PARAMETRIC SURFACES

I. COMPACTNESS

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We shall first give some definitions concerning parametric surfaces. Denote by H a closed circle (disk) and by M a variable point on it. Let $P = \Phi(M)$ be a continuous function on H whose value P is a point in three-dimensional space. The symbols $\Phi(E)$, $\Phi^{-1}(P)$, where E is a set of points on H and P a point in the three-dimensional space, will have their usual meaning. $\Phi^{-1}(P)$ is a closed set. *Any saturated continuum in $\Phi^{-1}(P)$ or any point of $\Phi^{-1}(P)$ that does not belong to such continua is called a Φ-element of H.* Thus to any continuous function $\Phi(M)$ corresponds a representation of H in the form of the sum ΣQ of Φ-elements. *The set of the pairs (P, Q), where Q runs through all Φ-elements of H and, for any Q, $P = \Phi(Q)$, is called a parametric surface, and any pair (P, Q) is called a point of the parametric surface.* We shall often speak of a point $\Phi(M)$ of the parametric surface, by which we shall mean either the point (P, Q), where $P = \Phi(M)$ and Q is the Φ-element containing M, or the point $P = \Phi(M)$ of the three-dimensional space. The exact meaning will always be clear from the context. If there are exactly k points of the parametric surface whose first member is P_0 we say that P_0 *is a point of multiplicity k.* If $k = 1$, P_0 *is a simple point.*

Let $X_1 = (P_1, Q_1)$ and $X_2 = (P_2, Q_2)$ be a pair of points of the parametric surface Π defined by the equation $P = \Phi(M)$, and c any curve in H joining Q_1 and Q_2.

We define the Π-distance between X_1 and X_2 as the lower bound of the diameter of the Φ-image of c, that is by the equation

$$D_\Pi(X_1, X_2) = \text{l.bd.}\{d\Phi(c)\}$$

(we shall always use d for the diameter of a set, and D with an appropriate suffix for the distance).

Given $\delta > 0$, a set $E(\delta, \Pi)$ of points of Π such that any pair of them are at Π-distance not less than δ from each other and that any point of Π is at a Π-distance less than δ from at least one of them, is called a δ-set of Π.

LEMMA 1*. *If E is a set of points on the parametric surface Π such that the Π-distance between any pair of them is not less than a positive number δ, then $\mathfrak{N}E$† is bounded.*

For let Π be defined by the equation $P = \Phi(M)$, $M \in H$. By uniform continuity there exists $h > 0$ such that for any $M \in H$ and for the circle $c(M, h)$ in the plane of H

$$d\Phi\{c(M, h)\, H\} < \delta.$$

Thus if $\Phi(M_1), \ldots, \Phi(M_n)$ are different points of E then no circle $c(M_i, h)$ contains any M_j $(j \neq i)$. Therefore the circles $c(M_i, \frac{1}{2}h)$ are non-overlapping. On the other hand, all

† $\mathfrak{N}E$ denotes the number of elements of E.

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of them are included in the circle of radius $1+\frac{1}{2}h$ (the radius of H being taken as 1). Hence

$$\mathfrak{N}E \leqslant \frac{(1+\frac{1}{2}h)^2}{(\frac{1}{2}h)^2},$$

which proves the lemma.

COROLLARY. *Any δ-set on a parametric surface is finite.*

Given two parametric surfaces Π, Π' defined by the functions $\Phi(M)$, $\Phi'(M)$, denote by $\Psi(M)$ a function transforming topologically H into itself; the lower bound of $\max. D_{\text{Euc.}}\{\Phi(M), \Phi'(\Psi(M))\}$ for all functions $\Psi(M)$ is called the Fréchet distance between Π and Π' and is denoted by $D_F(\Pi, \Pi')$. If $D_F(\Pi, \Pi') = 0$ we say that the surfaces are identical.

A sequence Π_n of parametric surfaces is said to be convergent if to any $\eta > 0$ corresponds $n_0 = n_0(\eta)$ such that for any $n' > n_0$, $n'' > n_0$, $D_F(\Pi_{n'}, \Pi_{n''}) < \eta$.

A sequence $\{\Pi_n\}$ is said to converge to a parametric surface Π if $\lim D_F(\Pi_n, \Pi) = 0$.

From the above definitions we easily deduce

THEOREM 1. *A necessary and sufficient condition for a sequence $\{\Pi_n\}$ of parametric surfaces to be convergent is that it should be representable by the equations $P = \Phi_n(M)$, where $\{\Phi_n(M)\}$ is a uniformly convergent sequence of functions.*

We shall now study certain dissections of H into domains.

