_i.

2.5 Minors

We use notation $[i] = \{1, \dots, i\}$ and $u[i] = \{u(1), \dots, u(i)\}$ for $u \in S_k$. If $I, J \subseteq [k]$ are sets of the same cardinality and $M \in \mathrm{GL}(k)$, we denote by $\Delta_{I,J}(M)$ the determinant of the $|I| \times |J|$-submatrix of M obtained by deleting the rows (resp. columns) not belonging to I (resp. J). For $i = 1, \dots, k$, we have the principal minor

$$\Delta_i(M) := \Delta_{[i],[i]}(M), \tag{2.4}$$

that is, $\Delta_i(M)$ is the determinant of the upper-left justified $i \times i$-submatrix of M. Thus, for example, $\Delta_1(M) = m_{11}$ and $\Delta_k(M) = \det(M)$. It is a classical result that a matrix admits an LU decomposition if and only if all its principal minors are nonzero.

More generally, we have the following.

Lemma 2.7 a) *For $w \in S_k$ and $M \in \mathrm{GL}(k)$, we have that $\mathcal{F}(M) \pitchfork \mathcal{F}(w)$ if and only if the minors*

$$\Delta_{ww_0[i],[i]}(M) = (-1)^{\ell(w)}\Delta_{[i],[i]}(w_0w^{-1}M)$$

are nonzero for all $i = 1, \dots, k$.

b) *If $\mathcal{F}^{\mathrm{std}} \xrightarrow{w} \mathcal{F}(M)$, then the minors $\Delta_{w[i],[i]}(M)$ are nonzero for $i = 1, \dots, k$.*

c) *If $\Delta_{w[i],[i]}(M) \neq 0$ for every $i = 1, \dots, k$, and $\mathcal{F}^{\mathrm{std}} \xrightarrow{v} \mathcal{F}(M)$, then $v \not< w$.*

Proof Part (a) follows from Lemma 2.4 and (2.4).

For part (b), observe that the condition $\mathcal{F}^{\mathrm{std}} \xrightarrow{w} \mathcal{F}(M)$ implies

$$\mathcal{F}(ww_0) \xrightarrow{w_0w^{-1}} \mathcal{F}^{\mathrm{std}} \xrightarrow{w} \mathcal{F}(M)$$

so by Corollary 2.3(a), we get $\mathcal{F}(ww_0) \pitchfork \mathcal{F}(M)$. Now, the statement follows from (a).

For part (c), we have that $\mathcal{F}(ww_0) \pitchfork \mathcal{F}(M)$ and $\mathcal{F}(ww_0) \xrightarrow{w_0w^{-1}} \mathcal{F}^{\mathrm{std}} \xrightarrow{v} \mathcal{F}(M)$. By Corollary 2.3(b) and using the fact that w_0 is the longest element in S_k, we have $w_0 = (w_0w^{-1}) \star v$. But if $v < w$, then $(w_0w^{-1}) \star v < w_0$, so we obtain $v \not< w$. ∎

Definition 2.8 For $w \in S_k$, we define the subset

$$\mathcal{U}(w) := \{\mathcal{F} \in \mathcal{F}\ell_k \mid \mathcal{F} \pitchfork \mathcal{F}(w)\} \subseteq \mathcal{F}\ell_k.$$

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

Lemma 2.9 *For all $w \in S_k$, the subset $\mathcal{U}(w)$ is open in $\mathcal{F}\ell_k$ and moreover*

$$\mathcal{F}\ell_k = \bigcup_{w \in S_k} \mathcal{U}(w).$$

Proof By Lemma 2.7, $\mathcal{U}(w)$ is given by the nonvanishing of the minors $\Delta_{ww_0[i],[i]}$, $i = 1, \ldots, k$, so it is clearly open in $\mathcal{F}\ell_k$. Now, consider a flag $\mathcal{F} = \mathcal{F}(M) \in \mathcal{F}\ell_k$, and pick any element $v \in S_k$. We have $\mathcal{F}(M) \xrightarrow{w_1} \mathcal{F}(v)$ for some $w_1 \in S_k$, and let $w_0 = w_1 w_2$ be a length-additive decomposition. We get

$$\mathcal{F}(M) \xrightarrow{w_1} \mathcal{F}(v) \xrightarrow{w_2} \mathcal{F}(vw_2)$$

and by Corollary 2.3(a), this implies that $\mathcal{F}(M) \xrightarrow{w_0} \mathcal{F}(vw_2)$, that is, $\mathcal{F}(M) \in \mathcal{U}(vw_2)$. ∎

Lemma 2.10 *Let $M \in \mathfrak{gl}(k)$. Let $i, j = 1, \ldots, k-1$ and assume $i \neq j$. Let $I \subseteq [k]$ be a set with $|I| = i$. Then, $\Delta_{I,[i]}(MB_j(z)) = \Delta_{I,[i]}(M)$.*

Proof By the Cauchy–Binet formula,

$$\Delta_i(MB_j(z)) = \sum_{\substack{J \subseteq [k] \\ |J| = i}} \Delta_{I,J}(M)\Delta_{J,[i]}(B_j(z)).$$

Since $i \neq j$, we have that $\Delta_{J,[i]}(B_j(z)) \neq 0$ if and only if $J = [i]$, in which case $\Delta_{[i],[i]}(B_j(z)) = 1$. Thus,

$$\Delta_{I,[i]}(MB_j(z)) = \Delta_{I,[i]}(M)\Delta_{[i],[i]}(B_j(z)) = \Delta_{I,[i]}(M).$$

∎

3 Cluster algebras

3.1 Definition

We recall the definition of a cluster algebra [8]. For this article, we will only need to restrict ourselves to the skew-symmetric case.

An *ice quiver* Q is a quiver with finite vertex set Q_0, that is, a finite directed graph with which we allow multiple edges between vertices but no loops nor directed two cycles. We specify that a special subset Q_0^f of the vertices of Q is declared to be *frozen* whereas an element in $Q_0 \backslash Q_0^f$ is declared to be *mutable*.

Given that Q is an ice quiver, with $|Q_0| = n + m$, where we distinguish n vertices as mutable and m vertices as frozen. We consider a field $\mathcal{F}$ of transcendence degree $n + m$ over $\mathbb{C}$. A *seed* $\Sigma = (Q, \mathbf{x})$ consists of:

(1) the ice quiver Q, and
(2) a set $\mathbf{x} = \{x_i \mid i \in Q_0\}$ that is a transcendental basis for $\mathcal{F}$, that is, $\mathcal{F} = \mathbb{C}(x_i \mid i \in Q_0)$. We say that $\mathbf{x}$ is the set of *cluster variables* of Σ.

Given a seed $\Sigma = (Q, \mathbf{x})$ and a mutable vertex $k \in Q_0$, the *mutation* of Σ in the direction k is the seed $\mu_k(\Sigma) = (\mu_k(Q), \mu_k(\mathbf{x}))$, where:

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

(1) $\mu_k(\mathbf{x}) = (\mathbf{x}\backslash\{x_k\}) \cup \{x_k'\}$, where $x_k' \in \mathcal{F}$ is defined by

$$x_k x_k' = \prod_{i \to k} x_i + \prod_{k \to j} x_j.$$

(2) $\mu_k(Q)$ has the same vertex set as Q, but the arrows change via the following three-step procedure:
   (a) Reverse all arrows incident with k.
   (b) For any pair of arrows $i \to k \to j$ in Q, create a new arrow $i \to j$.
   (c) If the previous two steps have created any 2-cycles, then remove the arrows which form a maximal collection of disjoint 2-cycles.

We call two seeds Σ and Σ' *mutation equivalent* if there is a finite sequence of mutations from one seed to the other. Mutation at a fixed vertex k is an involutive operation and therefore, mutation equivalence is well-defined. We will denote by $\mathtt{mut}(\Sigma)$ the set of all seeds which are mutation equivalent to Σ.

Definition 3.1 Let Σ be a seed. The *cluster algebra* $A(\Sigma)$ is the subalgebra of the field $\mathcal{F}$ generated by the sets of cluster variables in all seeds mutation equivalent to Σ, as well as by x_k^{-1} for $k \in Q_0^f$. We say that a commutative algebra A *admits a cluster structure* if there exists a seed Σ such that $A \cong A(\Sigma)$. Similarly, we say that an affine algebraic variety X admits a cluster structure if the coordinate algebra $\mathbb{C}[X]$ admits a cluster structure.

3.2 Quasi-cluster morphisms

It is possible that there exist two non-mutation equivalent seeds Σ, Σ' such that $A(\Sigma) \cong A \cong A(\Sigma')$, that is, the cluster structures for a commutative algebra A are generally not unique.

Example 3.2 Let Q be the quiver

$$a \to 1 \to b,$$

where the frozen vertices are shown in blue. Let x_a, x_1, x_b be the corresponding cluster variables, then the associated cluster algebra $A(\Sigma)$ is defined as

$$A(\Sigma) = \mathbb{C}[x_1, x_1', x_a^{\pm 1}, x_b^{\pm 1}]/(x_1 x_1' = x_a + x_b).$$

Now, let Q' be the quiver

$$a \to 1 \qquad b.$$

Let y_a, y_1, y_b be the corresponding cluster variables, then $A(\Sigma')$ is

$$A(\Sigma') = \mathbb{C}[y_1, y_1', y_a^{\pm 1}, y_b^{\pm 1}]/(y_1 y_1' = 1 + y_a).$$

Although these quivers are not mutation equivalent, there is an isomorphism between their cluster algebras given by the assignment $y_1 \mapsto x_1 x_b^{-1}$, $y_a \mapsto x_a x_b^{-1}$, $y_b \mapsto x_b$, and $y_1' \mapsto x_1'$.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

Let $A(\Sigma)$ and $A(\Sigma')$ be the cluster algebras associated with the seeds Σ and Σ', respectively. As demonstrated in Example 3.2, the algebras $A(\Sigma)$ and $A(\Sigma')$ may be isomorphic even if the seeds Σ and Σ' are not mutation equivalent. Following Fraser [9], see also [21, Section 5.2], we define an interesting class of morphisms between cluster algebras of perhaps non-mutation-equivalent seeds.

Given a seed Σ and a mutable vertex i, we define the exchange ratio $\widehat{y}_i$ as the ratio

$$\widehat{y}_i = \frac{\prod_{j\to i} x_j^{\#\{j\to i\}}}{\prod_{i\to j} x_j^{\#\{i\to j\}}}.$$

Definition 3.3 [9, 10] Let $A(\Sigma)$ and $A(\Sigma')$ be cluster algebras of rank $n+m$, each with m frozen variables. Let $\mathbf{x} = \{x_1, \dots, x_{n+m}\}$ be the cluster variables of Σ, and $\mathbf{x} = \{x_1', \dots x_{n+m}'\}$ be the cluster variables of Σ'. A *quasi-cluster isomorphism* is an algebra isomorphism $f : A(\Sigma) \to A(\Sigma')$ satisfying the following conditions:

(1) For each frozen variable $x_j \in \mathbf{x}$, $f(x_j)$ is a Laurent monomial in the frozen variables of $\mathbf{x}'$.
(2) For each mutable variable $x_i \in \mathbf{x}$, $f(x_i)$ coincides with x_i', up to multiplication by a Laurent monomial in the frozen variables of $\mathbf{x}'$.
(3) The exchange ratios are preserved, that is, for each mutable variable x_i of Σ, $f(\widehat{y}_i) = \widehat{y}_i'$.

Remark 3.4 By [9, Corollary 4.5], if a quasi-cluster isomorphism as in Definition 3.3 exists, then the mutable parts of the quivers Q and Q' coincide, that is, the quivers Q and Q' coincide after deleting all frozen variables (and arrows incident to them). See also [21, Section 5.2].

One can check that the map in Example 3.2 is a quasi-cluster isomorphism. We remark that if properties (1)–(3) are stable under mutations, that is, if (1)–(3) hold for two seeds Σ, Σ', then it also holds for $\mu_i(\Sigma), \mu_i(\Sigma')$ for every mutable vertex i (see [21, Section 5.2]). While a quasi-cluster isomorphism $f : A(\Sigma) \to A(\Sigma')$ does not send cluster variables to cluster variables, it does preserve all the cluster-theoretic geometric information such as cluster tori, and induces an isomorphism between upper cluster algebras (see [5, Section 2.4]).

To finish this section, let us see how quasi-cluster morphisms can be obtained from maps between exchange matrices. For this, let us first observe that the information of an ice quiver Q with n mutable vertices $1, \dots, n$ and m frozen ones $n+1, \dots, n+m$ can be codified into an *extended exchange matrix*, that is, an $(n+m) \times n$-matrix $\widetilde{B}$, defined by

$$\widetilde{b}_{i,j} = \#\{\text{arrows } i \to j\} - \#\{\text{arrows } j \to i\}.$$

Note that the top part (formed by the first n rows) of $\widetilde{B}$ is a skew-symmetric matrix B, called the *principal part* of $\widetilde{B}$.

Lemma 3.5 [21, Theorem 5.7] *Let $\Sigma = (Q, \mathbf{x}), \Sigma' = (Q', \mathbf{z})$ be two seeds, both with n mutable and m frozen variables, and extended exchange matrices $\widetilde{B}, \widetilde{B}'$. Assume that there exists an $(n+m) \times (n+m)$ integer matrix R of block triangular form*

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

$$R = \begin{pmatrix} \mathbf{1}_n & 0 \\ P & Q \end{pmatrix} \tag{3.1}$$

such that $R\widetilde{B} = \widetilde{B}'$. *If* $\det(Q) = \pm 1$, *then the assignment*

$$\Phi(x_j) = \prod_{i=1}^{n+m} z_i^{r_{i,j}} \tag{3.2}$$

extends to a quasi-cluster isomorphism $\Phi : A(\Sigma) \to A(\Sigma')$. *Moreover, if* $\Phi : A(\Sigma) \to A(\Sigma')$ *is a quasi-cluster isomorphism, then there exists a seed* Σ'' *mutation equivalent to* Σ' *and a matrix* R *of the form (3.1) such that* $R\widetilde{B} = \widetilde{B}''$ *and the quasi-cluster isomorphism is given by (3.2).*

4 Braid and double Bott–Samelson varieties

In this section, we recall the definition of the braid variety, both via configurations of flags and as an affine variety given by an explicit set of equations.

4.1 Definition via flags

Recall that Br_k^+ is the positive braid monoid on k strands, with generators $\sigma_1, \dots, \sigma_{k-1}$.

Definition 4.1 Let $\beta = \sigma_{i_1} \dots \sigma_{i_r} \in \mathrm{Br}_k^+$ be a positive braid. The *braid variety* $X(\beta)$ is the variety consisting of $(r+1)$-tuples of flags $(\mathcal{F}^0, \dots, \mathcal{F}^r)$ satisfying the following relative position conditions:

(1) $\mathcal{F}^0 = \mathcal{F}^{\mathrm{std}}$ and $\mathcal{F}^r = \mathcal{F}^{\mathrm{ant}}$.

(2) For every $j = 1, \dots, r$, $\mathcal{F}^{j-1} \xrightarrow{s_{i_j}} \mathcal{F}^j$.

Let us remark that, up to a canonical isomorphism, $X(\beta)$ depends only on the braid β and not on its presentation as a product of generators, this follows from Lemma 2.2. We also remark that $X(\beta)$ is nonempty if and only if we have $\delta(\beta) = w_0$, in which case $X(\beta)$ is a smooth, affine algebraic variety of dimension $r - \binom{k}{2} = r - \ell(w_0)$ (see [2, 3]).

Remark 4.2 We could define the braid variety by replacing condition (1) of Definition 4.1 by $\mathcal{F}^0 = \mathcal{F}^{\mathrm{std}}$ and $\mathcal{F}^r = \mathcal{F}(\delta(\beta))$. With this definition, we always have that $X(\beta)$ is affine, smooth, nonempty and of dimension $r - \ell(\delta(\beta))$. However, by [2, Lemma 3.4], we lose no generality by assuming that $\delta(\beta) = w_0$. We will always assume that $\delta(\beta) = w_0$.

Definition 4.3 Let $\beta = \sigma_{i_1} \dots \sigma_{i_r} \in \mathrm{Br}_k^+$ be a positive braid. The *double Bott–Samelson variety* $\mathrm{BS}(\beta)$ is the variety consisting of $(r+1)$-tuples of flags $(\mathcal{F}^0, \dots, \mathcal{F}^r)$ satisfying the following relative position conditions:

(1) $\mathcal{F}^0 = \mathcal{F}^{\mathrm{std}}$ and $\mathcal{F}^r \pitchfork \mathcal{F}^{\mathrm{ant}}$.

(2) For every $j = 1, \dots, r$, $\mathcal{F}^{j-1} \xrightarrow{s_{i_j}} \mathcal{F}^j$.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

Note that the definitions of $X(\beta)$ and $\mathrm{BS}(\beta)$ are superficially very similar, with the crucial difference that in $X(\beta)$ we require $\mathcal{F}^r = \mathcal{F}^{\mathrm{ant}}$ and in $\mathrm{BS}(\beta)$ we require $\mathcal{F}^r \pitchfork \mathcal{F}^{\mathrm{ant}}$. We can, nevertheless, realize every double Bott–Samelson variety as a braid variety in at least two different ways.

Lemma 4.4 *We have isomorphisms*

$$\varphi_1 : \mathrm{BS}(\beta) \to X(\beta\Delta), \qquad \varphi_2 : \mathrm{BS}(\beta) \to X(\Delta\beta).$$

Proof If we have a chain of flags $(\mathcal{F}^0, \dots, \mathcal{F}^r) \in \mathrm{BS}(\beta)$ then, since $\mathcal{F}^r \pitchfork \mathcal{F}(w_0)$ by Lemma 2.2, there exists a unique chain $\mathcal{F}^r \xrightarrow{s_{a_1}} \mathcal{F}^{r+1} \xrightarrow{s_{a_2}} \dots \xrightarrow{s_{a_{\ell(w_0)}}} \mathcal{F}(w_0)$, where $\Delta = \sigma_{a_1} \dots \sigma_{a_{\ell(w_0)}}$, so setting $\varphi_1(\mathcal{F}^0, \dots, \mathcal{F}^r) = (\mathcal{F}^0, \dots, \mathcal{F}^r, \mathcal{F}^{r+1}, \dots, \mathcal{F}(w_0))$ we obtain an element of $X(\beta\Delta)$. Conversely, if $(\mathcal{F}^0, \dots, \mathcal{F}^r, \mathcal{F}^{r+1}, \dots, \mathcal{F}(w_0)) \in X(\beta\Delta)$, it is easy to see that $(\mathcal{F}^0, \dots, \mathcal{F}^r) \in \mathrm{BS}(\beta)$, so φ_1 is an isomorphism.

