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Subgroup closed Fitting classes

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1. *Introduction.* In (1) we showed that a subgroup closed Fitting formation is a primitive saturated formation, and in (2) we showed that a subgroup closed and metanilpotent Fitting class is a formation. Whether or not a subgroup closed Fitting class is always a formation is a question that has plagued us ever since. The purpose of this paper is to prove

THEOREM 1·1. *A subgroup closed Fitting class of nilpotent length three is a formation.*

It follows, of course, that a subgroup closed Fitting class of nilpotent length three is a primitive saturated formation. It is worth noting that much of the proof given below does not rely on nilpotent length three and the method can be used to dispose of special cases in nilpotent length four.

The techniques used are an extension of those employed in (1) and (2) to which, and the references contained in them, we refer the reader for definitions and the basic concepts and results concerning Fitting classes.

2. *Preliminaries.* In this section we collect together a number of results of a general nature about subgroup closed Fitting classes.

In lemma 3·5 of (2) we showed that a subgroup closed Fitting class $\mathfrak{X}$ has a small measure of Q-closure in that central factor groups of $\mathfrak{X}$-groups are also in $\mathfrak{X}$. The next results extend this measure of a Q-closure a little.

LEMMA 2·1. *Let $\mathfrak{X}$ be an* S*-closed Fitting class and G a group in $\mathfrak{X}$. Suppose that G has an abelian normal subgroup N and a normal subgroup K such that*

(i) $G = KC_G(N)$ *and*

(ii) $KN/N \in \mathfrak{X}$.

Then $G/N \in \mathfrak{X}$.

Proof. Since $C_G(N) \in \mathrm{S}\mathfrak{X} = \mathfrak{X}$ we have $C_G(N)/N \in \mathfrak{X}$ by lemma 3·5 of (3). Hence

$$G/N = (K/N)(C_G(N)/N) \in \mathrm{N}_0\mathfrak{X} = \mathfrak{X}.$$

COROLLARY 2·2. *Let $\mathfrak{X}$ be an* S*-closed Fitting class and G a group in $\mathfrak{X}$. Suppose that G has minimal normal subgroups $N_1, \ldots, N_t$ and a normal subgroup K such that*

(i) $G = KC_G(N_i) \quad (1 \leqslant i \leqslant t)$,

(ii) $\mathrm{Q}\{K\} \subseteq \mathfrak{X}$.

Then

$$G \Big/ \prod_{i=1}^{t} N_i \in \mathfrak{X}.$$

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Proof. From the lemma $G/N_1 \in \mathfrak{X}$. But then, since for $2 \leqslant i \leqslant t$, $N_1 \leqslant C_G(N_i)$,

$$C_G(N_i)/N_1 \leqslant C_{G/N_1}(N_i N_1/N_1),$$

and

$$G/N_1 = (KN_1/N_1)(C_G(N_i)/N_1),$$

it follows that the minimal normal subgroups $N_i N_1/N_1$ of G/N_1 and the normal subgroup $KN_1/N_1 (\cong K/K \cap N_1)$ satisfy conditions (i) and (ii). Hence, by induction,

$$\begin{aligned} G\Big/\prod_{i=1}^{t} N_i &\cong (G/N_1)\Big/\Big(\prod_{i=1}^{t} N_i\Big)\Big/N_1 \\ &= (G/N_1)\Big/\prod_{i=2}^{t} N_i N_1/N_1 \\ &\in \mathfrak{X}. \end{aligned}$$

LEMMA 2·3. *Let A be a group in an* s*-closed Fitting class $\mathfrak{X}$. Suppose that $A = BC$ where B is a normal p-subgroup for some prime p and $B \cap C = 1$. Suppose further that N is a minimal normal subgroup of A contained in B and that $|A : C_A(N)| = r$ for some prime r. Then $A/N \in \mathfrak{X}$.*

Proof. Observe that we must have $r \neq p$, $|C : C_C(N)| = r$ and $N \leqslant \zeta_1(B)$. Now let R be a Sylow r-subgroup of C. Then $NR \in \mathfrak{s}\mathfrak{X} = \mathfrak{X}$ and since NR is not nilpotent it follows from theorem 3·1 of (2) and theorem 1 of Hawkes (4) that $\mathfrak{S}_p \mathfrak{S}_r \subseteq \mathfrak{X}$.