A finitely connected plane domain whose boundary consists of a finite number of disjoint Jordan curves is called a regular domain.

LEMMA 1. *Given a bounded finitely connected domain G and $\epsilon > 0$, there exists a regular domain $G' \subset G$ such that every point of $G - G'$ is within ϵ of the boundary of G.*

LEMMA 2. *If the total boundary of a bounded finitely connected domain G contains Jordan arcs whose every point is a prime end, then there exists a regular domain $G' \subset G$, whose boundary contains any finite number of these arcs, and such that all points of $G - G'$ are within ϵ from the part of the boundary of G, that does not belong to that of G'.*

LEMMA 3. *Given a finite number of non-overlapping regular domains in H, the number of complementary domains having a common boundary with more than two of the above domains or with two of them and with H itself is finite.*

Lemma 3 can be deduced from the simpler

LEMMA 3·1. *Given $k > 2$ non-overlapping regular domains $G_1, G_2, \ldots, G_k$ there are at most two complementary domains of $\bar{G}_1 + \bar{G}_2 + \ldots + \bar{G}_k$ that have a boundary in common with each of the domains $G_1, G_2, \ldots, G_k$.*

A representation of H in the form of a finite sum

$$H = \sum_{i=1}^{k} \bar{G}_i,$$

where $G_1, \ldots, G_k$ are regular disjoint domains is called a regular dissection of H.

Correspondingly, *a representation of the parametric surface Π, $P = \Phi(M)$, in the form*

$$\Pi = \Sigma\Phi(\bar{G}_i),$$

is called a regular dissection of Π.

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Two regular dissections of the disk H

$$H = \sum_1^k \bar{G}_i \quad \text{and} \quad H = \sum_1^k \bar{G}'_i,$$

are called homoeomorphic if there exists a topological transformation of H into itself, such that every G_i is transformed into G'_i.

THEOREM 2. *Given a parametric surface Π, $P = \Phi(M)$, and a δ-set E of n points on Π, there exist regular dissections*

$$H = \sum_1^m \bar{G}_i, \quad \Pi = \sum_1^m \Phi(\bar{G}_i) \quad (m \leqslant n),$$

such that $$E \subset \Sigma\Phi(G_i),$$

and that $$E\Phi(G_i) \neq 0, \quad d\Phi(G_i) < 50\delta$$

for every i.

We shall call such a dissection a *dissection corresponding to the set E*.

We may assume the coordinate axes in the three-dimensional space to be taken so that none of the points of the δ-set lies on any of the planes $x = 3k'\delta, y = 3k''\delta, z = 3k'''\delta$, where k', k'', k''' are any integers. We shall consider the systems of cubes of side 9δ formed by the planes

$$x = 3(3k' + l)\,\delta, \quad y = 3(3k'' + l)\,\delta, \quad z = 3(3k''' + l)\,\delta,$$

for $l = 0, 1, 2$. Any one of these cubes and any one of these planes will be called a Γ_l-cube and a Γ_l-plane, and the set of all Γ_l-planes will be denoted by (Γ_l). If $\Phi(M_0)$ is an interior point of a Γ_0-cube, we denote by $K(M_0)$ the saturated domain in H containing the point M_0 and such that the whole of $\Phi(K(M_0))$ lies inside the same Γ_0-cube as $\Phi(M_0)$. For two different points M_0, M_1 of H the domains $K(M_0)$, $K(M_1)$ are either coincident or disjoint. Let $G'_1, \ldots, G'_m$ $(m \leqslant n)$ be all the different domains $K(M)$ whose Φ-images contain points of E. The δ-neighbourhood of any point $\Phi(M)$ in Π contains a point of E. If $\Phi(M)$ is distant not less than δ from (Γ_0), then the δ-neighbourhood is within the same Γ_0-cube as $\Phi(M)$ itself and thus is included in $\Phi\{K(M)\}$. Hence $\Phi\{K(M)\}$ contains a point of E and $K(M)$ is one of the domains G'_j. We have arrived at the conclusion:

Any point of $\Phi\{H^0 - \Sigma G'_j\}$ is within δ from (Γ_0).