Let us now construct the map $\varphi_2^{-1} : X(\Delta\beta) \to \mathrm{BS}(\beta)$. Take an element

$$\left(\mathcal{F}^0, \dots, \mathcal{F}^{\ell(w_0)}, \mathcal{F}^{\ell(w_0)+1}, \dots, \mathcal{F}^{\ell(w_0)+r} = \mathcal{F}(w_0)\right) \in X(\Delta\beta).$$

Note that $\mathcal{F}^{\ell(w_0)} \in \mathsf{B}w_0\mathsf{B}/\mathsf{B}$, so it is of the form $\mathcal{F}(U_1 w_0 U_2)$, where U_1, U_2 are upper triangular and U_1 is unique provided it has 1's on the diagonal. Now consider the sequence of flags

$$w_0 U_1^{-1} \left(\mathcal{F}^{\ell(w_0)}, \dots, \mathcal{F}^{\ell(w_0)+r}\right). \tag{4.1}$$

Note that

$$w_0 U_1^{-1} \mathcal{F}^{\ell(w_0)+r} = w_0 U_1^{-1} \mathcal{F}(w_0) \pitchfork \mathcal{F}^{\mathrm{ant}}$$

so (4.1) defines a point in $\mathrm{BS}(\beta)$, and we define it to be $\varphi_2^{-1}(\mathcal{F}^0, \dots, \mathcal{F}^{\ell(w_0)+r})$.

To construct the map φ_2, take an element $(\mathcal{F}^0, \dots, \mathcal{F}^r) \in \mathrm{BS}(\beta)$. Since $\mathcal{F}^r \pitchfork \mathcal{F}(w_0)$, we have $\mathcal{F}^r = \mathcal{F}((w_0 V_1 w_0) V_2)$, where V_1, V_2 are upper triangular and V_1 is unique provided it has 1's on the diagonal. Note that $V_1^{-1} w_0 \mathcal{F}^r = \mathcal{F}(w_0 V_2) = \mathcal{F}(w_0)$, and that $\mathcal{F}^{\mathrm{std}} \xrightarrow{w_0} V_1^{-1} w_0 \mathcal{F}^0$. So there exist unique flags $\widetilde{\mathcal{F}}^1, \dots, \widetilde{\mathcal{F}}^{\ell(w_0)-1}$ so that

$$\varphi_2(\mathcal{F}^0, \dots, \mathcal{F}^r) = \left(\mathcal{F}^{\mathrm{std}}, \widetilde{\mathcal{F}}^1, \dots, \widetilde{\mathcal{F}}^{\ell(w_0)-1}, V_1^{-1} w_0 \mathcal{F}^0, \dots, V_1^{-1} w_0 \mathcal{F}^r\right) \in X(\Delta\beta).$$

It is straightforward to see that φ_2 and φ_2^{-1} are indeed inverse maps. ∎

4.2 Definition via equations

For a positive braid $\beta = \sigma_{i_1} \dots \sigma_{i_r}$, define the matrix

$$B_\beta(z_1, \dots, z_r) = B_{i_1}(z_1) \dots B_{i_r}(z_r).$$

This allows us to present the braid variety $X(\beta)$ explicitly via equations.

Corollary 4.5 *Let $\beta = \sigma_{i_1} \dots \sigma_{i_r}$ be a braid, and assume the Demazure product is $\delta(\beta) = w_0$. Then*

$$X(\beta) = \{(z_1, \dots, z_r) : w_0 B_\beta(z_1, \dots, z_r) \text{ is upper-triangular}\}.$$

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

Indeed, given $(z_1, \dots, z_r)$, by Lemma 2.5, one gets the sequence of flags $\mathcal{F}^0 = \mathcal{F}^{\mathrm{std}}$, $\mathcal{F}^1 = \mathcal{F}(B_{i_1}(z_1))$, $\mathcal{F}^2 = \mathcal{F}(B_{i_1}(z_1)B_{i_2}(z_2))$, and so on. To obtain a similar description for the double Bott–Samelson variety $\mathrm{BS}(\beta)$, we use Lemma 2.4.

Corollary 4.6 *Let β be a positive braid. The double Bott–Samelson variety is*

$$\mathrm{BS}(\beta) = \{(z_1, \dots, z_r) \in \mathbb{C}^r \mid B_\beta(z_1, \dots, z_r) \text{ admits an LU decomposition}\}.$$

Equivalently, using principal minors we have

$$\mathrm{BS}(\beta) = \{(z_1, \dots, z_r) \in \mathbb{C}^r \mid \Delta_{[i],[i]}(B_\beta(z)) \neq 0 \text{ for every } i = 1, \dots, k\}$$

so that $\mathrm{BS}(\beta)$ is a principal open subvariety of $\mathbb{C}^r$. If $w \in S_k$, then $\mathrm{BS}(\underline{w})$ coincides with the double Bruhat cell of w, that is, the open Richardson variety $R(e, w)$.

The following lemmas give some useful examples of explicit braid matrices.

Lemma 4.7 *Consider the partial Coxeter element $\mathbf{c}(i) = \sigma_{k-1} \dots \sigma_{k-i}$. Then,*

$$B_{\mathbf{c}(i)}(z_1, \dots, z_i) = \begin{pmatrix} \mathbf{1}_{(k-i-1)\times(k-i-1)} & \mathbf{0}_{(k-i-1)\times 1} & \mathbf{0}_{(k-i-1)\times 1} & \dots & \mathbf{0}_{(k-i-1)\times 1} \\ \mathbf{0}_{1\times(k-i-1)} & z_i & -1 & \dots & 0 \\ \mathbf{0}_{1\times(k-i-1)} & z_{i-1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{1\times(k-i-1)} & z_1 & 0 & \dots & -1 \\ \mathbf{0}_{1\times(k-i-1)} & 1 & 0 & \dots & 0 \end{pmatrix},$$

where $\mathbf{1}_{(k-i-1)\times(k-i-1)}$ is the identity matrix of size $k-i-1$, $\mathbf{0}_{1\times(k-i-1)}$ is the row vector of size $k-i-1$ with only 0's, and similarly for $\mathbf{0}_{(k-i-1)\times 1}$.

Proof By induction on i, the case $i = 1$ is simply the definition of the braid matrix $B_{k-1}(z_1)$ and the induction step is a simple computation. ∎

The next result is analogous to [4, Equation (2.5)] and can be shown by induction using Lemma 4.7 above.

Lemma 4.8 *Consider the braid word $\Delta = (\sigma_{k-1} \dots \sigma_1)(\sigma_{k-1} \dots \sigma_2) \dots (\sigma_{k-1}\sigma_{k-2})\sigma_{k-1}$. Then,*

(4.2)

$$B_\Delta(z_1, \dots, z_{\binom{k}{2}}) = \begin{pmatrix} z_{k-1} & -z_{2k-3} & z_{3k-6} & \dots & (-1)^{k-1} z_{\binom{k}{2}-1} & (-1)^k z_{\binom{k}{2}} & (-1)^{k+1} \\ z_{k-2} & -z_{2k-4} & z_{3k-7} & \dots & (-1)^{k-1} z_{\binom{k}{2}-2} & (-1)^k & 0 \\ z_{k-3} & -z_{2k-5} & z_{3k-8} & \dots & (-1)^{k-1} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ z_2 & -z_k & 1 & \dots & 0 & 0 & 0 \\ z_1 & -1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

In particular, $w_0 B_\Delta(z_1, \dots, z_{\binom{k}{2}})$ is a lower-triangular matrix, and $B_\Delta(z_1, \dots, z_{\binom{k}{2}})w_0$ is an upper-triangular matrix.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

Lemma 4.9 *Let $(z_1, \dots, z_r) \in X(\beta)$ and let $\widetilde{\Delta}$ be any braid of length $\ell(w_0)$ such that $\pi(\widetilde{\Delta}) = w_0$. Assume $\delta(\beta) = w_0$. Then, there exist unique functions $y_1, \dots, y_{\ell(w_0)} \in \mathbb{C}[X(\beta)]$ such that the matrix*

$$B_\beta(z_1, \dots, z_r) B_{\widetilde{\Delta}}(y_1 \dots, y_{\ell(w_0)})$$

is diagonal.

Proof Since any two reduced expressions for w_0 are related by braid moves, it is enough to prove this for a single reduced braid lift $\widetilde{\Delta}$ of w_0. For

$$\widetilde{\Delta} = (\sigma_{k-1} \dots \sigma_1)(\sigma_{k-1} \dots \sigma_2)(\sigma_{k-1}\sigma_{k-2})(\sigma_{k-1}),$$

this follows immediately from Lemma 4.8. ∎

We can obtain a description in coordinates of the isomorphism $\varphi_2 : \mathrm{BS}(\beta) \to X(\Delta\beta)$.

Corollary 4.10 *There exist unique functions $p_1, \dots, p_{\ell(w_0)} \in \mathbb{C}[\mathrm{BS}(\beta)]$ such that the map*

$$\varphi_2 : \mathrm{BS}(\beta) \to X(\Delta\beta), \qquad \varphi_2(z_1, \dots, z_r) = \left(p_1, \dots, p_{\ell(w_0)}, z_1, \dots, z_r\right)$$

is an isomorphism.

Proof If $(z_1, \dots, z_r) \in \mathrm{BS}(\beta)$, then the braid matrix $B_\beta(z_1, \dots, z_r)$ admits an LU decomposition, say $B_\beta(z_1, \dots, z_r) = LU$, equivalently, $w_0 L^{-1} B_\beta(z_1, \dots, z_r) = w_0 U$. By Lemma 4.8, we can find unique functions $p_1, \dots, p_{\ell(w_0)}$ such that $w_0 L^{-1} = B_\Delta(p_1, \dots, p_{\ell(w_0)})$ and $\varphi_2 : \mathrm{BS}(\beta) \to X(\Delta\beta)$ is well-defined.

Note that the inverse map is given by

$$\varphi_2^{-1}(z_1, \dots, z_{\ell(w_0)}, z_{\ell(w_0)+1}, \dots, z_{\ell(w_0)+r}) = (z_{\ell(w_0)+1}, \dots, z_{\ell(w_0)+r}).$$

Indeed, if $B_\Delta(z_1, \dots, z_{\ell(w_0)}) B_\beta(z_{\ell(w_0)+1}, \dots, z_{\ell(w_0)+r}) = w_0 U_1$ for some upper triangular matrix U_1, then $B_\beta(z_{\ell(w_0)+1}, \dots, z_{\ell(w_0)+r}) = B_\Delta^{-1}(z_1, \dots, z_{\ell(w_0)}) w_0 U_1$, which has an LU-decomposition by Lemma 4.8. It is straightforward to verify that $\varphi_2, \varphi_2^{-1}$ are indeed inverses of each other. ∎

4.3 Cluster structure

Let us now describe the cluster structure on braid varieties obtained in [2] (see also [12, 13]). This cluster structure generalizes the one obtained for double Bott–Samelson varieties in [26], that we will also describe.

4.3.1 Cluster structure on braid varieties: Variables

We will define an initial cluster $\mathbf{x}_{X(\beta)}$ first by means of its non-vanishing locus. For each $m = 1, \dots, r$, let $\beta_m = \sigma_{i_1} \dots \sigma_{i_m}$, so that $\beta_r = \beta$.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

Definition 4.11 The left-to-right inductive torus (also known as Deodhar torus) $\vec{\mathbb{T}}_\beta \subseteq X(\beta)$ consists of those elements $(\mathcal{F}^0 = \mathcal{F}^{\mathrm{std}}, \mathcal{F}^1, \dots, \mathcal{F}^r = \mathcal{F}^{\mathrm{ant}})$ such that

$$\mathcal{F}^0 \xrightarrow{\delta(\beta_m)} \mathcal{F}^m$$

for all $m = 1, \dots, r$.

We can give a system of coordinates on $\vec{\mathbb{T}}_\beta$ as follows.

Lemma 4.12 [12, Proposition 2.12] *The torus $\vec{\mathbb{T}}_\beta$ is given by the non-vanishing of the minors*

$$\Delta_{\delta(\beta_m)[i_m],[i_m]}\left(B_{\beta_m}(z_1, \dots, z_m)\right). \tag{4.3}$$

Proof If $(z_1, \dots, z_r) \in \vec{\mathbb{T}}_\beta$, then all minors (4.3) are nonzero by Lemma 2.7(b). Conversely, assume all minors (4.3) are nonzero. We show by induction on m that $\mathcal{F}^{\mathrm{std}} \xrightarrow{\delta(\beta_m)} \mathcal{F}(B_{\beta_m}(z_1, \dots, z_m))$. The base of induction is $m = 0$, where β_m is the identity braid and $B_{\beta_m}(z_1, \dots, z_m)$ is the identity matrix. For brevity, we will denote $\mathcal{F}^m := \mathcal{F}(B_{\beta_m}(z_1, \dots, z_m))$.

Assume $\mathcal{F}^0 \xrightarrow{\delta(\beta_m)} \mathcal{F}^m$. Note that we have

$$\mathcal{F}^0 \xrightarrow{\delta(\beta_m)} \mathcal{F}^m \xrightarrow{s_{i_{m+1}}} \mathcal{F}^{m+1}.$$

If $\delta(\beta_{m+1}) = \delta(\beta_m)s_{i_{m+1}}$, then using Corollary 2.3(a), we obtain $\mathcal{F}^0 \xrightarrow{\delta(\beta_{m+1})} \mathcal{F}^{m+1}$, as needed. If, on the other hand, $\delta(\beta_{m+1}) = \delta(\beta_m)$, then using Corollary 2.3(b), we have that $\mathcal{F}^0 \xrightarrow{\nu} \mathcal{F}^{m+1}$ with $\nu \le \delta(\beta_m) \star s_{i_{m+1}} = \delta(\beta_m)$. Thus, by Lemma 2.7(c), to conclude that $\nu = \delta(\beta_m) = \delta(\beta_{m+1})$, it is enough to show that $\Delta_{\delta(\beta_m)[j],[j]}\left(B_{\beta_m}(z_1, \dots, z_m)B_{s_{i_{m+1}}}(z_{m+1})\right)$ is nonzero for all $j = 1, \dots, k$. But by Lemma 2.10, if $j \neq i_{m+1}$, then

$$\Delta_{\delta(\beta_m)[j],[j]}\left(B_{\beta_m}(z_1, \dots, z_m)B_{s_{i_{m+1}}}(z_{m+1})\right) = \Delta_{\delta(\beta_m)[j],[j]}\left(B_{\beta_m}(z_1, \dots, z_m)\right) \neq 0,$$

where the last inequality follows from the induction assumption. If $j = i_{m+1}$, it is our assumption that $\Delta_{\delta(\beta_m)[i_{m+1}],[i_{m+1}]}\left(B_{\beta_m}(z_1, \dots, z_m)B_{s_{i_{m+1}}}(z_{m+1})\right) \neq 0$, and the result follows. ∎

One might then expect that the minors (4.3) are the cluster variables in an initial seed for $X(\beta)$. However, this does not work because in general (4.3) is not an irreducible polynomial in $z_1, \dots, z_m$. However, the irreducible factors of (4.3) are in fact the cluster variables defining the torus $\vec{\mathbb{T}}_\beta$ and the minors (4.3) are cluster monomials. Moreover, one can obtain (4.3) from the cluster variables in an upper unitriangular fashion, and we only need to consider the minors (4.3) for those m so that $\delta(\beta_m) = \delta(\beta_{m-1})$. To summarize, for each $m = 1, \dots, r$ such that $\delta(\beta_m) = \delta(\beta_{m-1})$, the minor (4.3) has a unique irreducible factor that has not appeared in such a minor for a smaller index, this irreducible factor appears with multiplicity one, and it is a cluster variable. See [2, 12, 13] and also [19] for details.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

The quiver Q forming a seed with the cluster described above can be obtained using the *Lusztig cycles* in the left-to-right inductive weave of β. Since we will not need this level of detail we will not go into it, and instead refer the reader to [2, Section 4].

4.3.2 Cluster structure on double Bott–Samelson varieties: Variables

We use the isomorphism $\varphi_2 : \mathrm{BS}(\beta) \to X(\Delta\beta)$ from Lemma 4.4 in order to translate the cluster structure on $X(\Delta\beta)$ described above to $\mathrm{BS}(\beta)$. We remark that the resulting cluster structure is the one obtained by Shen and Weng in [26], which predates the construction of cluster structures on general braid varieties (see [2, Section 4.8 and Proposition 4.20]).

First, we make some simplifications. By Corollary 4.10, we have that φ_2 sends $(z_1, \dots, z_r)$ to $(p_1, \dots, p_{\ell(w_0)}, z_1, \dots, z_r)$, where $\mathbf{p} = \{p_1, \dots, p_{\ell(w_0)}\}$ are uniquely defined functions of $\mathbf{z} = \{z_1, \dots, z_r\}$.

Second, $\delta((\Delta\beta)_m) = \delta((\Delta\beta)_{m-1})$ if and only if $m > \ell(w_0)$, and in this case, the Demazure product is precisely w_0. We denote $j = m - \ell(w_0)$, so that the m-th letter in $\Delta\beta$ is i_j.