Set

$$W = B \operatorname{twr}_C(C \times D),$$

where D is of order r. (The reader is referred to B. H. Neumann (5) for an account of the twisted wreath product.) We claim that $W \in \mathfrak{X}$. For, if Π is the base group of W then $\Pi C \in \mathfrak{R}_0\{A\} \subseteq \mathfrak{X}$, and $\Pi D \in \mathfrak{S}_p \mathfrak{S}_r \subseteq \mathfrak{X}$ so

$$W = (\Pi C)(\Pi D) \in \mathfrak{N}_0 \mathfrak{X} = \mathfrak{X}.$$

If $GF(p)D = T \oplus N_1 \oplus \ldots \oplus N_k$ where T is the one-dimensional trivial $GF(p)D$-module and the N_i are irreducible and faithful for D we have, as $C \times D$-module over $GF(p)$ (using $\#$ to denote outer tensor product),

$$\begin{aligned} N^D &\cong N \# GF(p)D \\ &= N \# (T \oplus N_1 \oplus \ldots \oplus N_k) \\ &\cong N \# T \oplus N \# N_1 \oplus \ldots \oplus N \# N_k. \end{aligned} \tag{2·4}$$

Each $N \# N_i$ decomposes as a direct sum of irreducible $C \times D$-modules, say

$$N \# N_i = \bigoplus_{j=1}^{k(i)} M_{ij},$$

where the kernels $K_{ij} = \ker M_{ij}$ have the property that

$$C \times D = K_{ij} D \quad (1 \leqslant j \leqslant k(i),\ 1 \leqslant i \leqslant k).$$

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Moreover we may regard each M_{ij} as a minimal normal subgroup of W and they, together with the subgroup $K = \Pi D$, satisfy the hypotheses of Corollary 2·2. Hence writing

$$M = \prod_{i,j} M_{ij}$$

we have $$W/M \in \mathfrak{X}.$$

Put $B_1 = \Pi/M$ and then $A_1 = B_1 C \in \mathfrak{X}$. Now if $d \in D$, $N^d \cap M = 1$ for, if not, $N^d \leqslant M$, since N^d is irreducible for C, which means that $N^D \leqslant M$ contrary to (2·4). It follows that B_1 is a central product of r copies of B and, from (2·4), that the amalgamated subgroup $N_1 = N^D/M \cong N$ as C-module over $GF(p)$. It follows that A_1, B_1, C and N satisfy the hypotheses of the present lemma. We may therefore repeat the above construction, using A_1 instead of A, and then again successively. We conclude that for each $n \geqslant 0$ there exists a C-operator group B_n which is a central product of r^n copies of B in which, moreover, the central factors admit C, and such that $B_n C \in \mathfrak{X}$.

We may therefore construct an onto C-homomorphism $\theta_n : B^{r^n} \to B$ (where B^{r^n} denotes the direct product of r^n copies of B), with

$$\ker \theta_n = \langle (a, 1, \ldots, 1, ae_i, 1, \ldots, 1) : a \in N, 2 \leqslant i \leqslant r^n \rangle,$$

the e_i being non-zero C-endomorphisms of N. By choosing n sufficiently large we may assume that p of the e_i's are equal, say $e_2 = e_3 = \ldots = e_{p+1} = e$ by re-numbering if necessary. But then, for all $a \in N$,

$$(1, \underbrace{ae, ae, \ldots, ae}_{p \text{ terms}}, 1, \ldots, 1) = \prod_{i=2}^{p+1} (a, 1, \ldots, 1, ae_i, 1, \ldots, 1)$$