Observe that if the Φ-image of a complementary domain of G'_{j_0} contains a point of E then it contains the whole of $\Phi(G'_j)$ to which the point of E belongs. A complementary domain of G'_{j_0} whose Φ-image does not contain points of E is called empty. The sum of $\overline{G'^0_j}$ and of the closures of all the empty complementary domains of G'_j will be denoted by G''_j. The domain G''_j is of connectivity less than m. The Φ-image of the whole interior boundary of G''_j is on the surface of the Γ_0-cube q_j containing $\Phi(G'_j)$, and so is the Φ-image of the part of the outer boundary of G''_j that belongs to H^0. If an arc of the boundary C of H belongs to the boundary of G''_j, its Φ-image need not be on the surface of the cube q_j. In fact the boundary of G''_j may contain a finite or an infinite number of different open arcs of C, whose Φ-images are inside the cube q_j, but the Φ-images of the end-points are on the surface of the cube. Let λ be such an open

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arc of C and $\bar{M}$ a point of λ. As $\bar{M}$ belongs to the outer boundary of G_j'', it also belongs to the boundary of G_j'. $\Phi(\bar{M})$ being inside the cube q_j, there exists a $\delta_0 > 0$ such that every point of $\Phi\{c^0(\bar{M}, \delta_0) H^0\}$ is inside the cube. On the other hand, $c^0(\bar{M}, \delta_0) H^0$ contains points of G_j'. Consequently $G_j' + c^0(\bar{M}, \delta_0) H^0$ is a domain whose Φ-image is inside the cube. As G_j' is a saturated domain satisfying this condition $c^0(\bar{M}, \delta_0) H^0 \subset G_j' \subset G_j''$. Hence $\bar{M}$ is a prime-end of the boundary of G_j'' and so is every point of λ. Suppose first that the outer boundary C_j of G_j'' has a finite number of such arcs $\lambda_1, \ldots, \lambda_k$ (or no such arc). By Lemma 2 there exists a regular domain $G_j''' \subset G_j''$ whose boundary contains all the arcs $\lambda_1, \ldots, \lambda_k$ and such that every point of $G_j'' - G_j'''$ is within an arbitrary $\eta > 0$ from $C_j - \sum_1^k \lambda_i$. Choose $\eta > 0$ so that for any pair M', M'' of points of H within η from each other the points $\Phi(M')$, $\Phi(M'')$ are within $\frac{1}{2}\delta$ from each other. As the whole of $\Phi(C_j - \sum_1^k \lambda_i)$ is on the surface of q_j the whole of $\Phi(G_j'' - G_j''')$ is within $\frac{1}{2}\delta$ from the surface and *a fortiori* within $\frac{1}{2}\delta$ from (Γ_0). Suppose now that C_j contains an infinite sequence $\{\lambda_n\}$ of arcs of the above kind. Obviously there exists a k such that, for any $n > k$, $\Phi(\lambda_n)$ is within $\frac{1}{2}\delta$ of the surface of q_j. As before, we can define a regular domain $G_j''' \subset G_j''$ whose boundary contains $\lambda_1, \ldots, \lambda_k$ and such that every point of $G_j'' - G_j'''$ is within η of $C_j - \sum_1^k \lambda_i$. As all the points of $\Phi(C_j - \sum_1^k \lambda_i)$ are within $\frac{1}{2}\delta$ of the surface of q_j, all the points of $\Phi(G_j'' - G_j''')$ are within δ of the surface of q_j and of the set (Γ_0). We may assume that no $\Phi(G_j'' - G_j''')$ contains any point of E.

Thus corresponding to any domain G_j'' a regular domain $G_j''' \subset G_j''$ can be defined so that all the points of $\Phi(G_j'' - G_j''')$ are within δ of (Γ_0).

This, together with the previously proved similar result about $\Phi(H^0 - \Sigma \overline{G_j'})$, leads to the conclusion

Any point of $\Phi(H^0 - \Sigma \overline{G_j'''})$ is within δ from (Γ_0).

For any point $M \in H^0 - \Sigma \overline{G_j'''}$, such that $\Phi(M)$ is distant not less than δ from (Γ_1), define the saturated domain $K(M) \subset H^0 - \Sigma G_j'''$ such that the whole of $\Phi\{K(M)\}$ is within the same Γ_1-cube. There is a point $\Phi(M_0) \in E$ whose Π-distance from $\Phi(M)$ is less than δ. Let b be an arc in H joining M and M_0 and such that $d\Phi(b) < \delta$, and let M_1 be the first point of intersection of b with the boundary of a domain G_j''', as we move along b from M to M_0. Denote by $\cup MM_1$ the part of b between M and M_1. The whole of $\cup MM_1$ apart from the point M_1 is included in $K(M)$. Define r so that

$$d\Phi\{c(M_1, r) H^0\} < \delta - d\Phi(\cup MM_1),$$

then Φ-image of every point of $c(M_1, r) H^0$ is distant less than δ from $\Phi(M)$.

Let g be the saturated domain of $c^0(M_1, r)$ $(H^0 - \Sigma \overline{G_j'''})$ that contains the part of $\cup MM_1$ near M_1. The boundary of g inside the circle consists of one or more Jordan arcs belonging to the boundaries of one or more domains G_j'''.