Therefore, the cluster variables are the irreducible factors of

$$\Delta_{w_0[i_j],[i_j]}\left(B_{(\Delta\beta)_m}\left(p_1, \dots, p_{\ell(w_0)}, z_1, \dots, z_j\right)\right), \qquad j = 1, \dots, \ell(\beta).$$

Note, however, that

$$\begin{aligned}\Delta_{w_0[i_j],[i_j]}\left(B_{(\Delta\beta)_m}(\mathbf{p}, z_1, \dots, z_j)\right) &= \Delta_{w_0[i_j],[i_j]}\left(B_\Delta(\mathbf{p})B_{\beta_j}(z_1, \dots, z_j)\right)\\ &= \Delta_{w_0[i_j],[i_j]}\left(w_0L(\mathbf{p})B_{\beta_j}(z_1, \dots, z_j)\right)\\ &= \Delta_{[i_j],[i_j]}\left(L(\mathbf{p})B_{\beta_j}(z_1, \dots, z_j)\right)\\ &= \Delta_{[i_j],[i_j]}\left(B_{\beta_j}(z_1, \dots, z_j)\right),\end{aligned}$$

where we have used that $L(\mathbf{p})$ is lower triangular with 1's on the diagonal, and Lemma 4.8. We conclude that the cluster variables in $\mathrm{BS}(\beta)$ are the irreducible factors of

$$x_j := \Delta_{[i_j],[i_j]}(B_{\beta_j}(z_1, \dots, z_j)). \tag{4.4}$$

However, by [26, Lemma 3.30], the function x_j is already irreducible. We summarize the previous discussion in the following result.

Lemma 4.13 [26] *The cluster variables in* $\mathrm{BS}(\beta)$ *are as in (4.4), with* $j = 1, \dots, r$.

Since $\Delta_{[i],[i]}(U) \neq 0$ for any matrix $U \in \mathsf{B}(k)$ and any $i = 1, \dots, k$, we immediately obtain the following.

Corollary 4.14 *The common nonvanishing locus of* x_j, $j = 1, \dots, r$, *is*

$$\vec{\mathbb{T}}_{\Delta\beta} = \left\{(\mathcal{F}^0, \mathcal{F}^1, \dots, \mathcal{F}^r) \in \mathrm{BS}(\beta) \mid \mathcal{F}^i \pitchfork \mathcal{F}(w_0) \text{ for all } i\right\}.$$

For $s = 1, \dots, k-1$, let $\mathrm{last}(s)$ be such that $i_{\mathrm{last}(s)} = s$ and $i_j \neq s$ for $j > \mathrm{last}(s)$. In words, $\mathrm{last}(s)$ is the position of the rightmost appearance of σ_s in β. If s does not

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

appear in β, we define $\mathrm{last}(s)$ to be a formal symbol $\oslash$. If $\mathrm{last}(s) \neq \oslash$, then by Lemma 2.10, we have

$$x_{\mathrm{last}(s)} = \Delta_{[s],[s]}(B_{\beta_{\mathrm{last}(s)}}(z_1, \ldots, z_{\mathrm{last}(s)})) = \Delta_{[s],[s]}(B_\beta(z_1, \ldots, z_r)),$$

so that, by the definition of $\mathrm{BS}(\beta)$, $x_{\mathrm{last}(s)}$ is invertible in $\mathbb{C}[\mathrm{BS}(\beta)]$. In fact, the frozen variables in $\mathrm{BS}(\beta)$ are $\{x_{\mathrm{last}(s)} \mid s = 1, \ldots, k-1, \mathrm{last}(s) \neq \oslash\}$. For convenience and future use, we set $x_\oslash = 1$.

Let us describe the frozen variables in terms of the LU-decomposition of $B_\beta(z_1, \ldots, z_r)$. Let $B_\beta(z_1, \ldots, z_r) = LU$, where L has 1's in the diagonal and the diagonal of U is $\mathrm{diag}(u_1, \ldots, u_k)$, where each $u_i \in \mathbb{C}[\mathrm{BS}(\beta)]$ is a function on $\mathrm{BS}(\beta)$. Note that

$$\det(B_\beta(z_1, \ldots, z_r)) = 1 = u_1 \ldots u_k$$

so that each u_i is a unit in $\mathbb{C}[\mathrm{BS}(\beta)]$ and, moreover,

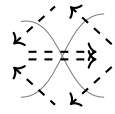
$$x_{\mathrm{last}(s)} = \Delta_{[s],[s]}(B_\beta(z_1, \ldots, z_r)) = u_1 \ldots u_s. \tag{4.5}$$

Expressing u's in terms of the frozen variables, we have $u_1 = x_{\mathrm{last}(1)}$, $u_s = x_{\mathrm{last}(s)} x_{\mathrm{last}(s-1)}^{-1}$ for $s = 2, \ldots, k-1$ and $u_k = u_1^{-1} \ldots u_{k-1}^{-1} = x_{\mathrm{last}(k-1)}^{-1}$.

4.3.3 Cluster structure on double Bott–Samelson varieties: Quiver

Next, we describe the quiver for double Bott–Samelson variety $\mathrm{BS}(\beta)$.

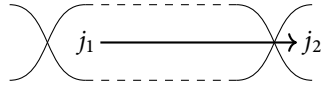
The quiver Q_β can be read directly from the braid diagram of β drawn horizontally. The vertices of Q_β are in bijection with the connected components of $\mathbb{R}^2$ minus the braid diagram of β which are bounded on the left: each connected component corresponds to the crossing directly to its left. Around each crossing, we have the following configuration of half arrows



which produces a quiver that possibly has oriented 2-cycles, and the quiver Q_β is obtained after removing a maximal collection of disjoint oriented 2-cycles.

To see that half arrows add up to an integer number of arrows, it is useful to rephrase the construction of Q_β. The vertices of the quiver are in bijection with the letters of β. Let us say that the vertex j has color i and write $\mathrm{clr}(j) = i$ if $i_j = i$. The arrows in Q_β are of two types: *mixed* (between vertices of different colors) and *unmixed* (between vertices of the same color).

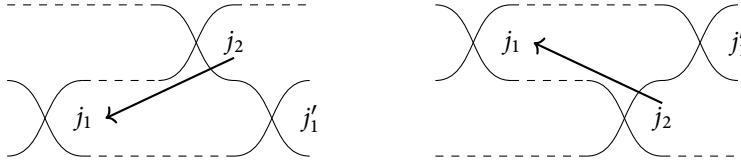
If $j_1 < j_2$ are vertices of the same color, then there is an unmixed arrow $j_1 \to j_2$ if and only if there does not exist $j_1 < k < j_2$ of the same color as j_1, j_2. Put it more succinctly: there is an unmixed arrow pointing right between two consecutive appearances of the same color. These are all the unmixed arrows.



https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

Let us now describe the mixed arrows. Assume $j_1 < j_2$ have different colors. If there is a mixed arrow $j_2 \to j_1$, then $\mathrm{clr}(j_2) = \mathrm{clr}(j_1) \pm 1$. Now, let us say that $j_1 < j_2$ and $\mathrm{clr}(j_2) = \mathrm{clr}(j_1) \pm 1$. Then, there is an arrow $j_2 \to j_1$ if and only if there exists $j_1 < j_2 < j_1'$ such that:

- $\mathrm{clr}(j_1) = \mathrm{clr}(j_1')$ and, moreover, j_1 and j_1' are consecutive appearances of this color;
- there does not exist $j_2 < k < j_1'$ of the same color as j_2.



Finally, the frozen vertices are $\{\mathrm{last}(k) | k = 1, \dots, n-1\}$.

5 Splicing braid varieties

5.1 The splicing map

We now describe splicing maps for general braid varieties. Let us fix the braid $\beta = \sigma_{i_1} \dots \sigma_{i_r}$. Furthermore, let $\beta^1 = \sigma_{i_1} \dots \sigma_{i_{r_1}}$ and $\beta^2 = \sigma_{i_{r_1+1}} \dots \sigma_{i_r}$ so that $\beta = \beta^1\beta^2$. For fixed $w \in S_k$, we consider the subset

$$\mathcal{U}_{r_1,w}(\beta) := \left\{ \left(\mathcal{F}^0, \dots, \mathcal{F}^r\right) \in X(\beta) \mid \mathcal{F}^{r_1} \pitchfork \mathcal{F}(w_0 w) \right\}. \tag{5.1}$$

By Lemma 2.9, $\mathcal{U}_{r_1,w} \subseteq X(\beta)$ is open and

$$X(\beta) = \bigcup_{w \in S_k} \mathcal{U}_{r_1,w}.$$

Note, however, that for a fixed $w \in S_k$, the open set $\mathcal{U}_{r_1,w}$ may be empty. Below, we will obtain necessary and sufficient conditions for $\mathcal{U}_{r_1,w}(\beta)$ to be nonempty (see Corollary 5.5). Assume for the time being that $\mathcal{U}_{r_1,w}(\beta) \neq \varnothing$ and let $(\mathcal{F}^0, \dots, \mathcal{F}^r) \in \mathcal{U}_{r_1,w}(\beta)$. We fix a reduced expression for w_0 that has a reduced word for w as a prefix:

$$w_0 = s_{a_1} \dots s_{a_{\ell(w_0)}}, \quad w = s_{a_1} \dots s_{a_{\ell(w)}}. \tag{5.2}$$

By Lemma 2.2, there exists a unique collection of flags (all of them coordinate flags) $\widetilde{\mathcal{F}}^1, \dots, \widetilde{\mathcal{F}}^{\ell(w_0)} = \mathcal{F}^{\mathrm{std}}$ so that $\widetilde{\mathcal{F}}^{\ell(w)} = \mathcal{F}(w_0 w)$ and we have the following configuration of flags:

$$\begin{array}{ccccccccccc}
\mathcal{F}^{\mathrm{std}} & \xrightarrow{s_{i_1}} & \mathcal{F}^1 & \xrightarrow{s_{i_2}} & \cdots & \xrightarrow{s_{i_{r_1}}} & \mathcal{F}^{r_1} & \xrightarrow{s_{i_{r_1+1}}} & \cdots & \xrightarrow{s_{i_r}} & \mathcal{F}^{\mathrm{ant}} \\
\| & & & & & & & & & & \| \\
\mathcal{F}^{\mathrm{std}} & \xleftarrow{s_{a_{\ell(w_0)}}} & \widetilde{\mathcal{F}}^{\ell(w_0)-1} & \longleftarrow & \cdots & \longleftarrow & \mathcal{F}(w_0 w) & \xleftarrow{s_{a_{\ell(w)}}} & \cdots & \xleftarrow{s_{a_1}} & \mathcal{F}^{\mathrm{ant}}
\end{array}$$

In particular, $\widetilde{\mathcal{F}}^{\ell(w)} = \mathcal{F}(w_0 w)$. Since $\mathcal{F}^{r_1} \pitchfork \mathcal{F}(w_0 w)$, there exist

(1) $g_1 \in \mathrm{GL}(k)$ such that $g_1 \mathcal{F}^{r_1} = \mathcal{F}^{\mathrm{ant}}$, $g_1 \mathcal{F}(w_0 w) = \mathcal{F}^{\mathrm{std}}$.
(2) $g_2 \in \mathrm{GL}(k)$ such that $g_2 \mathcal{F}^{r_1} = \mathcal{F}^{\mathrm{std}}$, $g_2 \mathcal{F}(w_0 w) = \mathcal{F}^{\mathrm{ant}}$.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

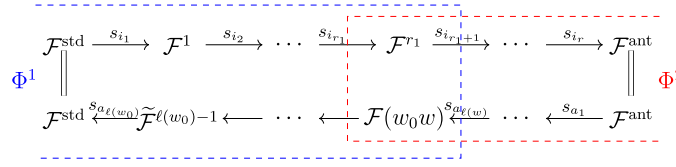
Remark 5.1 Given that $\mathsf{B} \cap w_0 \mathsf{B} w_0$ is the subgroup of diagonal matrices, the elements g_1, g_2 are unique up to multiplication on the left by a diagonal matrix. We will clarify this ambiguity in the course of the proof of Theorem 5.2 below.

Now, we define

$$\Phi^1\left(\mathcal{F}^0, \dots, \mathcal{F}^r\right) = g_1\left(\mathcal{F}(w_0 w), \dots, \mathcal{F}_{\mathrm{std}}, \mathcal{F}^1, \dots, \mathcal{F}^{r_1}\right) \in X\left(\underline{w^{-1} w_0} \beta^1\right), \tag{5.3}$$

$$\Phi^2\left(\mathcal{F}^0, \dots, \mathcal{F}^r\right) = g_2\left(\mathcal{F}^{r_1}, \mathcal{F}^{r_1+1}, \dots, \mathcal{F}^r, \widetilde{\mathcal{F}}^1, \dots, \mathcal{F}(w_0 w)\right) \in X\left(\beta^2 \underline{w}\right). \tag{5.4}$$

Schematically, and up to the translation by g_1 and g_2, the maps Φ_1 and Φ_2 are defined as follows:



(5.5)

Theorem 5.2 *The map* $\Phi_{r_1,w} = (\Phi^1, \Phi^2) : \mathcal{U}_{r_1,w}(\beta) \to X\left(\underline{w^{-1} w_0} \beta^1\right) \times X\left(\beta^2 \underline{w}\right)$ *is an isomorphism of algebraic varieties.*

Proof We break down the proof in several steps. In what follows, we fix the reduced expression (5.2) for w_0. Let $y_1, \dots, y_{\ell(w_0)} \in \mathbb{C}[X(\beta)]$ be the functions constructed in Lemma 4.9 for the lift $\underline{w_0}$. We denote $\mathbf{y} = \{y_1, \dots, y_{\ell(w_0)}\}$ and

$$\mathbf{y}^L = \{y_{\ell(w)+1}, \dots, y_{\ell(w_0)}\},\ \mathbf{y}^R = \{y_1, \dots, y_{\ell(w)}\}.$$

Also, if $\mathbf{z} = (z_1, \dots, z_r) \in X(\beta)$, we will set

$$\mathbf{z}^L = \{z_1, \dots, z_{r_1}\},\ \mathbf{z}^R = \{z_{r_1+1}, \dots, z_r\},$$

and $M = M(\mathbf{z}^L) = B_{\beta^1}(z_1, \dots, z_{r_1})$. Note that, by definition, $\mathbf{z} \in \mathcal{U}_{r_1,w}(\beta)$ if and only if $\mathcal{F}(M(\mathbf{z}^L)) \pitchfork \mathcal{F}(w_0 w)$. Also, $M B_{\beta^2}(\mathbf{z}^R) = B_\beta(\mathbf{z})$.

In the course of defining the maps Φ^1, Φ^2, we glossed over the fact that the elements g_1, g_2 are only defined up to left multiplication by a diagonal matrix. Thus, in the first two steps, we carefully choose g_1 and g_2.

Step 1: Explicit construction of the map Φ^2. Assume $\mathbf{z} \in \mathcal{U}_{r_1,w}(\beta)$. By Lemma 2.4, M admits a decomposition

$$M = M(\mathbf{z}^L) = (w_0 w w_0) L(\mathbf{z}) U(\mathbf{z}) = (w_0 w w_0) L U, \tag{5.6}$$

where $L = L(\mathbf{z})$ is lower-triangular with 1's on the diagonal and $U = U(\mathbf{z})$ is upper-triangular. Note that the entries of $L(\mathbf{z})$ and $U(\mathbf{z})$ are rational functions on $\mathbb{C}[X(\beta)]$, which are regular on $\mathcal{U}_{r_1,w}(\beta)$. Now, we consider the sequence of matrices

$$M\left(I, B_{i_{r_1+1}}(z_{r_1+1}), \dots, B_{\beta^2}\left(\mathbf{z}^R\right), B_{\beta^2}\left(\mathbf{z}^R\right) B_{a_1}(y_1), \dots, B_{\beta^2}\left(\mathbf{z}^R\right) B_{\underline{w}}\left(\mathbf{y}^R\right)\right), \tag{5.7}$$

which projects to the flags on the red part of the diagram (5.5). Multiplying these matrices on the left by $((w_0 w w_0) L(\mathbf{z}))^{-1}$, we obtain the sequence of matrices

$$U\left(I, B_{i_{r_1+1}}(z_{r_1+1}), \dots, B_{\beta^2}\left(\mathbf{z}^R\right), B_{\beta^2}\left(\mathbf{z}^R\right) B_{a_1}(y_1), \dots, B_{\beta^2}\left(\mathbf{z}^R\right) B_{\underline{w}}\left(\mathbf{y}^R\right)\right), \tag{5.8}$$

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

where $U = U(\mathbf{z})$ as in (5.6). Since $\mathcal{F}(B_\beta(\mathbf{z})\, B_{\underline{w}}(\mathbf{y}^R)) = \mathcal{F}(w_0 w)$, we have that there exists an upper triangular matrix U_1 such that

$$B_\beta(\mathbf{z})\, B_{\underline{w}}(\mathbf{y}^R) = M B_{\beta^2}\left(\mathbf{z}^R\right) B_{\underline{w}}\left(\mathbf{y}^R\right) = w_0 w U_1,$$

so that $U B_{\beta^2}\left(\mathbf{z}^R\right) B_{\underline{w}}\left(\mathbf{y}^R\right) = L^{-1} w_0 U_1 \in w_0 \mathsf{B}$, that is, $\mathcal{F}(U B_{\beta^2}\left(\mathbf{z}^R\right) B_{\underline{w}}\left(\mathbf{y}^R\right)) = \mathcal{F}(w_0)$ and the projection of (5.8) to the flag variety defines an element of $X(\beta^2 \underline{w})$, that we define to be Φ^2. Note that the element g_2 from (5.4) (see also Remark 5.1) is precisely

$$g_2 = ((w_0 w w_0) L(\mathbf{z}))^{-1}. \tag{5.9}$$

In coordinates, we "slide the matrix U to the right" using Lemma 2.6 to obtain regular functions $\widetilde{\mathbf{z}}^R = \{\widetilde{z}_{r_1}, \widetilde{z}_{r_1+1}, \ldots, \widetilde{z}_r\}, \widetilde{\mathbf{y}}^R = \{\widetilde{y}_1, \ldots, \widetilde{y}_{\ell(w)}\}$ on $U_{r_1,w}$ such that

$$U B_{\beta^2}\left(\mathbf{z}^R\right) B_{\underline{w}}\left(\mathbf{y}^R\right) = B_{\beta^2 \underline{w}}\left(\widetilde{\mathbf{z}}^R, \widetilde{\mathbf{y}}^R\right) U'$$

and we define $\Phi^2(\mathbf{z}) = (\widetilde{z}_{r_1+1}, \ldots, \widetilde{z}_r, \widetilde{y}_1, \ldots, \widetilde{y}_{r_1}) \in X\left(\beta^2 \underline{w}\right)$.

Step 2: Explicit construction of the map Φ^1. The map Φ^1 is constructed analogously to Φ^2, as follows. We now consider the sequence of matrices

$$\left(B_\beta(\mathbf{z}) B_{\underline{w}}(\mathbf{y}^R), \ldots, B_\beta(\mathbf{z}) B_{\underline{w_0}}(\mathbf{y}) = U_2 | B_{i_1}(z_1), \ldots, M\right) \tag{5.10}$$

that project down to the flags in the blue part of the diagram (5.5). The vertical line indicates the fact that, while the flags $\mathcal{F}^{\mathrm{std}}$ on the left extreme of (5.5) are equal, the matrix $B_\beta(\mathbf{z}) B_{\underline{w_0}}(\mathbf{y})$ does not need to be the identity matrix; we only know that $B_\beta(\mathbf{z}) B_{\underline{w_0}}(\mathbf{y})$ is an upper triangular matrix U_2. In order to circumvent this problem, we write the matrices $\left(B_{i_1}(z_1), \ldots, B_{\beta^1}(\mathbf{z}^L) = M\right)$ as $U_2 U_2^{-1}\left(B_{i_1}(z_1), \ldots, B_{\beta^1}(\mathbf{z}^L)\right)$ and slide the upper triangular matrix U_2^{-1} to the right, so we obtain regular functions $\mathbf{z}'^L = \{z'_1, \ldots, z'_{r_1}\}$ on $\mathcal{U}_{r_1,w}(\beta)$ so that the sequence of matrices

$$\left(B_\beta(\mathbf{z}) B_{\underline{w}}(\mathbf{y}^R), \ldots, B_\beta(\mathbf{z}) B_{\underline{w_0}}(\mathbf{y}) = U_2, U_2 B_{i_1}(z'_1), \ldots, U_2 B_{\beta^1}(\mathbf{z}'^L)\right) \tag{5.11}$$

project to the same flags as the sequence (5.10). Note that $U_2 B_{\beta^1}(\mathbf{z}'^L) = M(\mathbf{z}) U_3(\mathbf{z})$, where $U_3(\mathbf{z})$ is an upper triangular matrix, so we have a decomposition

$$U_2 B_{\beta^1}(z'_1, \ldots, z'_{r_1}) = (w_0 w w_0) L(\mathbf{z}) U(\mathbf{z}) U_3(\mathbf{z}), \tag{5.12}$$

where the matrix $L(\mathbf{z})$ is the same as in the decomposition (5.6). So we can multiply all matrices in the sequence (5.11) by $w_0((w_0 w w_0) L(\mathbf{z}))^{-1}$ and now we proceed as in Step 1. Note that the element g_1 from (5.3) (see also Remark 5.1) is given by

$$g_1 = w_0((w_0 w w_0) L(\mathbf{z}))^{-1}. \tag{5.13}$$

Steps 1 and 2 show that $\Phi_{r_1,w}$ is indeed a regular map. Note that, if g_1 and g_2 denote the translating elements as in (5.3) and (5.4), we obtain that $g_2 = w_0 g_1$.

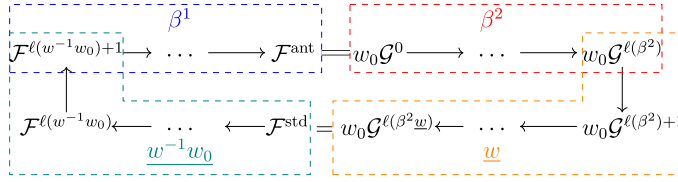
https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

Step 3. Construction of $\Phi_{r_1,w}^{-1}$. Let us construct the map $\Phi_{r_1,w}^{-1}$. For this, we take

$$\mathcal{F} = \left(\mathcal{F}^0 = \mathcal{F}^{\mathrm{std}}, \mathcal{F}^1, \dots, \mathcal{F}^{\ell(w^{-1}w_0)+\ell(\beta^1)} = \mathcal{F}^{\mathrm{ant}}\right) \in X\left(\underline{w^{-1}w_0}\beta^1\right),$$
$$\mathcal{G} = \left(\mathcal{G}^0 = \mathcal{F}^{\mathrm{std}}, \mathcal{G}^1, \dots, \mathcal{G}^{\ell(\beta^2)+\ell(w)} = \mathcal{F}^{\mathrm{ant}}\right) \in X\left(\beta^2\underline{w}\right),$$

and arrange these flags as follows:

(5.14)



Since $\underline{w} \cdot \underline{w^{-1}w_0}$ is a reduced lift of the longest element w_0, we see (moving on the orange and then teal parts of (5.14)) that the flags $w_0\mathcal{G}^{\ell(\beta^2)}$ and $\mathcal{F}^{\ell(w^{-1}w_0)+1}$ are transverse. So we can find $g \in \mathrm{GL}(k)$ such that $g\mathcal{F}^{\ell(w^{-1}w_0)+1} = \mathcal{F}^{\mathrm{std}}$ and $gw_0\mathcal{G}^{\ell(\beta^2)} = \mathcal{F}^{\mathrm{ant}}$. Translating all the flags in (5.14) by g, we see that:

(1) The flags on the top row constitute an element of $X(\beta^1\beta^2) = X(\beta)$.

(2) Since $g\mathcal{F}^{\ell(w^{-1}w_0)+1} = \mathcal{F}^{\mathrm{std}}$, $gw_0\mathcal{G}^{\ell(\beta^2)} = \mathcal{F}^{\mathrm{ant}}$, and $\underline{w} \cdot \underline{w^{-1}w_0}$ is a reduced lift of w_0, the translations by g of all the flags in the bottom row are coordinate flags, so the flag $g\mathcal{F}^{\mathrm{std}}$ is $\mathcal{F}(w_0w)$ and $g\mathcal{F}^{\mathrm{ant}} \pitchfork g\mathcal{F}^{\mathrm{std}} = \mathcal{F}(w_0w)$, that is, the element of $X(\beta)$ obtained in (1) belongs to $\mathcal{U}_{r_1,w}(\beta)$.

It only remains to specify the element $g \in \mathrm{GL}(k)$ uniquely, as it is only defined up to multiplication on the left by a diagonal matrix. For this, we specify concrete matrices that project to the flags in (5.14). For the flags in $\mathcal{F}$, these are the braid matrices $B_{i_1}(z_1), \dots, B_{\beta^1}(z_1, \dots, z_{r_1}) = w_0V$, while the matrices for the flags in $\mathcal{G}$ are the braid matrices translated by the upper triangular matrix V. Once we have specified M_1 and M_2 such that $\mathcal{F}^{\ell(w^{-1}w_0)+1} = \mathcal{F}(M_1)$ and $w_0\mathcal{G}^{\ell(\beta^2)} = \mathcal{F}(M_2)$, the fact that $\mathcal{F}(M_1) \pitchfork \mathcal{F}(M_2)$ means that $M_2^{-1}M_1$ belongs to $\mathrm{B}w_0\mathrm{B}$ and thus there exists a unique decomposition $M_2^{-1}M_1 = U'w_0V'$ with U' unipotent. We then take $g = w_0(U')^{-1}M_2^{-1}$. We leave details to the reader. ∎

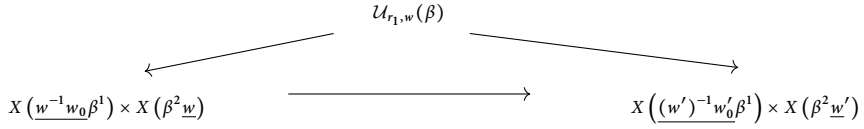
Remark 5.3 The proof of Theorem 5.2 can be simplified by taking a slightly different realization of the braid variety $X(\beta)$ using weighted flags as in [12, 13] (see also [1]). Let $\mathrm{U} \subseteq \mathrm{GL}(k)$ be the subgroup of upper uni-triangular matrices. A weighted flag is an element of $\mathrm{GL}(k)/\mathrm{U}$. Two weighted flags $h\mathrm{U}$ and $h'\mathrm{U}$ are said to be in strong relative position $w \in S_k$ if there exists $g \in \mathrm{GL}(k)$ such that $g(h\mathrm{U}) = \mathrm{U}$ and $g(h'\mathrm{U}) = w\mathrm{U}$. We denote this by $h\mathrm{U} \xrightarrow{w} h'\mathrm{U}$. Similarly, $h\mathrm{U}$ and $h'\mathrm{U}$ are said to be in weak relative position $w \in S_k$ if there exist $g \in \mathrm{GL}(k)$ and a diagonal matrix t such that $g(h\mathrm{U}) = \mathrm{U}$ and $g(h'\mathrm{U}) = tw\mathrm{U}$. We denote this relation by $h\mathrm{U} \overset{w}{\dashrightarrow} h'\mathrm{U}$. Given a braid $\beta = \sigma_{i_1} \dots \sigma_{i_r}$, we then have an isomorphism

$$X(\beta) \cong \left\{h_0\mathrm{U} \xrightarrow{s_{i_1}} h_1\mathrm{U} \xrightarrow{s_{i_2}} \dots \xrightarrow{s_{i_r}} h_r\mathrm{U} \overset{w_0}{\dashrightarrow} h_0\mathrm{U}\right\}/\mathrm{GL}(k),$$

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

where $\mathrm{GL}(k)$ acts diagonally on all weighted flags $h_1\mathsf{U}, \ldots, h_r\mathsf{U}$. The set $\mathcal{U}_{r_1,w}(\beta)$ is then defined to consist of those chains of weighted flags such that $h_{r_1}\mathsf{U} \overset{w_0}{\dashrightarrow} w_0 w\mathsf{U}$. An advantage of working with weighted flags is that, since $\mathsf{U} \cap (w_0 \mathsf{B} w_0) = \{\mathbf{1}\}$, now we have uniqueness of the translating element in all the arguments used above.

Remark 5.4 A priori, the construction of the map $\Phi_{r_1,w}$ depends on the choice of the reduced expression (5.2). However, one can check that different choices of the reduced expression lead to, essentially, the same map. More precisely, if we have two reduced expressions $\underline{w}, \underline{w}'$ of w, choose reduced expressions $\underline{w_0}, \underline{w_0}'$ that contain $\underline{w}, \underline{w}'$ as a prefix, respectively. We have canonical isomorphisms $X(\beta^2 \underline{w}) \to X(\beta^2 \underline{w}')$ and $X(\underline{w^{-1}w_0}\beta^1) \to X(\underline{(w')^{-1}w_0}'\beta^1)$ such that the following diagram commutes:



In particular, the flags in the bottom part of the diagram (5.14) are determined (for a given reduced expression (5.2)) by $\mathcal{F}^{\ell(w^{-1}w_0)+1}$ and $w_0 \mathcal{G}^{\ell(\beta^2)}$. Furthermore, the element g is completely determined by these two flags.

From Theorem 5.2 and its proof, we obtain the following result.

Corollary 5.5 *The open set $\mathcal{U}_{r_1,w}(\beta) \subseteq X(\beta)$ is nonempty if and only if*

$$\delta\left(\underline{w^{-1}w_0}\beta^1\right) = w_0 = \delta\left(\beta^2 \underline{w}\right).$$

By Lemma 2.1, this happens if and only if $\delta(\beta^1) \geq w_0 w w_0$ and $\delta(\beta^2) \geq w_0 w^{-1}$.

5.2 Conjectural properties

We conjecture that the set $\mathcal{U}_{r_1,w}(\beta)$ and the map $\Phi_{r_1,w} = \Phi_1 \times \Phi_2$ satisfy various desirable cluster-theoretic properties. To state this conjecture precisely, we need some notation. If Q is an ice quiver, we denote by Q^{uf} the quiver obtained by deleting the frozen vertices and all arrows adjacent to them. We also denote by Q_1 the ice quiver from the cluster structure on $\mathbb{C}[X(\underline{w^{-1}w_0}\beta^1)]$, and by Q_2 the ice quiver associated with $\mathbb{C}[X(\beta^2 \underline{w})]$.

Conjecture 5.6 *Assume $\mathcal{U}_{r_1,w}(\beta)$ is nonempty, and consider the cluster structure on $\mathbb{C}[X(\beta)]$ from Section 4.3. There exists a seed $\Sigma = (Q, \mathbf{x})$ satisfying the following properties:*

(1) *There exist cluster variables $x_{a_1}, \ldots, x_{a_s} \in \mathbf{x}$ such that $\mathcal{U}_{r_1,w}(\beta)$ is the common nonvanishing locus of $x_{a_1}, \ldots, x_{a_s}$.*
(2) *The variety $\mathcal{U}_{r_1,w}(\beta)$ admits a cluster structure, and an initial seed $\widehat{\Sigma} = (\widehat{Q}, \widehat{\mathbf{x}})$ is given by freezing the variables $x_{a_1}, \ldots, x_{a_s}$ in the seed $\Sigma = (Q, \mathbf{x})$.*
(3) *The quiver $\widehat{Q}^{\mathrm{uf}}$ is mutation equivalent to the disjoint union of the quivers Q_1^{uf} and Q_2^{uf}.*

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

(4) *Note that the variety* $X\left(\underline{w^{-1}w_0}\beta^1\right) \times X\left(\beta^2\underline{w}\right)$ *admits a natural product cluster structure. Then, the map* $\Phi_{r_1,w} : \mathcal{U}_{r_1,w}(\beta) \to X\left(\underline{w^{-1}w_0}\beta^1\right) \times X\left(\beta^2\underline{w}\right)$ *is a quasi-cluster isomorphism.*

The items (1)–(4) in Conjecture 5.6 are not independent.

Lemma 5.7 *Assume (1) of Conjecture 5.6 holds. Then, we have that (4)* $\Rightarrow$ *(3)* $\Rightarrow$ *(2). Moreover, (3) implies that the cluster structures on* $X\left(\underline{w^{-1}w_0}\beta^1\right) \times X\left(\beta^2\underline{w}\right)$ *and* $\mathcal{U}_{r_1,w}(\beta)$ *are abstractly quasi-cluster isomorphic (but it does not guarantee that the map* $\Phi_{r_1,w}$ *is a quasi-cluster isomorphism).*

Proof Assume (1). Quasi-cluster isomorphisms do not affect the mutable part of the cluster structure, see Remark 3.4, so (4) implies (3).

Assume (3) holds. By [2, Theorem 7.13], the cluster structures on $\mathbb{C}[X(\underline{w^{-1}w_0}\beta^1]$ and $\mathbb{C}[X(\beta^2\underline{w})]$ are locally acyclic, so the cluster structure given by the seed $\widehat{\Sigma}$ is also locally acyclic (see [23, Proposition 3.10]). From here, (2) follows from Lemma 3.4 and Theorem 4.1 in [23].

Let us show that, in the presence of (1), (3) already implies that the cluster structures on $X(\underline{w^{-1}w_0}\beta^1) \times X(\beta^w\underline{w})$ and $\mathcal{U}_{r_1,w}(\beta)$ are quasi-cluster isomorphic. For this, we use exchange matrices and Lemma 3.5 above. We have the following extended exchange matrices:

- $\widetilde{B} = \begin{pmatrix} B \\ \hline C \end{pmatrix}$, the extended exchange matrix for the cluster structure on $X(\beta)$. By (1) and (2), the extended exchange matrix $\widetilde{B}^\circ$ for the cluster structure on $\mathcal{U}_{r_1,\beta}$ is given by deleting some columns on $\widetilde{B}$ and freezing (moving to the bottom) the corresponding rows. By [2, Corollary 8.5], the exchange matrix $\widetilde{B}$ has really full rank, meaning that its rows span $\mathbb{Z}^n$, where n is the number of mutable variables in $X(\beta)$. Note that it follows that $\widetilde{B}^\circ$ has really full rank as well.
- $\widetilde{B}_1 = \begin{pmatrix} B_1 \\ \hline C_1 \end{pmatrix}$, the extended exchange matrix for the cluster structure on $X(\underline{w^{-1}w_0}\beta^1)$. We also have the extended exchange matrix $\widetilde{B}_2$ for the cluster structure on $X(\beta^2\underline{w})$. Note that the extended exchange matrix for the cluster structure on the product variety $X(\underline{w^{-1}w_0}\beta^1) \times X(\beta^2\underline{w})$ has the form

$$\widetilde{B}^\times = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \\ \hline C_1 & 0 \\ 0 & C_2 \end{pmatrix}.$$

Note that by (3), we can assume that, maybe after mutations, the matrix $\widetilde{B}^\circ$ has the following form:

$$\widetilde{B}^\circ = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \\ \hline C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

And our job is to produce a matrix R as in Lemma 3.5 such that $R\widetilde{B}^{\times} = \widetilde{B}^{\circ}$. We consider first the following matrices:

$$\widetilde{B}_1^{\times} := \begin{pmatrix} B_1 \\ 0 \\ \hline C_1 \\ 0 \end{pmatrix}, \qquad \widetilde{B}_1^{\circ} = \begin{pmatrix} B_1 \\ 0 \\ \hline C_{11} \\ C_{12} \end{pmatrix}.$$

Since $\widetilde{B}_1$ has really full rank, the same is true for $\widetilde{B}_1^{\times}$. So we can express the rows of C_{11} and C_{12} as an integer linear combination of the rows of B_1 and C_1. This means that we can find a matrix $R_1 = \begin{pmatrix} \mathbf{1} & 0 \\ P_1 & Q_1 \end{pmatrix}$ so that $R_1\widetilde{B}_1^{\times} = \widetilde{B}_1^{\circ}$. Note that Q_1 is invertible over $\mathbb{Z}$, since the matrix $\widetilde{B}_1^{\circ}$ has really full rank. Similarly, we can find $R_2 = \begin{pmatrix} \mathbf{1} & 0 \\ P_2 & Q_2 \end{pmatrix}$ such that $R_2\widetilde{B}_2^{\times} = \widetilde{B}_2^{\circ}$. It is now easy to see that

$$R = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ P_1 & 0 & Q_1 & 0 \\ 0 & P_2 & 0 & Q_2 \end{pmatrix}$$

satisfies the required properties. ∎

Remark 5.8 Let f, f_1, f_2 denote the numbers of frozen variables for $X(\beta)$, $X(\underline{w^{-1}w_0}\beta^1)$ and $X(\beta^2\underline{w})$, respectively. By [14, Theorem 1.3], the group $\mathbb{C}[X]^{\times}$ of global invertible functions on a cluster variety X is an abelian group generated by monomials in frozens and nonzero scalars. In particular, the number of frozen variables does not depend on a choice of a seed and is an invariant of a cluster variety. Theorem 5.2 and the inclusion

$$\mathbb{C}[X(\beta)]^{\times} \hookrightarrow \mathbb{C}[\mathcal{U}_{r_1,w}]^{\times}$$

imply the inequality

$$f_1 + f_2 \geq f. \tag{5.15}$$

Assuming Conjecture 5.6, we can compute the number s of cluster variables $x_{a_1}, \ldots, x_{a_s}$ that need to be frozen in (1) or (2). By comparing the number of frozen variables for $\mathcal{U}_{r_1,w}(\beta)$ and their image under $\Phi_{r_1,w}$, we arrive at the equation $f + s = f_1 + f_2$, so

$$s = f_1 + f_2 - f.$$

We can verify (1)–(3) of Conjecture 5.6 in the extreme cases of $w = e$ and $w = w_0$.