As the whole open arc $\cup MM_1$ is in $K(M)$, $K(M) + g$ is a domain whose Φ-image is in the same Γ_1-cube as that of M. But $K(M)$ is the saturated domain of that kind; therefore $g \subset K(M)$. Thus $K(M)$ has a common boundary along an arc with at least one G_j''' and, for such G_j''', $\overline{K(M) + G_j'''}^0$ is a domain. Add each $K(M)$ to one of those G_j''' for

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which $\overline{K(M)+G_j'''}^0$ is a domain and denote by G_j^{iv} $(j=1,2,\ldots,m)$ the domains G_j''' enlarged in this way.

As before, the sum of $\overline{G_j^{\text{iv}}}^0$ and of the closures of all the empty complementary domains of G_j^{iv} will be denoted by G_j^{v} and regular domains $G_j^{\text{vi}} \subset G_j^{\text{v}}$ will be defined so that every point of $\Phi(G_j^{\text{v}} - G_j^{\text{vi}})$ be distant less than δ from (Γ_1).

All the points of $\Phi(H^0 - \Sigma\overline{G_j^{\text{vi}}})$ *are within* δ *both from* (Γ_0) *and from* (Γ_1).

We proceed now in a similar way. We take any $M \in H^0 - \Sigma\overline{G_j^{\text{vi}}}$ such that $\Phi(M)$ is distant not less than δ from (Γ_2) and form the saturated domain $K(M)$ whose Φ-image is inside the same Γ_2-cube as $\Phi(M)$.

Form the domains G_j^{vii}, G_j^{viii} and the regular domains G_j^{ix} just as the domains G_j^{iv}, G_j^{v}, G_j^{vi} were formed before. All the points of $\Phi(H^0 - \Sigma\overline{G_j^{\text{ix}}})$ are within δ from (Γ_0), (Γ_1), (Γ_2) and thus all of them are included in cubes of sides 2δ and centres at the point of intersection of three planes, one from each set (Γ_0), (Γ_1), (Γ_2).

We can obviously assume that every pair of domains H^0, G_j^{ix} $(j=1,\ldots,m)$ either have no common boundary or that the common boundary consists of a finite number of non-degenerate arcs. Then the set $H^0 - \Sigma\overline{G_j^{\text{ix}}}$ is a finite number of domains. Distribute them as before among the domains G_j^{ix} and denote the enlarged domains by G_j, which are obviously regular. We have

$$H = \Sigma\bar{G}_j.$$

Let us now estimate the diameters of $\Phi(G_j)$. First every $\Phi(G_j')$ is inside a Γ_0-cube q_j. Then all points of $\Phi(G_j'')$, and *a fortiori* of $\Phi(G_j''')$, are either inside the cube q_j or within δ of its surface. All the points of $\Phi(G_j^{\text{iv}})$ are within the eight Γ_1-cubes, including parts of q_j. They form a cube r_j of side 18δ. $\Phi(G_j^{\text{v}})$, and *a fortiori* $\Phi(G_j^{\text{vi}})$, is either inside r_j or within δ of its surface. $\Phi(G_j^{\text{vii}})$ is inside 27 Γ_2-cubes including the cube r_j; they form a cube s_j of side 27δ. Finally $\Phi(G_j)$ is either inside s_j or within δ of its surface and thus

$$d\Phi(G_j) < (27\sqrt{3}+2)\,\delta < 50\delta.$$

Thus the domains G_j satisfy the conditions of the theorem and the theorem is proved.

Let $H = \sum_1^n \bar{G}_i$ be a regular dissection of H into simply connected domains. Write $\mathrm{Fr}G_i = C_i$. Any saturated connected set of CC_i (C is the boundary of H) is called a saturated arc of intersection. It is either an arc or a point. Denote by $\theta(n)$ the maximum number of saturated arcs of intersection for all possible representations $H = \sum_1^n \bar{G}_i$.

LEMMA 4. $\theta(n) = 2(n-1)$ *for* $n > 1$.

There is at least one C_i whose intersection with C consists of a single saturated arc. Let it be C_n. Write $H' = \overline{H - \bar{G}_n}$. We have

$$H' = \sum_1^{n-1} \bar{G}_1 \quad \text{and} \quad \mathrm{Fr}H' = C' = C - CC_n + C_n - CC_n.$$

The number of saturated arcs of intersection of C' with $C_1, \ldots, C_{n-1}$ is at most $\theta(n-1)$. The maximum number of saturated arcs of intersection of C with $C_1, \ldots, C_{n-1}$ occurs if the arc $\overline{C_n - CC_n}$ lies inside one of the saturated arcs of intersection of C' with $C_1, \ldots, C_{n-1}$, since two parts of such an arc outside $\overline{C_n - CC_n}$ will form two separate saturated

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arcs of intersection with C. Thus $\theta(n) = 1+\theta(n-1)+1 = 2+\theta(n-1)$, and the lemma follows from the fact that $\theta(2) = 2$.