Proposition 5.9 *Assume $w = e$ or $w = w_0$. Then, (1)–(3) of Conjecture 5.6 hold.*

Proof Assume $w = w_0$, so that $\mathcal{U}_{r_1,w_0}(\beta)$ is given by the condition that $\mathcal{F}^{r_1}$ is transverse to $\mathcal{F}^{\mathrm{std}}$, and $\mathcal{U}_{r_1,w_0}(\beta)$ is nonempty if and only if $\delta(\beta^1) = w_0$. In this

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

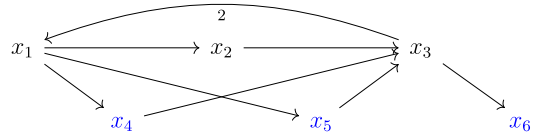
case, the seed Σ predicted by Conjecture 5.6 is the left-to-right inductive seed, whose corresponding cluster torus is precisely the Deodhar torus $\mathbb{T}_\beta$. It follows from [2, Section 7.1] that a cluster variable x_a is nowhere vanishing on $\mathcal{U}_{r_1,w_0}(\beta)$ if and only if the Lusztig cycle associated with x_a intersects the horizontal slice of the inductive weave right after obtaining $\delta(\beta^1) = w_0$, and that these are the cluster variables predicted by Conjecture 5.6(1). By construction, cf. [2, Section 4] the mutable part of the quiver obtained after freezing these cluster variables is equal to the disjoint union of the mutable parts of the quivers Q_1 and Q_2, so (3) is valid and by Lemma 5.7(2) also holds. Moreover, again by Lemma 5.7, the cluster structures on $\mathcal{U}_{r_1,w_0}$ and $X(\beta^1) \times X(\beta^2 w_0)$ are abstractly quasi-cluster isomorphic. The case $w = e$ is similar, taking the right-to-left inductive weave instead. ■

Example 5.10 Consider the braid word $\beta = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_2\sigma_3\sigma_1\sigma_2\sigma_2\sigma_1\sigma_3\sigma_2$, that was considered in [2, Section 11.4]. As explained in *loc. cit.*, for the left-to-right inductive seed, we have the cluster variables

$$x_1 = z_5,\ \ x_2 = -z_6z_7 + z_5z_8,\ \ x_3 = -z_6z_7z_9 + z_5z_8z_9 - z_5,\ \ x_4 = -z_6z_9 + z_5z_{10},$$
$$x_5 = -z_7z_9 + z_5z_{11},$$

$$x_6 = z_6z_7z_{10}z_{11} - z_5z_8z_{10}z_{11} - z_6z_7z_9z_{12} + z_5z_8z_9z_{12} - z_8z_9 + z_7z_{10} + z_6z_{11} - z_5z_{12} + 1,$$

the variables x_4, x_5, x_6 are frozen and the quiver Q is



Let us take $r_1 = 9$, so that $\beta^1 = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_2\sigma_3\sigma_1\sigma_2\sigma_2$ and $\beta^2 = \sigma_1\sigma_3\sigma_2$, and $w = w_0$, so that $\mathcal{U}_{9,w_0}(\beta)$ is given by the condition that $\mathcal{F}^9$ is transverse to $\mathcal{F}^{\mathrm{std}}$. The flag $\mathcal{F}^9$ is the flag associated with the matrix $B_{\beta^1}(z_1, \ldots, z_9)$:

$$\begin{pmatrix} -z_4z_5 + z_2z_7 + 1 & z_4z_6z_9 - z_2z_8z_9 + z_2 & -z_4z_6 + z_2z_8 & -z_4 \\ -z_3z_5 + z_1z_7 & z_3z_6z_9 - z_1z_8z_9 + z_1 + z_9 & -z_3z_6 + z_1z_8 - 1 & -z_3 \\ z_7 & -z_8z_9 + 1 & z_8 & 0 \\ z_5 & -z_6z_9 & z_6 & 1 \end{pmatrix}. \tag{5.16}$$

So $\mathcal{F}^9$ is transverse to $\mathcal{F}^{\mathrm{std}}$ if and only if the lower-left justified minors of (5.16) are nonzero. These are

$$\left|z_5\right| = x_1, \quad \begin{vmatrix} z_7 & -z_8z_9 + 1 \\ z_5 & -z_6z_9 \end{vmatrix} = x_3,$$

$$\begin{vmatrix} -z_3z_5 + z_1z_7 & z_3z_6z_9 - z_1z_8z_9 + z_1 + z_9 & -z_3z_6 + z_1z_8 - 1 \\ z_7 & -z_8z_9 + 1 & z_8 \\ z_5 & -z_6z_9 & z_6 \end{vmatrix} = x_1,$$

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

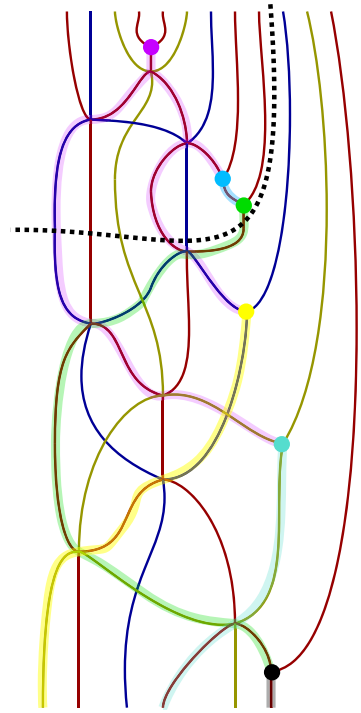


Figure 1: The left-to-right inductive weave for the braid $\beta = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_2\sigma_3\sigma_1\sigma_2\sigma_2\sigma_1\sigma_3\sigma_2$. Taking $r_1 = 9$, the part of the weave above the dotted line is an inductive weave for β^1, while the part of the weave below the dotted line is an inductive weave for $\underline{w_0}\beta^2$, and the braid varieties $X(\underline{w_0}\beta^2)$ and $X(\beta^2\underline{w_0})$ are quasi-cluster isomorphic.

so that $\mathcal{U}_{9,w_0}(\beta)$ is the cluster variety associated with the seed

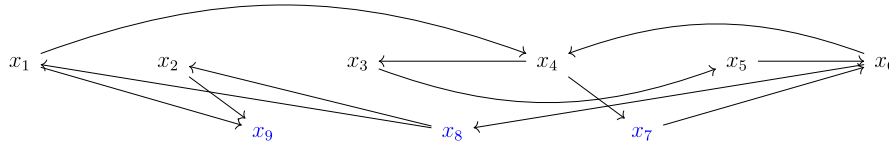
$$\begin{array}{ccccc} x_1 & \longrightarrow & x_2 & \longrightarrow & x_3 \\ & & & & \\ x_4 & & x_5 & & x_6. \end{array}$$

Note that the top row of this seed is isomorphic to a seed for $X(\beta^1)$, while the bottom part of the seed is isomorphic to a seed for $X(\underline{w_0}\beta^2)$, so (1)–(3) of Conjecture 5.6 hold.

Combinatorially, splicing amounts to cutting the left-to-right inductive weave along the dotted line in Figure 1.

Example 5.11 Now consider the braid word $\beta = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_2\sigma_3\sigma_1\sigma_1\sigma_2\sigma_3\sigma_2\sigma_2\sigma_3\sigma_2\sigma_1$. The quiver for the left-to-right inductive weave is

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press



and the extended exchange matrix is

$$\widetilde{B} = \left(\begin{array}{cccccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 & 0 \end{array}\right). \tag{5.17}$$

Now, let us take $r = 11$. The flag $\mathcal{F}^{11}$ is associated with the matrix $B_{\beta_{11}}(z_1, \ldots, z_{11})$, and a computation (using, e.g., Sage) shows that the lower-left justified minors of this matrix are

$$x_1x_2, \qquad x_5, \qquad x_4, \qquad 1.$$

So that the cluster structure on the open set $\mathcal{U}_{11,e}$ is obtained by freezing the cluster variables x_1, x_2, x_4, x_5. Note that there are no arrows between x_3 and x_6, so the mutable part of the quiver indeed becomes disconnected after freezing.

Freezing the corresponding rows, and deleting the corresponding columns, in (5.17), we obtain the following extended exchange matrix:

$$\widetilde{B}^\circ = \begin{array}{c} {\scriptstyle 3} \\ {\scriptstyle 6} \\ \hline {\scriptstyle 1} \\ {\scriptstyle 2} \\ {\scriptstyle 4} \\ {\scriptstyle 5} \\ {\scriptstyle 7} \\ {\scriptstyle 8} \\ {\scriptstyle 9} \end{array} \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 1 & -1 \\ -1 & 1 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \end{array}\right). \tag{5.18}$$

On the other hand, the extended exchange matrices for the cluster structures on $X(\beta^1)$, $X(\Delta\beta^2)$, and $X(\beta^1) \times X(\Delta\beta^2)$ are, respectively:

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

$$\widetilde{B}_1 = \left(\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ 1 \\ -1 \end{array}\right), \qquad \widetilde{B}_2 = \left(\begin{array}{c} 0 \\ \hline 1 \\ -1 \\ 0 \end{array}\right), \qquad \widetilde{B}^{\times} = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \end{array}\right).$$

We verify that $\widetilde{B}^{\circ}$ and $\widetilde{B}^{\times}$ define quasi-cluster equivalent cluster structures. Following the strategy of Lemma 3.5, we need to find a 9×9 invertible integer matrix R of the form $R = \begin{pmatrix} \mathbf{1}_2 & 0 \\ P & Q \end{pmatrix}$ such that $R\widetilde{B}^{\circ} = \widetilde{B}^{\times}$. It is easy to see that taking $P = 0$ and

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

works. Note, however, that the induced quasi-cluster isomorphism preserves the mutable variables x_3, x_6, so it is unlikely to coincide with the isomorphism $\Phi_{11,e}$.

5.3 Open Richardson varieties

We apply our results to the setting of open Richardson varieties in the flag variety. Recall that if $u \le w \in S_k$ are permutations, the open Richardson variety is

$$R(u,w) := \left\{ \mathcal{F} \in \mathcal{F}\ell_k \mid \mathcal{F}^{\mathrm{std}} \xrightarrow{w} \mathcal{F} \xrightarrow{u^{-1}w_0} \mathcal{F}^{\mathrm{ant}} \right\}.$$

The variety $R(u,w)$ is nonempty provided $u \le w$, in which case it is an affine, smooth variety of dimension $\ell(w) - \ell(u)$. In fact, we have an isomorphism

$$X\left(\underline{w} \cdot \underline{u^{-1}w_0}\right) \to R(u,w), \tag{5.19}$$

which takes a sequence of flags $(\mathcal{F}^0 = \mathcal{F}^{\mathrm{std}}, \mathcal{F}^1, \ldots, \mathcal{F}^{\mathrm{ant}})$ to the flag $\mathcal{F}^{\ell(w)}$ (cf. [2, Theorem 3.14]).

Proposition 5.12 *Let $u \le v \le w \in S_k$. Then, there exists an open embedding*

$$\Psi_{u,v,w} : R(u,v) \times R(v,w) \to R(u,w) \tag{5.20}$$

whose image is the set

$$\mathcal{U}_{u,v,w} = \{\mathcal{F} \in R(u,w) \mid \mathcal{F} \pitchfork \mathcal{F}(vw_0)\}. \tag{5.21}$$

Moreover, if $u, v, w \in S_k$ are permutations so that $\mathcal{U}_{u,v,w}$ is nonempty, then $u \le v \le w$.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

Proof We use the isomorphism (5.19) together with a special case of the maps from Theorem 5.2, so we work with the variety $X\left(\underline{w}\cdot\underline{u^{-1}w_0}\right)\cong R(u,w)$. In the setting of Section 5.1, we set $r_1=\ell(w)$, so that $\beta^1=\underline{w}$ and $\beta^2=\underline{u^{-1}w_0}$. Consider $v^*:=w_0vw_0$. Then,

$$\mathcal{U}_{r_1,v^*}=\left\{\left(\mathcal{F}^0=\mathcal{F}^{\mathrm{std}},\dots,\mathcal{F}^{\mathrm{ant}}\right)\in X\left(\underline{w}\cdot\underline{u^{-1}w_0}\right)\mid \mathcal{F}^{\ell(w)}\pitchfork\mathcal{F}(w_0v^*)\right\}$$

maps isomorphically onto $\mathcal{U}_{u,v,w}$ upon the isomorphism $X\left(\underline{w}\cdot\underline{u^{-1}w_0}\right)\cong R(u,w)$. On the other hand, by Theorem 5.2, we have

$$\mathcal{U}_{r_1,v^*}\cong X\left(\underline{(v^*)^{-1}w_0}\cdot\underline{w}\right)\times X\left(\underline{u^{-1}w_0}\cdot\underline{v^*}\right).$$

Using cyclic rotation isomorphisms (see [2, Section 5.5]), we moreover have

$$X\left(\underline{u^{-1}w_0}\cdot\underline{v^*}\right)\cong X\left(\underline{v}\cdot\underline{u^{-1}w_0}\right)\cong R(u,v),$$

while

$$X\left(\underline{(v^*)^{-1}w_0}\cdot\underline{w}\right)\cong X\left(\underline{w}\cdot\underline{v^{-1}w_0}\right)\cong R(v,w).$$

So $\mathcal{U}_{r_1,v^*}\cong R(u,v)\times R(v,w)$ and we obtain the embedding (5.20). The last claim of the statement of the proposition follows from Corollary 5.5. ∎

Remark 5.13 In [6, Definition 3.1], an open embedding $R(u,v)\times R(v,w)\to R(u,w)$ is obtained. We do not know if it coincides with the map constructed in Proposition 5.12.

Remark 5.14 Note that if $\beta=\beta^1\beta^2$, with β^1 and β^2 both reduced words, then the condition $\delta(\beta)=w_0$ forces $\beta^1=\underline{w}$ and $\beta^2=\underline{v^{-1}w_0}$ for some $v\le w$. Thus, any splicing map with β^1 and β^2 both reduced is equivalent to one of the form (5.20).

We can use Proposition 5.12 to give a (non-explicit) formula for the number of frozen variables on Richardson varieties. For $v\le w$, write $f_{v,w}$ for the number of frozen variables on the Richardson variety $R(v,w)$.

Corollary 5.15 *Let $v\le w\in S_k$. Let $s_{v,w}$ be the number of distinct irreducible factors appearing in the minors $\Delta_{v[i],[i]}(B_{\underline{w}}(z_1,\dots,z_{\ell(w)}))$ which are not themselves principal minors of the matrix $B_{\underline{w}}(z_1,\dots,z_{\ell(w)})$. Then,*

$$f_{v,w}=f_{e,w}-f_{e,v}+s_{v,w}. \tag{5.22}$$

Proof We have an embedding $R(e,v)\times R(v,w)\to R(e,w)$ whose image is the set $\mathcal{U}_{e,v,w}$ of flags $\mathcal{F}\in R(e,w)$ that are transverse to $\mathcal{F}(vw_0)$. The set of all flags in $R(e,w)$ is parameterized by those matrices $B_{\underline{w}}(z_1,\dots,z_{\ell(w)})$ with nonvanishing principal minors, and a flag $B_{\underline{w}}(z_1,\dots,z_{\ell(w)})$ is transverse to $\mathcal{F}(w_0v)$ if and only if $\Delta_{v[i],[i]}(B_{\underline{w}}(z_1,\dots,z_{\ell(w)})$ is nonzero for every $i=1,\dots,k$ (cf. Lemma 2.7(a)). So

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

$s_{v,w}$ is precisely the number of irreducible elements in $\mathbb{C}[R(e,w)]$ that we have to localize in order to obtain $\mathbb{C}[\mathcal{U}_{e,v,w}]$ and we get

$$f_{e,v} + f_{v,w} = f_{e,w} + s_{v,w},$$

which proves the result. ∎

Remark 5.16 For $w \in S_k$, the number $f_{e,w}$ is simply the number of different simple generators of S_k appearing in one (equivalently, any) reduced expression for w. So by (5.22), the computation of $f_{v,w}$ is equivalent to that of $s_{v,w}$, which still remains a challenging problem.