Remark. It is obvious that the lemma remains true if the domains G_i are subject only to the condition to be regular finitely connected domains.

Denote by $\phi(n)$ the number of all topologically different regular dissections $H = \sum_1^n G_i$.

THEOREM 3. *For any n, $\phi(n)$ is finite.*

We may assume, without loss of generality, that C_n, the outer boundary of G_n, has a single saturated arc of intersection with C. The arc $C_n - CC_n$—call its end-points A and B—divides H into two parts, H' bounded by C_n and H'' bounded by $AB = C_n - CC_n$ and $C - CC_n$. If G_n is a simply connected domain then $H' = G_n$, otherwise $H' = \Sigma' \bar{G}_i$, where Σ' is extended over $k > 1$ terms. Then $H'' = \Sigma'' \bar{G}_i$, the summation being extended over $n-k$ terms. The number of topologically different cases for H' is not greater than $\phi(k)$. As a matter of fact it is less than $\phi(k)$ since no G_i apart from G_n has a common boundary with $\mathrm{Fr}\, H'$. The number of topologically different cases for H'' is not more than $\phi(n-k)$. The frontier of H'' consists of at most $2(n-k-1)$ saturated arcs of intersections, with the boundaries of those G_i that belong to H''. There are less than $4^2(n-k-1)^2$ topologically different positions of A and B with respect to these arcs. Hence

$$\phi(n) \leqslant \sum_{k=1}^{n-1} 4^2(n-k-1)^2 \phi(k)\phi(n-k) < 4^2 n^3 \phi^2(n-1),$$

which proves the theorem.

We shall now study aggregates of parametric surfaces. An aggregate of parametric surfaces is said to be bounded if all the surfaces are included in a fixed sphere. We shall consider only bounded aggregates and we shall omit the word 'bounded'.

Given a set E we shall denote by $\mathfrak{N}E$ the number of elements of E. *An aggregate A of parametric surfaces Π is said to be uniformly bounded with respect to neighbourhoods if for any constant $\delta > 0$, $\mathfrak{N}E(\delta, \Pi)$ is bounded on the whole of A.* Given a parametric surface Π and $\delta > 0$, neither a set $E(\delta, \Pi)$ nor $\mathfrak{N}E(\delta, \Pi)$ is defined uniquely. Then the question arises whether the definition of an aggregate A that is uniformly bounded with respect to neighbourhoods does not depend upon the choice of *sets* $E(\delta, \Pi)$. As a $\frac{1}{2}\delta$-neighbourhood cannot include a pair of points whose distance is greater than or equal to δ, we have for any pair of sets $E(\delta, \Pi)$, $E(\frac{1}{2}\delta, \Pi)$

$$\mathfrak{N}E(\delta, \Pi) \leqslant \mathfrak{N}E(\tfrac{1}{2}\delta, \Pi).$$

Hence if the aggregate A of parametric surfaces is uniformly bounded with respect to neighbourhoods for a particular choice of sets $E(\delta, \Pi)$, it remains so for any other choice.

δ-sets of similar neighbourhoods. Suppose that a sequence $\{\Pi_n\}$ of parametric surfaces have a sequence $\{E(\delta, \Pi_n)\}$ of δ-sets of constant number ν of terms, the points of $E(\delta, \Pi_n)$ being $A_{n1}, A_{n2}, \ldots, A_{n\nu}$. We say that *the δ-sets are of similar neighbourhoods* if whenever A_{ni} belongs to the $k\delta$-neighbourhood, k integral, of A_{nj} in Π_n for one value of n, $A_{n'i}$ belongs to the $k\delta$-neighbourhood of $A_{n'j}$ in $\Pi_{n'}$, for any n'.

Dissections similar with respect to δ-sets. Given a sequence of parametric surfaces Π_n defined by the equations $P = \Phi_n(M)$, the regular dissection of every Π_n defined by

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a regular dissection of H, $H = \sum_{i=1}^{\mu_n} G_{ni}$, and a sequence of δ-sets $E(\delta, \Pi_n)$, we say that the regular dissections are similar with respect to the δ-sets if the following conditions are satisfied:

(i) $\mu_n = \mu$ is independent of n.

(ii) The sets $E(\delta, \Pi_n)$ are of similar neighbourhoods.

(iii) All points of $E(\delta, \Pi_n)$ are interior points of the sets $\Phi_n(G_{nj})$.

(iv) $A_{1i} \in \Phi_1(G_{1j})$ implies $A_{ni} \in \Phi_n(G_{nj})$ for all n.

Oscillation of a sequence of δ-sets $E(\delta, \Pi_n)$ with a constant number of points. If $D_{\text{Euc.}}(A_{ni}, A_{n'i}) \leqslant \eta$ for any n, n', i we say that the oscillation of the sequence does not exceed η.

An aggregate A of parametric surfaces is said to be compact if any sequence of surfaces belonging to A contains a convergent subsequence.

THEOREM 4. *A necessary and sufficient condition for an aggregate of parametric surfaces to be compact, is that it should be uniformly bounded with respect to neighbourhoods.*

We shall first prove that the condition is necessary. Suppose that the aggregate A is not uniformly bounded with respect to neighbourhoods. Then a sequence $\{\Pi_n\}$ of parametric surfaces in A and a $\delta_0 > 0$ can be found such that $\mathfrak{N}E(\delta_0, \Pi_n) \to \infty$.

LEMMA 5. *Given two parametric surfaces Π, Π' by the equation*

$$P = \Phi(M), \quad P = \Phi'(M),$$

such that

$$D_{\text{Euc.}}\{\Phi(M), \Phi'(M)\} < \epsilon$$

for all M, then for any M_1, $M_2 \in H$

$$-2\epsilon \leqslant D_{\Pi}\{\Phi(M_1), \Phi(M_2)\} - D_{\Pi}\{\Phi'(M_1), \Phi'(M_2)\} \leqslant 2\epsilon.$$

The lemma follows immediately from the fact that for any arc c in H

$$-2\epsilon < d\Phi(c) - d\Phi'(c) < 2\epsilon.$$

We shall prove that the sequence $\{\Pi_n\}$ does not converge.

Suppose that it does; then there exists n_0 such that for any $n > n_0$ $D_F(\Pi_n, \Pi_{n_0}) < \frac{1}{4}\delta_0$. By Lemma 1* there exists a ν_0 such that $\mathfrak{N}E < \nu_0$ for any set E of points of Π_{n_0} distant (in the Π_{n_0} sense) more than $\frac{1}{2}\delta_0$, from one another. Take $n > n_0$ such that

$$\mathfrak{N}E(\delta_0, \Pi_n) = \nu > \nu_0$$

and let $\Psi(M)$, $\Psi_0(M)$ define Π_n, Π_{n_0}, and satisfy the inequality

$$D_{\text{Euc.}}\{\Psi(M), \Psi_0(M)\} < \tfrac{1}{4}\delta_0$$

for all $M \in H$. Let further $\Psi(M_1)$, $\Psi(M_2)$, ..., $\Psi(M_\nu)$ be all the points of $E(\delta_0, \Pi_n)$. We have

$$D_{\Pi_n}\{\Psi(M_i), \Psi(M_j)\} \geqslant \delta_0$$

for any $i \gtrless j$. Hence, by Lemma 5,

$$D_{\Pi_{n_0}}\{\Psi_0(M_i), \Psi_0(M_j)\} > \tfrac{1}{2}\delta_0.$$

Thus we have a set of $\nu > \nu_0$ points $\Psi_0(M_1)$, $\Psi_0(M_2)$, ..., $\Psi_0(M_\nu)$ on Π_{n_0} distant more than $\frac{1}{2}\delta_0$ from one another. The contradiction proves our statement. We conclude at once that no subsequence of $\{\Pi_n\}$ converges, since for any subsequence $\{\Pi_{i_n}\}$ $\mathfrak{N}E(\delta_0, \Pi_{i_n}) \to 0$. Thus A is not compact.

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We shall now pass to the proof of the sufficiency of the condition.

Let $\{\Pi_n\}$ be a sequence in A given by the sequence of equations $P = \Phi_n(M)$, and $\{\delta_s\}$ a sequence of positive numbers rapidly decreasing to zero and such that, for any s, δ_s/δ_{s+1} is an integer greater than 4. We define the sets $E(\delta_s, \Pi_n)$ for all s and n. We assume that the points of $E(\delta_s, \Pi_n)$ are A^s_{ni} $(1 \leqslant i \leqslant \mathfrak{N}E(\delta_s, \Pi_n))$, and that for any s $E(\delta_s, \Pi_n) \subset E(\delta_{s+1}, \Pi_n)$ and $A^{s+1}_{ni} = A^s_{ni}$ for $i \leqslant \mathfrak{N}E(\delta_s, \Pi_n)$.