Example 5.17 Let us consider $k = 4$, $v = s_2$, and $w = s_3s_2s_1s_2s_3$. We have

$$B_{\underline{w}}(z_1, \dots, z_5) = \begin{pmatrix} z_3 & -z_4 & z_5 & -1 \\ z_2 & -1 & 0 & 0 \\ z_1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $v[i] \neq [i]$ if and only if $i = 2$, so we only need to check the irreducible factors of $\Delta_{[1,3],[1,2]}(B_{\underline{w}}(z_1, \dots, z_5)) = z_1z_4$. Neither z_1 nor z_4 is a principal minor, so we obtain $s_{v,w} = 2$ and $f_{v,w} = 3 - 1 + 2 = 4$. Since $\dim(R(v,w)) = 4$, we obtain that $R(v,w) \cong (\mathbb{C}^\times)^4$ is a four-dimensional torus. The left-to-right inductive initial seed for the cluster structure on $R(e,w)$ is

$$\boxed{z_3} \longrightarrow z_2 \longrightarrow \boxed{z_4z_2 - z_3} \longrightarrow z_1 \longrightarrow \boxed{z_1z_5 + z_3 - z_2z_4}$$

Note that mutating at z_2 we obtain $z_2' = z_4$, so there is a seed in $R(e,w)$ with mutable variables z_1, z_4.

Remark 5.18 Note that Corollary 5.15 is the easiest to apply when v is a simple transposition s_i: in this case, we only need to find irreducible factors of a single minor $\Delta_{s_i[i],[i]}$, since $\Delta_{s_i[j],[j]}$ is a principal minor for $j \neq i$.

Example 5.17 can be generalized as follows. Let $k = 2i$, and let $w = [k, 2, 3, \dots, k-1, 1]$, that is, w is the transposition $(1k)$. Let $v = s_2s_4 \dots s_{k-2}$. Up to signs in the pivotal 1's, the matrix $B_{\underline{w}}$ has the following form:

$$B_{\underline{w}} = \begin{pmatrix} p_{k-1} & p_k & p_{k+1} & \dots & p_{2k-3} & 1 \\ p_{k-2} & 1 & 0 & \dots & 0 & 0 \\ p_{k-3} & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_1 & 0 & 0 & \dots & 1 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

where $p_1, \dots, p_{2k-3}$ are irreducible polynomials in the z-variables. Among the minors $\Delta_{v[j],[j]}$, the ones that are not already principal minors are (up to signs)

$$\Delta_{v[2],[2]} = p_{k-3}p_k, \ \Delta_{v[4],[4]} = p_{k-5}p_{k+2}, \dots, \Delta_{v[2(i-1)],[2(i-1)]} = p_1p_{2k-2},$$

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

so each of these minors contributes with two irreducible factors. It follows that the number of frozen variables in $R(v,w)$ is

$$f_{v,w} = k-1-(i-1)+2(i-1) = k-1+i-1 = k-1+\frac{k}{2}-1,$$

which coincides with $\dim(R(v,w))$. In particular, $R(v,w)$ is a torus.

Remark 5.19 For $u,v \in S_k$ with $u \le v$, let $f_{u,v}$ denote the number of frozen variables in the cluster structure on $R(u,v)$. By (5.15), we have the inequality

$$f_{u,v} + f_{v,w} \ge f_{u,w}. \tag{5.23}$$

In forthcoming work of the second author [20], $f_{u,v}$ is shown to be equal to the coefficient of the second highest degree of the so-called R-polynomial $\mathrm{R}_{u,v}(q)$ associated with the pair (u,v) which has interesting combinatorial interpretations (cf. [24]). In particular, [20] provides an alternative combinatorial proof of (5.23).

Note that all the results so far in this section have been independent of the validity of Conjecture 5.6. The next result explores the consequences of Conjecture 5.6 in the Richardson setting.

Proposition 5.20 *Assume Conjecture 5.6 holds. Then, for every $u \le v \le w \in S_k$ and every $i = 1,\dots,k$, the minors $\Delta_{v[i],[i]}$ are cluster monomials in $R(u,w)$. Moreover, there exists a single cluster $\mathbf{x}$ so that all said minors are cluster monomials $\mathbf{x}$.*

Note that, fixing u and w, $\mathbf{x}$ may depend on the permutation v.

Proof If Conjecture 5.6(1) holds, then there exists a cluster $\mathbf{x}$ and cluster variables $x_{a_1},\dots,x_{a_s} \in \mathbf{x}$ such that $\mathcal{U}_{u,v,w}$ is the non-vanishing locus of $x_{a_1}\dots x_{a_s}$. Moreover, if Conjecture 5.6(2) holds, then $\mathcal{U}_{u,v,w}$ admits a cluster structure whose frozen variables are the frozen variables in $R(u,w)$, plus the variables $x_{a_1},\dots x_{a_s}$.

Now, by Lemma 2.7(a), for every $i = 1,\dots,k$, the minor $\Delta_{v[i],[i]}$ is nowhere vanishing on $\mathcal{U}_{u,v,w}$. Thus, by [14, Theorem 2.2], the restriction $\Delta_{v[i],[i]}|_{\mathcal{U}_{u,v,w}}$ is a monomial on the frozen variables of $\mathcal{U}_{u,v,w}$ that, as we have seen, are frozen variables of $R(u,w)$ plus some cluster variables in $\mathbf{x}$. Now, $\mathbb{C}[R(u,w)] \subseteq \mathbb{C}[\mathcal{U}_{u,v,w}]$ and the function $\Delta_{v[i],[i]}$ is regular on $R(u,w)$. Since $\Delta_{v[i],[i]}$ is a cluster monomial on $\mathcal{U}_{u,v,w}$, it is also a cluster monomial on $R(u,w)$. ∎

In the remainder of the article, we will construct splicing maps satisfying all the properties of Conjecture 5.6 in the special case of double Bott–Samelson varieties.

6 Splicing double Bott–Samelson varieties

6.1 Setup

Let β be a positive braid, and assume we have a decomposition $\beta = \beta^1\beta^2$, where $\beta^1 = \sigma_{i_1}\dots\sigma_{i_{r_1}}$ and $\beta^2 = \sigma_{i_{r_1+1}}\dots\sigma_{i_r}$. For each $s = 1,\dots,k-1$, let $\mathrm{last}^1(s) \in \{1,\dots,r_1\}$ be such that $i_{\mathrm{last}^1(s)} = s$, and $i_j \ne s$ for $\mathrm{last}^1(s) < j \le r_1$, that is, $\mathrm{last}^1(s)$ is the rightmost

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

appearance of σ_s in β^1. We will consider the cluster variable $x_{\text{last}^1(s)} \in \mathbb{C}[X(\beta)]$. If σ_s does not appear in β^1, we will simply set $x_{\text{last}^1(s)} = 1$.

6.2 Splicing

In the setup of Section 6.1, we have the following result.

Theorem 6.1 *Let $\mathcal{U}_{r_1}(\beta) \subseteq \mathrm{BS}(\beta)$ be the locus where none of the cluster variables $x_{\text{last}^1(s)}$ vanish for $s = 1, \dots, k-1$. Then:*

(1) *We have $(z_1, \dots, z_r) \in \mathcal{U}_{r_1}(\beta)$ if and only if $(z_1, \dots, z_{r_1}) \in \mathrm{BS}\left(\beta^1\right)$.*
(2) *We have an isomorphism of algebraic varieties*

$$\Phi_{r_1} : \mathcal{U}_{r_1}(\beta) \xrightarrow{\cong} \mathrm{BS}\left(\beta^1\right) \times \mathrm{BS}\left(\beta^2\right).$$

Proof Let $(z_1, \dots, z_r) \in \mathcal{U}_{r_1}(\beta)$. By Lemma 2.10 for $s = 1, \dots, k-1$, we have

$$\Delta_s(B_{\beta^1}(z_1, \dots, z_{r_1})) = x_{\text{last}^1(s)}$$

so that the condition $(z_1, \dots, z_r) \in \mathcal{U}_{r_1}(\beta)$ is equivalent to $\Delta_s(B_{\beta^1}(z_1, \dots, z_{r_1})) \neq 0$ for all s which is equivalent to $(z_1, \dots, z_{r_1}) \in \mathrm{BS}\left(\beta^1\right)$. Now, since $(z_1, \dots, z_{r_1}) \in \mathrm{BS}\left(\beta^1\right)$, we have an LU-decomposition

$$B_{\beta^1}(z_1, \dots, z_{r_1}) = L_1 U_1,$$

where L_1 has 1's on the diagonal. Since $(z_1, \dots, z_r) \in \mathrm{BS}(\beta)$, we also have an LU-decomposition $B_\beta(z_1, \dots, z_r) = LU$. Thus, we obtain

$$L_1 U_1 B_{\beta^2}(z_{r_1+1}, \dots, z_r) = LU. \tag{6.1}$$

Now, by Lemma 2.6, we can write

$$U_1 B_{\beta^2}(z_{r_1+1}, \dots, z_r) = B_{\beta^2}(z'_{r_1+1}, \dots, z'_r) U'_1 \tag{6.2}$$

for some change of variables $z'_{r_1+1}, \dots, z'_r$. It follows from (6.1) that $(z'_{r_1+1}, \dots, z'_r) \in \mathrm{BS}\left(\beta^2\right)$. The map $\Phi_{r_1} : (z_1, \dots, z_{r_1}, z_{r_1+1}, \dots, z_r) \mapsto ((z_1, \dots, z_{r_1}), (z'_{r_1+1}, \dots, z'_r))$ gives the desired isomorphism.

To construct the inverse map, suppose that we are given$((z_1, \dots, z_{r_1}), (z'_{r_1+1}, \dots, z'_r))$ such that $B_{\beta^1}(z_1, \dots, z_{r_1}) = L_1 U_1$ and $B_{\beta^2}(z'_{r_1+1}, \dots, z'_r) = L_2 U_2$. As above, the matrices L_1, L_2 are lower-triangular with 1 on diagonal and U_1, U_2 are upper triangular, all these matrices are uniquely determined by $((z_1, \dots, z_{r_1}), (z'_{r_1+1}, \dots, z'_r))$.

By (6.2), there exist $(z_{r_1+1}, \dots, z_r)$ and an upper-triangular matrix U''_1 such that

$$U_1^{-1} B_{\beta^2}(z'_{r_1+1}, \dots, z'_r) = B_{\beta^2}(z_{r_1+1}, \dots, z_r) U''_1,$$

which implies

$$B_{\beta^2}(z'_{r_1+1}, \dots, z'_r)(U''_1)^{-1} = U_1 B_{\beta^2}(z_{r_1+1}, \dots, z_r).$$

This determines the inverse map. We can check that the result lands in $\mathrm{BS}(\beta)$:

$$B_\beta(z_1, \dots, z_r) = B_{\beta^1}(z_1, \dots, z_{r_1}) B_{\beta^2}(z_{r_1+1}, \dots, z_r) = L_1 U_1 B_{\beta^2}(z_{r_1+1}, \dots, z_r) =$$

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

$$L_1 B_{\beta^2}(z'_{r_1+1}, \dots, z'_r)(U''_1)^{-1} = L_1 L_2 U_2 (U''_1)^{-1} = LU,$$

where $L = L_1 L_2$ and $U = U_2(U''_1)^{-1}$. ■

Remark 6.2 We constructed the map Φ_{r_1} using coordinates since we will need this in order to show cluster-theoretic properties of Φ_{r_1}. It is useful, however, to have a more conceptual understanding of it using flags (cf. Section 6.3 below). Note that $\mathcal{U}_{r_1}(\beta)$ is the locus of elements $(\mathcal{F}^0, \dots, \mathcal{F}^r) \in \mathrm{BS}(\beta)$ such that $\mathcal{F}^{r_1} \pitchfork \mathcal{F}(w_0)$. In particular, $\mathcal{F}^{r_1} = \mathcal{F}(L_1 U_1)$, and L_1 is uniquely determined provided it has 1's on the diagonal. Then,

$$\Phi_{r_1}\left(\mathcal{F}^0, \dots, \mathcal{F}^r\right) = \left(\left(\mathcal{F}^0, \dots, \mathcal{F}^{r_1}\right), L_1^{-1}\left(\mathcal{F}^{r_1}, \mathcal{F}^{r_1+1}, \dots, \mathcal{F}^r\right)\right) \in \mathrm{BS}\left(\beta^1\right) \times \mathrm{BS}\left(\beta^2\right).$$

Let Σ_β be the left inductive seed on $\mathrm{BS}(\beta)$ as described in Section 4.3, and let $\widehat{\Sigma}_\beta$ be the seed obtained upon freezing the variables $x_{\mathrm{last}^1(s)}$, $s = 1, \dots, k-1$. The following lemma says that the set $\mathcal{U}_{r_1}(\beta)$ together with the cluster variables $x_{\mathrm{last}^1(s)}$ satisfy the properties (1)–(3) of Conjecture 5.6.

Lemma 6.3 *We have* $A(\widehat{\Sigma}_\beta) \cong \mathbb{C}[\mathcal{U}_{r_1}(\beta)]$.

Proof From the description of the quiver Q_β in Section 4.3, it is easy to see that $\widehat{Q}_\beta^{\mathrm{uf}}$ is isomorphic to the disjoint union $Q_{\beta^1}^{\mathrm{uf}} \sqcup Q_{\beta^2}^{\mathrm{uf}}$, so the result follows as in the proof of Lemma 5.7 above. ■

6.3 Comparison to braid variety splicing

It is natural to ask whether the map Φ_{r_1} constructed in Theorem 6.1 is compatible with the braid variety splicing from Theorem 5.2. Recall the isomorphisms

$$\varphi_1 : \mathrm{BS}(\beta) \to X(\beta\Delta), \quad \varphi_2 : \mathrm{BS}(\beta) \to X(\Delta\beta)$$

from Lemma 4.4. We have the following results.

Lemma 6.4 *Let $\beta = \beta^1\beta^2$ be a positive braid. The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{BS}(\beta) \supseteq \mathcal{U}_{r_1} & \xrightarrow{\Phi_{r_1}} & \mathrm{BS}(\beta^1) \times \mathrm{BS}(\beta^2) \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2 \times \varphi_1} \\ X(\beta\Delta) \supseteq \mathcal{U}_{r_1,e} & \xrightarrow{\Phi_{r_1,e}} & X(\Delta\beta^1) \times X(\beta^2\Delta). \end{array}$$

Proof Let $(\mathcal{F}^0, \dots, \mathcal{F}^r) \in \mathcal{U}_{r_1}$, so that $\varphi_1(\mathcal{F}^0, \dots, \mathcal{F}^r) = (\mathcal{F}^0, \dots, \mathcal{F}^r, \widetilde{\mathcal{F}}^{r+1}, \dots, \widetilde{\mathcal{F}}^{r+\ell(w_0)})$, where $\widetilde{\mathcal{F}}^{r+1}, \dots, \mathcal{F}^{r+\ell(w_0)}$ are uniquely determined. Note that both $\mathcal{U}_{r_1}$ and $\mathcal{U}_{r_1,e}$ are defined by $\mathcal{F}^{r_1} \pitchfork \mathcal{F}(w_0)$, that is, the isomorphism φ_1 identifies $\mathcal{U}_{r_1}$ with $\mathcal{U}_{r_1,e}$.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

Let us write $\mathcal{F}^{r_1} = \mathcal{F}(L_1U_1)$, where L_1 is uniquely determined by the condition that it has 1's on the diagonal. Then,

(6.3)
$$\begin{aligned}(\varphi_2 \times \varphi_1) \circ \Phi_{r_1}(\mathcal{F}^0, \dots, \mathcal{F}^r) &= \left(\varphi_2\left(\mathcal{F}^0, \dots, \mathcal{F}^{r_1}\right), \varphi_1\left(L_1^{-1}\mathcal{F}^{r_1}, \dots, L_1^{-1}\mathcal{F}^r\right)\right) \\ &= \left(\left(\mathcal{G}^0, \dots, \mathcal{G}^{\ell(w_0)-1}, w_0L_1^{-1}\mathcal{F}^0, \dots, w_0L_1^{-1}\mathcal{F}^{r_1}\right),\right. \\ &\qquad \left.\left(L_1^{-1}\mathcal{F}^{r_1}, \dots, L_1^{-1}\mathcal{F}^r, \widetilde{\mathcal{G}}^1, \dots, \widetilde{\mathcal{G}}^{\ell(w_0)}\right)\right),\end{aligned}$$

where the flags $\mathcal{G}^0 = \mathcal{F}^{\mathrm{std}}, \dots, \mathcal{G}^{\ell(w_0)-1}$ and $\widetilde{\mathcal{G}}^1, \dots, \widetilde{\mathcal{G}}^{\ell(w_0)} = \mathcal{F}^{\mathrm{ant}}$ are uniquely determined. Indeed, to compute $\varphi_2(\mathcal{F}^0, \dots, \mathcal{F}^{r_1})$ by Lemma 4.4, we first write $\mathcal{F}^{r_1} = \mathcal{F}(L_1U_1) = \mathcal{F}((w_0U'w_0)U_1)$, where $U' = w_0L_1w_0$. Then $(U')^{-1}w_0\mathcal{F}^j = w_0L_1^{-1}\mathcal{F}^j$ for $j = 0, \dots, r_1$.

Now, we have to compare (6.3) with $\Phi_{r_1,e}(\mathcal{F}^0, \dots, \mathcal{F}^r, \widetilde{\mathcal{F}}^{r+1}, \dots, \widetilde{\mathcal{F}}^{r+\ell(w_0)})$. First, we factor $\beta\Delta = \beta^1(\beta^2\Delta)$. To get the $X(\beta^2\Delta)$ component, we need to find an element g_2 so that simultaneously $g_2\mathcal{F}^{r_1} = \mathcal{F}^{\mathrm{std}}$ and $g_2\widetilde{\mathcal{F}}^{r+\ell(w_0)} = \mathcal{F}^{\mathrm{ant}}$. By (5.9), we get $g_2 = L_1^{-1}$ and thus

$$\begin{aligned}\Phi^2\left(\mathcal{F}^0, \dots, \mathcal{F}^r, \widetilde{\mathcal{F}}^{r+1}, \dots, \widetilde{\mathcal{F}}^{r+\ell(w_0)}\right) &= g_2\left(\mathcal{F}^{r_1}, \mathcal{F}^{r_1+1}, \dots, \mathcal{F}^r, \widetilde{\mathcal{F}}^{r+1}, \dots, \widetilde{\mathcal{F}}^{r+\ell(w_0)}\right) \\ &= \left(L_1^{-1}\mathcal{F}^{r_1}, \dots, L_1^{-1}\mathcal{F}^r, \widetilde{\mathcal{G}}^1, \dots, \widetilde{\mathcal{G}}^{\ell(w_0)}\right) = \varphi_2 \circ \Phi_{r_1}(\mathcal{F}^0, \dots, \mathcal{F}^r)\end{aligned}$$

by the uniqueness of the flags $\widetilde{\mathcal{G}}^1, \dots, \widetilde{\mathcal{G}}^{\ell(w_0)}$.

Let us now examine the $X(\Delta\beta^1)$ component of $\Phi_{r_1,e}(\mathcal{F}^0, \dots, \mathcal{F}^r, \widetilde{\mathcal{F}}^{r+1}, \dots, \widetilde{\mathcal{F}}^{r+\ell(w_0)})$. For this, we obtain a unique sequence of flags going from $\widetilde{\mathcal{F}}^{r+\ell(w_0)} = \mathcal{F}^{\mathrm{ant}}$ to $\mathcal{F}^{\mathrm{std}}$ using the letters in Δ, say $\widehat{\mathcal{F}}^1, \dots, \widehat{\mathcal{F}}^{\ell(w_0)-1}$ and choose $g_1 = w_0L_1^{-1}$ by (5.13). Then

$$\begin{aligned}\Phi^1\left(\mathcal{F}^0, \dots, \mathcal{F}^r, \widetilde{\mathcal{F}}^{r+1}, \dots, \widetilde{\mathcal{F}}^{r+\ell(w_0)}\right) &= g_1\left(\mathcal{F}^{r+\ell(w_0)}, \widehat{\mathcal{F}}^1, \dots, \widehat{\mathcal{F}}^{\ell(w_0)-1}, \mathcal{F}^0, \dots, \mathcal{F}^{r_1}\right) \\ &= \left(\mathcal{G}^0, \dots, \mathcal{G}^{\ell(w_0)-1}, w_0L_1^{-1}\mathcal{F}^0, \dots, w_0L_1^{-1}\mathcal{F}^{r_1}\right) = \varphi_1 \circ \Phi_{r_1}(\mathcal{F}^0, \dots, \mathcal{F}^r)\end{aligned}$$

by the uniqueness of the flags $\mathcal{G}^0, \dots, \mathcal{G}^{\ell(w_0)-1}$. This finishes the proof. ∎

Lemma 6.5 *Let $\beta = \beta^1\beta^2$ be a positive braid. The following diagram commutes:*

$$\begin{array}{ccc}\mathrm{BS}(\beta) \supseteq \mathcal{U}_{r_1} & \xrightarrow{\Phi_{r_1}} & \mathrm{BS}(\beta^1) \times \mathrm{BS}(\beta^2) \\ \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_2 \times \varphi_1} \\ X(\Delta\beta) \supseteq \mathcal{U}_{r_1+\ell(w_0),w_0} & \xrightarrow{\Phi_{r_1+\ell(w_0),w_0}} & X(\Delta\beta^1) \times X(\beta^2\Delta).\end{array}$$

Note that at the bottom we use factorization $\Delta\beta = (\Delta\beta^1)\beta^2$ and $w = w_0$.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

Proof The composition $(\varphi_2 \times \varphi_1) \circ \Phi_{r_1}$ is unchanged, so we need to compute the composition $\Phi_{r_1+\ell(w_0),w_0} \circ \varphi_2$ and compare it with (6.3). We follow the notations $\mathcal{F}^{r_1} = \mathcal{F}(L_1U_1)$ from Lemma 6.4.

Given $(\mathcal{F}^0, \dots, \mathcal{F}^r) \in \mathrm{BS}(\beta)$, we have $\mathcal{F}^r = \mathcal{F}(L'U')$, where L' is unique provided that it has 1's on diagonal. Then, by Lemma 4.4, we get

$$\varphi_2(\mathcal{F}^0, \dots, \mathcal{F}^r) = \left(\mathcal{F}^{\mathrm{std}}, \dots, \widetilde{\mathcal{F}}^{\ell(w_0)-1}, w_0(L')^{-1}\mathcal{F}^0, \dots, w_0(L')^{-1}\mathcal{F}^r\right).$$

Next, we look at the $(\ell(w) + r_1)$-st flag:

$$w_0(L')^{-1}\mathcal{F}^{r_1} = w_0(L')^{-1}\mathcal{F}(L_1U_1) = \mathcal{F}(M), \quad M = w_0(L')^{-1}L_1U_1.$$

Next, according to (5.6), we need to write: $M = w_0LU$ so that $L = (L')^{-1}L_1$ and $U = U_1$. By (5.9) and (5.13), we compute

$$g_2 = (w_0L)^{-1} = L_1^{-1}L'w_0, \; g_1 = w_0L_1^{-1}L'w_0$$

and

$$g_2(w_0(L')^{-1}\mathcal{F}^j) = L_1^{-1}\mathcal{F}^j, \; g_1(w_0(L')^{-1}\mathcal{F}^j) = w_0L_1^{-1}\mathcal{F}^j.$$

Finally, by (5.3) and (5.4), we get

$$\Phi^1\left(\mathcal{F}^{\mathrm{std}}, \dots, \widetilde{\mathcal{F}}^{\ell(w_0)-1}, w_0(L')^{-1}\mathcal{F}^0, \dots, w_0(L')^{-1}\mathcal{F}^r\right) =$$
$$g_1\left(\mathcal{F}^{\mathrm{std}}, \dots, \widetilde{\mathcal{F}}^{\ell(w_0)-1}, w_0(L')^{-1}\mathcal{F}^0, \dots, w_0(L')^{-1}\mathcal{F}^{r_1}\right) =$$
$$\left(\mathcal{G}^0, \dots, \mathcal{G}^{\ell(w_0)-1}, w_0L_1^{-1}\mathcal{F}^0, \dots, w_0L_1^{-1}\mathcal{F}^{r_1}\right),$$

and

$$\Phi^2\left(\mathcal{F}^{\mathrm{std}}, \dots, \widetilde{\mathcal{F}}^{\ell(w_0)-1}, w_0(L')^{-1}\mathcal{F}^0, \dots, w_0(L')^{-1}\mathcal{F}^r\right) =$$
$$g_2\left(w_0(L')^{-1}\mathcal{F}^{r_1}, \dots, w_0(L')^{-1}\mathcal{F}^r, \widehat{\mathcal{F}}^1, \dots, \widehat{\mathcal{F}}^{\ell(w_0)}\right) =$$
$$\left(L_1^{-1}\mathcal{F}^r_1, \dots, L_1^{-1}\mathcal{F}^r, \widetilde{\mathcal{G}}^1, \dots, \widetilde{\mathcal{G}}^{\ell(w_0)}\right)$$

by the uniqueness of the flags $\mathcal{G}, \widetilde{\mathcal{G}}$. Therefore, $\Phi_{r_1+\ell(w_0),w_0} \circ \varphi_2(\mathcal{F}^0, \dots, \mathcal{F}^r)$ is also given by (6.3). ∎

In [17] we defined a subclass of open positroid varieties $S^{\circ,1}_{\lambda/\mu} \subset \mathrm{Gr}(k,n)$ called *skew shaped positroids* and labeled by skew Young diagrams λ/μ. By [17, Theorem 3.5.13], we have an isomorphism

$$S^{\circ,1}_{\lambda/\mu} \simeq X(\Delta\beta) \simeq \mathrm{BS}(\beta),$$

where β is a certain k-strand braid determined by λ/μ.

We also considered cutting the diagram λ/μ into two skew diagrams $\lambda^{a,L}/\mu^{a,L}$ and $\lambda^{a,R}/\mu^{a,R}$. These correspond to braids β_L and β_R and we showed that in fact $\beta = \beta_L\beta_R$. Then we defined a splicing map

$$S^{\circ,1}_{\lambda^L/\mu^L} \times S^{\circ,1}_{\lambda^R/\mu^R} \to \mathcal{U}_a \subset S^{\circ,1}_{\lambda/\mu}$$

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

or equivalently

$$X(\Delta\beta_L) \times X(\Delta\beta_R) \to \mathcal{U}_a \subset X(\Delta\beta). \tag{6.4}$$

Here, $\mathcal{U}_a$ is a certain open subset in $S^{\circ,1}_{\lambda/\mu}$. By [17, Theorem 5.6.1], the map (6.4) is a quasi-cluster equivalence. We invite the reader to explore more combinatorial and linear-algebraic properties of the map (6.4) in [17].

We claim that the map (6.4) agrees with the map Φ_r from Theorem 6.1 in this special case. We focus on [17, Definition 5.2.1]. The braid variety $X(\Delta\beta)$ parameterizes chains of (framed) flags

$$\Omega = \left[\mathcal{F}^W \overset{\Delta}{\dashrightarrow} \mathcal{F}^{W^{\mathrm{op}}} \overset{\beta_L}{\dashrightarrow} \mathcal{F}^a \overset{\beta_R}{\dashrightarrow} \mathcal{F}^0\right],$$

see [17] for unexplained notations. Such a chain belongs to $\mathcal{U}_a$ if and only if $\mathcal{F}^a \pitchfork \mathcal{F}^W$. In this case, the splicing map sends Ω to

$$\Omega^L = \left[\mathcal{F}^W \overset{\Delta}{\dashrightarrow} \mathcal{F}^{W^{\mathrm{op}}} \overset{\beta_L}{\dashrightarrow} \mathcal{F}^a\right], \quad \Omega^R = \left[\mathcal{F}^W \overset{\Delta}{\dashrightarrow} \mathcal{F}^a \overset{\beta_R}{\dashrightarrow} \mathcal{F}^0\right].$$

We can identify these with points in $X(\Delta\beta_L)$ and $X(\Delta\beta_R)$, respectively, up to multiplication by some $g \in \mathrm{GL}(k)$. Similarly to Lemmas 6.4 and 6.5, this agrees with Φ_{r_1} after applying the isomorphism $X(\Delta\beta_R) \simeq X(\beta_R\Delta)$.

6.4 Cluster-theoretic properties of the map Φ_{r_1}

By Lemma 6.3, the algebra $\mathbb{C}[\mathcal{U}_{r_1}(\beta)]$ admits a natural cluster structure. We then examine the cluster-theoretic properties of the map Φ_{r_1}. We consider the pullback

$$\Phi^*_{r_1} : \mathbb{C}\big[\mathrm{BS}(\beta^1) \times \mathrm{BS}(\beta^2)\big] \to \mathbb{C}[\mathcal{U}_{r_1}(\beta)]$$

that is an isomorphism of algebras. We will denote the coordinates on $\mathrm{BS}(\beta^1)$ by $z_1, \dots, z_{r_1}$ and the coordinates on $\mathrm{BS}(\beta^2)$ by $z_{r_1+1}, \dots, z_r$. Then the isomorphism $\Phi^*_{r_1}$ is determined by

$$\Phi^*_{r_1}(z_j) = z_j,\ j = 1, \dots, r_1, \qquad \Phi^*_{r_1}(z_j) = z'_j,\ j = r_1+1, \dots, r, \tag{6.5}$$

where $z'_{r_1+1}, \dots, z'_r$ are determined by (6.2).

On $\mathbb{C}[\mathcal{U}_{r_1}(\beta)]$ we have the cluster algebra structure obtained in Lemma 6.3. Since this comes comes from the open embedding $\mathcal{U}_{r_1}(\beta) \to \mathrm{BS}(\beta)$, we call this the *open cluster structure* on $\mathcal{U}_{r_1}(\beta)$, with quiver Q°_β. The quiver Q°_β is obtained from the quiver Q_β by freezing the vertices corresponding to the rightmost appearance of a crossing in β^1.

On the other hand, on $\mathbb{C}\left[\mathrm{BS}(\beta^1) \times \mathrm{BS}(\beta^2)\right]$, we have a natural cluster structure, with cluster variables $x^{(1)}_1, \dots, x^{(1)}_{r_1}$ (coming from $\mathrm{BS}(\beta^1)$) and $x^{(2)}r_1 + 1, \dots, x^{(2)}r$ (coming from $\mathrm{BS}(\beta^2)$). Thus, we obtain a cluster structure on $\mathbb{C}[\mathcal{U}_{r_1}(\beta)]$, with cluster variables being $\Phi^*_{r_1}\left(x^{(1)}_j\right)$, $\Phi^*_{r_1}\left(x^{(2)}\ell\right)$, $j = 1, \dots, r_1$, $\ell = r_1+1, \dots, r$. We call this the *product cluster structure* on $\mathbb{C}[\mathcal{U}_{r_1}(\beta)]$. We denote by $\widehat{Q}_\beta$ the quiver corresponding to this product cluster structure: it is simply the disjoint union of the quivers Q_{β^1} and Q_{β^2}.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

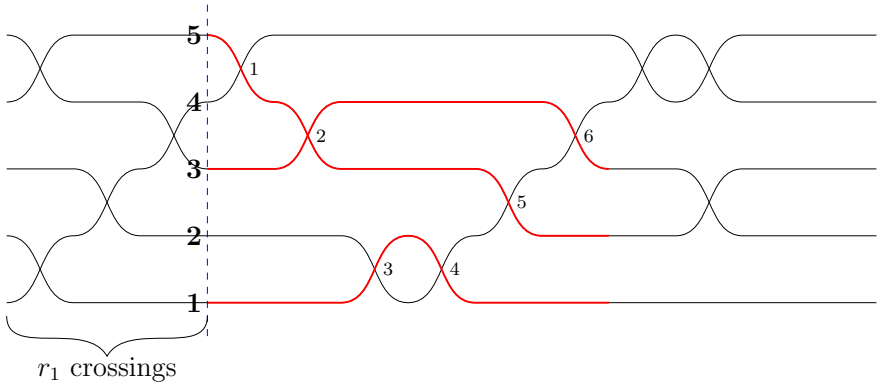


Figure 2: The monomial m_{r_1+6} equals $\left(u_3^{(1)}u_5^{(1)}u_1^{(1)}\right)^{-1}$.

Our goal is to relate the open and the product cluster structures. We start with some useful notations and definitions.

Definition 6.6 Let $B_{\beta^1}(z_1,\dots,z_{r_1}) = L_1U_1$ as above. We denote the diagonal elements of U_1 by $\left(u_1^{(1)},\dots,u_k^{(1)}\right)$.

Recall that, by (4.5), all $u_j^{(1)}$ are cluster monomials in the frozen variables $x_{\mathrm{last}^1(s)}$, $s = 1,\dots,k-1$. We can combine these to build more cluster monomials.

Definition 6.7 For $l = r_1+1,\dots,r$, we define

$$m_\ell = \prod_{t=1}^{i_\ell}\left(u^{(1)}_{s_{i_{r_1+1}}\dots s_{i_\ell}(t)}\right)^{-1}. \tag{6.6}$$

Let us interpret this pictorially. Draw the braid β^2 and look at the $\ell - r_1$-th crossing, counting from left to right. Right after this crossing, look at the bottom i_ℓ-strands, and follow them to the left. The end labels of the strands are precisely the subindices of the $u^{(1)}$-factors appearing in (6.6). See Figure 2 for an example.

Lemma 6.8 *Under the isomorphism* $\Phi^*_{r_1}:\mathbb{C}\left[\mathrm{BS}(\beta^1)\times\mathrm{BS}(\beta^2)\right]\to\mathbb{C}[\mathcal{U}_{r_1}(\beta)]$, *we have:*

(1) $\Phi^*_{r_1}\left(x_j^{(1)}\right) = x_j$ *for* $j = 1,\dots,r_1$.

(2) $\Phi^*_{r_1}\left(x^{(2)}\ell\right) = m_\ell x_\ell$ *for* $\ell = r_1+1,\dots,r$, *where* m_ℓ *is given by (6.6).*

Proof By (6.5), we have

$$\Phi^*_{r_1}\left(x_j^{(1)}\right) = \Phi^*_{r_1}\left(\Delta_{i_j}\left(B_{\beta^1_j}(z_1,\dots,z_j)\right)\right) = \Delta_{i_j}\left(B_{\beta_j}(z_1,\dots,z_j)\right) = x_j,$$

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

where $j = 1, \dots, r_1$. On the other hand, if $\ell = r_1 + 1, \dots, r$, we have

$$\Phi^*_{r_1}\left(x^{(2)}\ell\right) = \Phi^*_{r_1}\left(\Delta_{i_\ell}\left(B_{\beta^2_{\ell-r_1}}(z_{r_1+1}, \dots, z_\ell)\right)\right) = \Delta_{i_\ell}\left(B_{\beta^2_{\ell-r_1}}(z'_{r_1+1}, \dots, z'_\ell)\right)$$

and our job is to compare this to $x_\ell = \Delta_{i_\ell}\left(B_{\beta_\ell}(z_1, \dots, z_\ell)\right)$. Note however that

$$\begin{aligned}
x_\ell = \Delta_{i_\ell}\left(B_{\beta_\ell}(z_1, \dots, z_\ell)\right) &= \Delta_{i_\ell}\left(B_{\beta^1}(z_1, \dots, z_{r_1})B_{\beta^2_{\ell-r_1}}(z_{r_1+1}, \dots, z_\ell)\right) \\
&= \Delta_{i_\ell}\left(L_1 U_1 B_{\beta^2_{\ell-r_1}}(z_{r_1+1}, \dots, z_\ell)\right) \\
&= \Delta_{i_\ell}(L_1)\,\Delta_{i_\ell}\left(U_1 B_{\beta^2_{\ell-r_1}}(z_{r_1+1}, \dots, z_\ell)\right) \\
&= \Delta_{i_\ell}\left(U_1 B_{\beta^2_{\ell-r_1}}(z_{r_1+1}, \dots, z_\ell)\right) \\
&= \Delta_{i_\ell}\left(B_{\beta^2_{\ell-r_1}}(z'_{r_1+1}, \dots, z'_\ell)U'_1\right) \\
&= \Delta_{i_\ell}\left(B_{\beta^2_{\ell-r_1}}(z'_{r_1+1}, \dots, z'_\ell)\right)\Delta_{i_\ell}(U'_1) \\
&= \Phi^*_{r_1}\left(x^{(2)}\ell\right)\Delta_{i_\ell}(U'_1),
\end{aligned}$$

where we have used the Cauchy–Binet formula in the fourth and seventh equalities, and the fact that L_1 is uni-triangular in the fifth equality. Since U'_1 is upper triangular, its i_ℓ-th principal minor is just the product of the upper left i_ℓ-entries of the diagonal of U'_1. By Lemma 2.6, these coincide with the corresponding entries of U_1 up to permutation by $\pi(\beta^2_{\ell-r_1}) = s_{i_{r_1+1}} \dots s_{i_\ell}$. So

$$\Delta_{i_\ell}(U'_1) = m_\ell^{-1}, \quad x_\ell = \Phi^*_{r_1}\left(x^{(2)}\ell\right) m_\ell^{-1}$$

and the result follows. ∎

We are now ready to prove the following result.