For any n define a regular dissection of H

$$H = \sum_1^{\mu_n} \bar{G}_{ni}, \quad \mu_n \leqslant \mathfrak{N}E(\delta_1, \Pi_n),$$

with respect to $E'(\delta_1, \Pi_n)$ and the corresponding dissection of Π_n. By the boundedness of the aggregate A and of $\mathfrak{N}E(\delta_1, \Pi_n)$ and by Theorem 3 there exists a subsequence $\{\Pi_{i_n^1}\}$ of the sequence $\{\Pi_n\}$ such that:

(i) The dissections $H = \Sigma G^1_{i_n^1 i}$ are homoeomorphic and similar with respect to the sets $E(\delta_1, \Pi_{i_n^1})$.

(ii) The oscillation of the sequence $\{E(\delta_1, \Pi_{i_n^1})\}$ is not greater than $\frac{1}{2}$.

Define now regular dissections of H, $H = \Sigma G^2_{i_n^1 i}$, and of $\Pi_{i_n^1}$ with respect to the sets $E(\delta_2, \Pi_{i_n^1})$. As before there exists a subsequence $\{\Pi_{i_n^2}\}$ of the sequence $\{\Pi_{i_n^1}\}$ such that:

(i) The dissections $H = \Sigma \bar{G}^2_{i_n^2 i}$ are homoeomorphic and similar with respect to the sets $E(\delta_2, \Pi_{i_n^2})$.

(ii) The oscillation of the sequence $\{E(\delta_2, \Pi_{i_n^2})\}$ is not greater than $1/2^2$.

Similarly a subsequence $\{\Pi_{i_n^3}\}$ of $\{\Pi_{i_n^2}\}$, a subsequence $\{\Pi_{i_n^4}\}$ of $\{\Pi_{i_n^3}\}$, and so on, are defined. We shall prove that the diagonal sequence, that is, the sequence $\Pi_{i_1^1}, \Pi_{i_2^2}, \Pi_{i_3^3}, \ldots$, is convergent. Let $\Psi_1(M), \Psi_2(M), \ldots$ be functions defining topological transformations of $\sum_{i=1}^{\mu_1} \bar{G}^1_{i_1^1 i}, \sum_{i=1}^{\mu_2} \bar{G}^2_{i_2^2 i}, \ldots$, into $\sum_{i=1}^{\mu_1} \bar{G}^1_{i_2^1 i}, \sum_{i=1}^{\mu_2} \bar{G}^2_{i_3^2 i}, \ldots$. Then the equations

$$P = \Phi_{i_1^1}(M) = \Theta_1(M), \quad P = \Phi_{i_2^2}\{\Psi_1(M)\} = \Theta_2(M),$$

$$P = \Phi_{i_3^3}[\Psi_2\{\Psi_1(M)\}] = \Theta_3(M), \quad P = \Phi_{i_4^4}[\Psi_3[\Psi_2\{\Psi_1(M)\}]] = \Theta_4(M), \ldots$$

represent the surfaces $\Pi_{i_1^1}, \Pi_{i_2^2}, \ldots$. We shall prove that $\{\Theta_s(M)\}$ is a uniformly convergent sequence of functions. Take an arbitrary point M_0 of H. It belongs to one of $\bar{G}'_{i_1^1 j}$, say to $\bar{G}^1_{i_1^1 j^1}$, and let $A^1_{i_1^1 k^1}$ be a point of $E(\delta_1, \Pi_{i_1^1})$ belonging to $\Phi_{i_1^1}(G^1_{i_1^1 j^1})$. We have

$$D_{\Pi_{i_1^1}}\{\Theta_1(M_0), A^1_{i_1^1 k^1}\} < 50\delta_1. \tag{1}$$

By the definition of $\Psi_1(M)$ and by the construction of the homoeomorphic dissections we have

$$\Psi_1(M_0) \in G^1_{i_2^2 j^1} \quad \text{and} \quad A^1_{i_2^2 k^1} \in \Phi_{i_2^2}(G^1_{i_2^2 j^1}). \tag{2}$$

Hence

$$D_{\Pi_{i_2^2}}\{\Theta_2(M_0), A^1_{i_2^2 k^1}\} < 50\delta_1. \tag{3}$$

Let now

$$\Psi_1(M_0) \in \bar{G}^2_{i_2^2 j^2}, \quad A^2_{i_2^2 k^2} \in \Phi_{i_2^2}\{G^2_{i_2^2 j^2}\}; \tag{4}$$

then

$$D_{\Pi_{i_2^2}}\{\Theta_2(M_0), A^2_{i_2^2 k^2}\} < 50\delta_2. \tag{5}$$

By (3) and (5)

$$D_{\Pi_{i_2^2}}\{A^1_{i_2^2 k^1}, A^2_{i_2^2 k^2}\} < 50\delta_1 + 50\delta_2, \tag{6}$$

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and by the preservation of neighbourhoods

$$D_{\Pi_{i_l'}}\{A^1_{i_l^1 k^1}, A^2_{i_l^2 k^2}\} < 50\delta_1 + 50\delta_2 \quad \text{for} \quad l \geqslant 2. \tag{7}$$

By the definition of $\Psi_2(M)$, by the property of the homoeomorphic dissections concerning points of δ_2-sets and by (4),

$$\Psi_2\{\Psi_1(M_0)\} \in \bar{G}^2_{i_3^2 j^2}, \quad A^2_{i_3^2 k^2} \in \Phi_{i_3^2}\{G^2_{i_3^2 j^2}\}. \tag{8}$$

Let further

$$\Psi_2\{\Psi_1(M_0)\} \in \bar{G}^3_{i_3^3 j^3}, \quad A^3_{i_3^3 k^3} \in \Phi_{i_3^3}\{G^3_{i_3^3 j^3}\}. \tag{9}$$

By (8) and (9),

$$D_{\Pi_{i_3^2}}\{\Theta_3(M_0), A^2_{i_3^2 k^2}\} < 50\delta_2, \quad D_{\Pi_{i_3^3}}\{\Theta_3(M_0), A^3_{i_3^3 k^3}\} < 50\delta_3, \tag{10}$$

$$D_{\Pi_{i_3^2}}\{A^2_{i_3^2 k^2}, A^3_{i_3^3 k^3}\} < 50\delta_2 + 50\delta_3, \tag{11}$$

and by the preservation of neighbourhoods

$$D_{\Pi_{i_l^1}}\{A^2_{i_l^1 k^2}, A^3_{i_l^1 k^3}\} < 50\delta_2 + 50\delta_3 \quad \text{for} \quad l \geqslant 3. \tag{12}$$

Similarly for any positive integer s we shall find points $A^{s-1}_{i_s^{s-1} k^{s-1}}$, $A^s_{i_s^s k^s}$ such that

$$D_{\Pi_{i_s^s}}\{\Theta_s(M_0), A^{s-1}_{i_s^s k^{s-1}}\} < 50\delta_{s-1}, \tag{13}$$

$$D_{\Pi_{i_l^1}}\{A^{s-1}_{i_l^1 k^{s-1}}, A^s_{i_l^1 k^s}\} < 50\delta_{s-1} + 50\delta_s \quad \text{for} \quad l \geqslant s. \tag{14}$$

It follows from (13) and (14) that for any pair $s_0 < s$ of positive integers

$$D_{\Pi_{i_s^s}}\{\Theta_s(M_0), A^{s_0}_{i_s^s k^{s_0}}\} \leqslant D_{\Pi_{i_s^s}}\{\Theta_s(M_0), A^{s-1}_{i_s^s k^{s-1}}\} + D_{\Pi_{i_s^s}}\{A^{s-1}_{i_s^s k^{s-1}}, A^{s-2}_{i_s^s k^{s-2}}\} + \dots$$

$$+ D_{\Pi_{i_s^s}}\{A^{s_0+1}_{i_s^s k^{s_0+1}}, A^{s_0}_{i_s^s k^{s_0}}\} < 50(\delta_{s_0} + 2\delta_{s_0+1} + \dots + 2\delta_{s-1}) < 100\delta_{s_0} \tag{15}$$

(by the rapidity of convergence of δ_s to 0).

Let now s, s' be any pair of integers greater than s_0. We have

$$D_{\text{Euc.}}\{\Theta_s(M_0), \Theta_{s'}(M_0)\} \leqslant D_{\Pi_{i_s^s}}\{\Theta_s(M_0), A^{s_0}_{i_s^s k^{s_0}}\}$$

$$+ D_{\Pi_{i_{s'}^{s'}}}\{\Theta_{s'}(M_0), A^{s_0}_{i_{s'}^{s'} k^{s_0}}\} + D_{\text{Euc.}}\{A^{s_0}_{i_s^s k^{s_0}}, A^{s_0}_{i_{s'}^{s'} k^{s_0}}\} < 200\delta_{s_0} + \frac{1}{2^{s_0}}.$$

Thus the sequence of functions $\Theta_s(M)$ is uniformly convergent and the parametric surface defined by the equation

$$P = \Theta(M) = \lim \Theta_s(M)$$

is the limit surface of the sequence $\{\Pi_{i_s^s}\}$.

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