Theorem 6.9 *The product and the open cluster structures on $\mathcal{U}_{r_1}(\beta)$ are quasi-cluster equivalent.*

Proof The fact that the cluster variables in both structures differ by monomials in frozens follows directly from Lemma 6.8. Now, we need to verify that the exchange ratios in both cluster structures agree. The mutable cluster variables in both cluster structures are indexed by $\mathrm{Mut} := \{1, \dots, r\} \backslash \{\mathrm{last}^1(s), \mathrm{last}(s) \mid s = 1, \dots, k-1\}$.

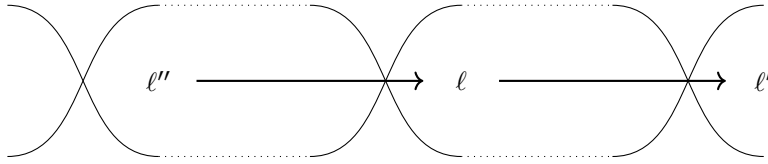
Let us take $j \in \mathrm{Mut}$ such that $j \le r_1$. We claim that if x_s is adjacent to x_j, then $s \le r_1$. We have two cases. If s and j are the same color (i.e., $i_s = i_j$), then either $s \le j \le r_1$, or $j < s \le \mathrm{last}^1(i_j) \le r_1$, where the last equation follows since there cannot be $j < t < s$ with $i_j = i_t$ and, x_j being mutable, j cannot be the last appearance of i_j in β^1. Now assume s and j are not of the same color. If there is an arrow $j \to s$, then $s \le j \le r_1$, so we assume that there is an arrow $s \to j$. So s must be strictly between j and $\mathrm{last}^1(j) \le r_1$. Thus, $s \le r_1$ and we have proven our claim. By Lemma 6.8(1), the exchange ratio associated with the cluster variable x_j in the open structure coincides with the exchange ratio associated with $\Phi^*_{r_1}\left(x^{(1)}_j\right)$ in the product structure.

Now, we take $\ell \in \mathrm{Mut}$ with $r_1 < \ell$. Our goal is to show that the exchange ratio associated with x_ℓ in the open structure coincides with the exchange ratio of $\Phi^*_{r_1}\left(x^{(2)}\ell\right)$ in the product structure. Since x_ℓ is not frozen, there exists $\ell < \ell' \le r$ such that

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

$\mathrm{clr}(\ell) = \mathrm{clr}(\ell')$. In particular, we have an arrow $\ell \to \ell'$ both in $\widehat{Q}_\beta$ and in $Q_{\beta^1} \sqcup Q_{\beta^2}$. We have several cases.

Case 1. The quiver Q°_β locally around ℓ looks as follows:



Since ℓ'' is not frozen in Q°_β, we have $r_1 < \ell''$, so that both arrows also belong to $\widehat{Q}_\beta$. So we have to show that $x_{\ell'}/x_{\ell''} = \Phi^*_{r_1}\left(x^{(2)}\ell'\right)/\Phi^*_{r_1}\left(x^{(2)}\ell''\right)$. By Lemma 6.8, this is equivalent to showing that $m_{\ell'} = m_{\ell''}$. Note that there are no crossings of color $\mathrm{clr}(\ell) \pm 1$ between any of the three crossings depicted on the figure, as otherwise the quiver Q°_β would have more arrows incident to either ℓ or ℓ''. It then follows from (6.6) that both $m_{\ell'}$ and $m_{\ell''}$ are equal to $\prod_{i \le i_\ell} \left(u^{(1)}_{s_{i_{r_1+1}} \dots s_{i_{\ell''}}(i)}\right)^{-1}$.

Case 2. The quiver $Q_{\beta^1} \sqcup Q_{\beta^2}$ locally around ℓ looks as follows:



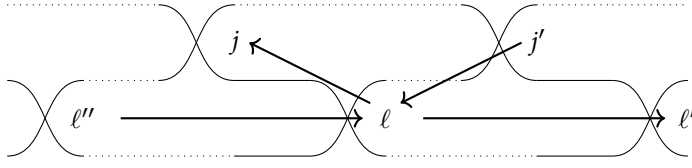
So that the quiver Q°_β locally around ℓ simply looks as $\ell'' \to \ell \to \ell'$, and $\ell'' = \mathrm{last}^1(\mathrm{clr}(\ell))$, so that ℓ'' is frozen. The exchange ratio in the product cluster structure is then simply $\Phi^*_{r_1}\left(x^{(2)}\ell'\right) = m_{\ell'}x_{\ell'}$. But $m_{\ell'} = \left(u_1 \dots u_{\mathrm{last}^1(\mathrm{clr}(\ell))}\right)^{-1} = x^{-1}_{\mathrm{last}^1(\mathrm{clr}(\ell))} = x^{-1}_{\ell''}$ and the result follows.

If the quiver Q°_β looks $\ell \to \ell'$ locally around ℓ, then

$$x_{\mathrm{last}^1(\mathrm{clr}(\ell))} = 1 = \left(u_1 \dots u_{\mathrm{last}^1(\mathrm{clr}(\ell))}\right)^{-1} = m_{\ell'},$$

so this case is similar.

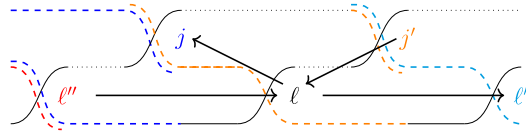
Case 3. The quiver Q°_β locally around ℓ looks as follows:



So we must show that $m_{\ell'}m_j = m_{\ell''}m_{j'}$. This follows easily from the pictorial interpretation of (6.6). Let us color the strands contributing to a monomial m_i with

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

the same color as i in the picture below:

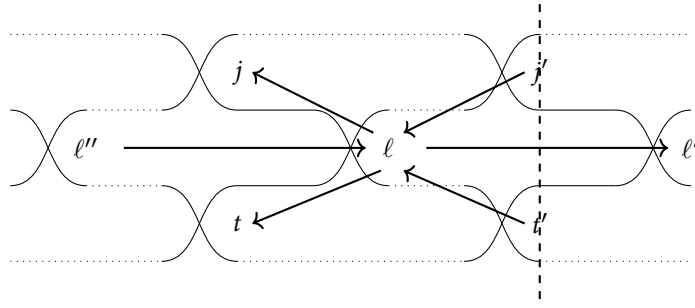


Note that we have used that there is no crossing of color $\mathrm{clr}(j')$ between j' and ℓ', no crossing of color $\mathrm{clr}(\ell)-1$ and no crossing of color $\mathrm{clr}(j)$ between j and ℓ. Note that every u factor appearing in $m_{\ell'}m_j$ cancels with one appearing in $m_{\ell''}m_{j'}$ and vice versa, so the result follows.

The case when $\ell'' < r_1$ (so that the quiver Q_{β^2} does not have the vertex ℓ'' above, and ℓ' becomes frozen in Q°_β) is similar, after noticing that in this case $\mathrm{last}^1(\mathrm{clr}(\ell)) = \ell''$.

We note that when the quiver locally looks like a horizontal reflection, we obtain a similar result.

Case 4. The quiver Q°_β locally around ℓ looks as follows:



In this case, we need to prove that

$$m_{\ell''}m_{j'}m_{t'} = m_{\ell'}m_j m_t,$$

or, equivalently, that

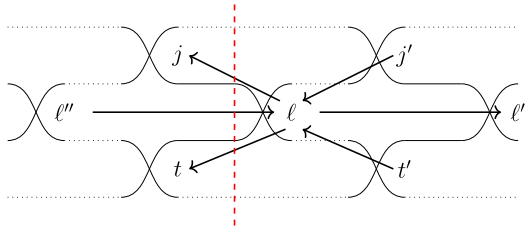
$$\frac{m_{j'}m_{t'}}{m_{\ell'}} = \frac{m_j m_t}{m_{\ell''}}.$$

Let $i = \mathrm{clr}(\ell)$ and consider the truncation of the braid β^2 on the dashed vertical line as depicted above. Let τ be the resulting permutation. Then

$$\frac{m_{j'}m_{t'}}{m_{\ell'}} = \prod_{s\le i}\left(u^{(1)}_{\tau^{-1}(s)}\right)^{-1}. \tag{6.7}$$

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

Now consider the truncation of the braid β^2 on the red dashed vertical line below:



and let τ_1 be the resulting permutation. Since there are no crossings of color $i = \mathrm{clr}(\ell)$ between ℓ and ℓ', we have that (6.7) can be rewritten as

$$\frac{m_{j'}m_{t'}}{m_{\ell'}} = \left(u^{(1)}_{\tau_1^{-1}(i+1)}\right)^{-1} \prod_{s<i} \left(u^{(1)}_{\tau_1^{-1}(s)}\right)^{-1}.$$

Which can easily seen to be $\frac{m_j m_t}{m_{\ell''}}$ and the result follows. Variations of this case (where, e.g., the t-crossing belongs to β^1) are proved similarly. ■

We now show that Example 1.10 given in the introduction is a quasi-cluster isomorphism.

Example 6.10 Recall Example 1.10. We verify the preservation of exchange ratios at each one of the vertices 10–13. Note that the preservation of exchange ratios at the mutable vertices 1–8 is immediate by construction. To lighten the notation, we denote $x'_k = \Phi^*_{r_1}\left(x^{(2)}_k\right)$.

- *Vertex 10:* In the open cluster structure, the exchange ratio is $\frac{x_6x_{13}}{x_7x_{14}}$. In the product cluster structure, this exchange ratio is $\frac{x'_{13}}{x'_{14}}$. According to Lemma 6.8, we have $x'_{14} = \left(u^{(1)}_2u^{(1)}_3u^{(1)}_4\right)^{-1} x_{14}$ and $x'_{13} = \left(u^{(1)}_2u^{(1)}_4\right)^{-1} x_{13}$. Now, we note that $\frac{x_6}{x_7} = \frac{u^{(1)}_3u^{(1)}_2u^{(1)}_1}{u^{(1)}_2u^{(1)}_1} = u^{(1)}_3$, which shows the desired equality.
- *Vertex 11:* In the open cluster structure, the exchange ratio is $\frac{x_7x_{12}}{x_9x_{13}}$ while in the product cluster structure, the exchange ratio is simply $\frac{x'_{12}}{x'_{13}}$. Now, according to Lemma 6.8,

$$x'_{12} = \left(u^{(1)}_4\right)^{-1} x_{12}, \qquad x'_{13} = \left(u^{(1)}_4u^{(1)}_2\right)^{-1} x_{13}, \qquad \text{so that} \qquad \frac{x'_{12}}{x'_{13}} = u^{(1)}_2\frac{x_{12}}{x_{13}}$$

and it remains to notice that $\frac{x_7}{x_9} = \frac{u^{(1)}_1u^{(1)}_2}{u^{(1)}_1} = u^{(1)}_2$.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

- *Vertex 12:* In the open cluster structure, the exchange ratio is $\frac{x_9x_{15}}{x_{11}x_{16}}$ and in the product cluster structure, it is $\frac{x'_{15}}{x'_{11}x'_{16}}$. Now,

$$\frac{x'_{15}}{x'_{11}x'_{16}} = \frac{\left(u_3^{(1)}u_4^{(1)}\right)^{-1}x_{15}}{\left(u_4^{(1)}u_1^{(1)}\right)^{-1}x_{11}\left(u_3^{(1)}\right)^{-1}x_{16}} = \frac{u_1^{(1)}x_{15}}{x_{11}x_{16}} = \frac{x_9x_{15}}{x_{11}x_{16}}.$$

- *Vertex 13:* We must verify that $\frac{x_{11}x_{14}}{x_{10}x_{15}}$ coincides with $\frac{x'_{11}x'_{14}}{x'_{10}x'_{15}}$:

$$\frac{x'_{11}x'_{14}}{x'_{10}x'_{15}} = \frac{\left(u_4^{(1)}u_1^{(1)}\right)^{-1}x_{11}\left(u_2^{(1)}u_3^{(1)}u_4^{(1)}\right)^{-1}x_{14}}{\left(u_4^{(1)}u_2^{(1)}u_1^{(1)}\right)^{-1}x_{10}\left(u_3^{(1)}u_4^{(1)}\right)^{-1}x_{15}} = \frac{x_{11}x_{14}}{x_{10}x_{15}}.$$

Acknowledgements The authors would like to thank Roger Casals, Mikhail Gorsky, Thomas Lam, Melissa Sherman-Bennett, David Speyer, Catharina Stroppel, and Daping Weng for the useful discussions.

References

[1] R. Casals, P. Galashin, M. Gorsky, L. Shen, M. Sherman-Bennett, and J. Simental, *Comparing cluster algebras on braid varieties*. Preprint, 2025. arXiv:2508.03816.
[2] R. Casals, E. Gorsky, M. Gorsky, I. Le, L. Shen, and J. Simental, *Cluster structures on braid varieties*. J. Amer. Math. Soc. **38**(2025), no. 2, 369–479.
[3] R. Casals, E. Gorsky, M. Gorsky, and J. Simental, *Positroid links and braid varieties*. J. Reine Angew. Math., to appear.
[4] R. Casals, E. Gorsky, M. Gorsky, and J. Simental, *Algebraic weaves and braid varieties*. Amer. J. Math. **146**(2024), no. 6, 1469–1576.
[5] M. Castronovo, M. Gorsky, J. Simental, and D. E Speyer, *Cluster deep loci and mirror symmetry*. Preprint, 2024. arXiv:2402.16970.
[6] J. N. Eberhardt and C. Stroppel, *Standard extension algebras I: Perverse sheaves and Fukaya calculus*. Preprint, 2023. arXiv:2310.09206.
[7] B. Elek and L. Jiang-Hua, *Bott-Samelson varieties and Poisson ore extensions*. Int. Math. Res. Not. **14**(2021), 10745–10797.
[8] S. Fomin and A. Zelevinsky, *Cluster algebras I. Foundations*. J. Amer. Math. Soc. **15**(2002), no. 2, 497–529.
[9] C. Fraser, *Quasi-homomorphisms of cluster algebras*. Adv. Appl. Math. **81**(2016), 40–77.
[10] C. Fraser and M. Sherman-Bennett, *Positroid cluster structures from relabeled plabic graphs*. Algebr. Combin. 5(2022), no. 3, 469–513.
[11] P. Galashin and T. Lam, *Positroids, knots, and q, t-Catalan numbers*. Duke Math. J. **173**(2024), no. 11, 2117–2195.
[12] P. Galashin, T. Lam, and M. Sherman-Bennett, *Braid variety cluster structures, II: General type*. Invent. Math. **243**(2026), 1079–1127.
[13] P. Galashin, T. Lam, M. Sherman-Bennett, and D. Speyer, *Braid variety cluster structures, I: 3D plabic graphs*. Preprint, 2022. arXiv:2210.04778.
[14] C. Geiss, B. Leclerc, and J. Schröer, *Factorial cluster algebras*. Doc. Math. **18**(2013), 249–274.
[15] K. R. Goodearl and M. T. Yakimov, *Cluster algebra structures on Poisson nilpotent algebras*. Mem. Amer. Math. Soc. **290**(2023), no. 1445, v+100.
[16] E. Gorsky and M. Hogancamp, *Hilbert schemes and y–ification of Khovanov–Rozansky homology*. Geom. Topol. **26**(2022), no. 2, 587–678.

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press

[17] E. Gorsky, S. Kim, T. Scroggin, and J. Simental, *Splicing skew shaped positroids*. Comb. Theory, to appear.
[18] E. Gorsky and T. Scroggin, *Splicing positroid varieties*. Algebr. Combin., to appear.
[19] G. Ingermanson, Cluster algebras of open Richardson varieties. Ph.D. thesis, ProQuest LLC, University of Michigan, 2019.
[20] S. Kim, In preparation.
[21] T. Lam and D. E. Speyer, *Cohomology of cluster varieties I: Locally acyclic case*. Algebra Number Theory **16**(2022), no. 1, 179–230.
[22] K. Mészáros, G. Musiker, M. Sherman-Bennett, and A. Vidinas, *Dimer face polynomials in knot theory and cluster algebras*. Selecta Math (N.S.), to appear.
[23] G. Muller, *Locally acyclic cluster algebras*. Adv. Math. **233**(2013), 207–247.
[24] L. Patimo, *A combinatorial formula for the coefficient of q in Kazhdan-Lusztig polynomials*. Int. Math. Res. Not. 5(2021), 3203–3223.
[25] T. Scroggin, *On the cohomology of two stranded braid varieties*. In: Contemporary mathematics, Algebraic structures in knot theory, volume 827 of Contemporary Mathematics, American Mathematical Society, Providence, RI, 2025, pp. 121–156.
[26] L. Shen and D. Weng, *Cluster structures on double Bott-Samelson cells*. Forum Math. Sigma **9**(2021), Article no. e66, 89 pp.
[27] M.-T. Q. Trinh, *From the Hecke category to the unipotent locus*. Preprint, 2021.

Department of Mathematics, University of California Davis, USA,
e-mail: egorskiy@ucdavis.edu syxkim@ucdavis.edu

Department of Mathematics, University of California Santa Barbara, USA
e-mail: tmscroggin@ucsb.edu

Instituto de Matemáticas, Universidad Nacional Autónoma de México, Mexico City, Mexico
e-mail: simental@im.unam.mx

https://doi.org/10.4153/S2976859426100034 Published online by Cambridge University Press