$$\in \ker \theta_n.$$

Let $$\bar{B} = \{(1, \underbrace{b, \ldots, b}_{p \text{ terms}}, 1, \ldots, 1) : b \in B\}$$

a subgroup of B^{r^n}, and put $\bar{N} = \bar{B} \cap N^{r^n}$. Both admit C, and $\bar{N} = \bar{B} \cap \ker \theta_n$. But now

$$A/N = BC/N \cong \bar{B}C/\bar{N}$$

$$\cong (\bar{B} \ker \theta_n / \ker \theta_n) C$$

which is isomorphic to a subgroup of $B_n C$, giving $A/N \in \mathfrak{X}$ as required.

Let $\mathfrak{X}$ be a class of groups, p a prime and H a group. We define a class of $GF(p)H$-modules by

$$\mathscr{M}(H) = \{M : MH \in \mathfrak{X}\}.$$

Though $\mathscr{M}(H)$ depends on $\mathfrak{X}$ and p we will usually be working with fixed $\mathfrak{X}$ and p and no confusion should arise.

LEMMA 2·4. (cf. lemma 2·1 in (1) and lemma 3·4 in (2)). *Let $\mathfrak{X}$ be an* S*-closed Fitting class and suppose that $\mathscr{M}(H)$ contains a module M_0. Then $\mathscr{M}(H)$ contains all modules M for which ker $M \geqslant$ ker M_0.*

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Proof. Observe that submodules of modules in $\mathscr{M}(H)$, direct sums of modules in $\mathscr{M}(H)$ and tensor products of modules in $\mathscr{M}(H)$ are also in $\mathscr{M}(H)$.

The proof of 2·2 in (1), using the fact that every module is a submodule of a free module in place of the fact that every module is a quotient of a free module, gives us that there is a normal subgroup L of H such that

$$\mathscr{M}(H) = \{M : L \leqslant \ker M\}.$$

Clearly $L \leqslant \ker M_0$, and the result follows.

LEMMA 2·5. *Let $\mathfrak{X}$ be a subgroup closed Fitting class and p a prime. Write*

$$(p, \mathfrak{X}) = \{H : \mathscr{M}(H) \text{ contains a module faithful for } H\}.$$

Then $(p, \mathfrak{X})$ is a subgroup closed Fitting class contained in $\mathfrak{X}$.

Proof. Suppose that $H \in (p, \mathfrak{X})$ and that K is a subgroup of H. Let $M \in \mathscr{M}(H)$ be faithful for H. Then $MK \leqslant MH \in \mathfrak{X}$ so $M_K \in \mathscr{M}(K)$. Since M_K is faithful, $K \in (p, \mathfrak{X})$.

Now assume that H is a product of normal subgroups N_1 and N_2 where each $N_i \in (p, \mathfrak{X})$. Let M be a module faithful for H. By Lemma 2·4 $\mathscr{M}(N_i)$ contains all N_i-modules; in particular $M_{N_i} \in \mathscr{M}(N_i)$. But then

$$MH = (MN_1)(MN_2) \in \mathfrak{N}_0\mathfrak{X} = \mathfrak{X}$$

giving $M \in \mathscr{M}(H)$ and $H \in (p, \mathfrak{X})$.

A crucial step in the proof of Theorem 1·1 is to show that it is enough to consider an s-closed Fitting class contained in a product of $\mathfrak{S}_p$'s. The following rather complicated sequence of results enables us to do this.

Let $\mathscr{U}$ be the set of all finite sequences $(p_1, p_2, \ldots, p_n)$ of primes in which for

$$1 \leqslant i \leqslant n-1,$$

$p_i \neq p_{i+1}$. For $u = (p_1, p_2, \ldots, p_n) \in \mathscr{U}$ define $l(u) = n$ and $\hat{u} = (p_2, p_3, \ldots, p_n)$. Also define $\mathscr{U}_n = \{u \in \mathscr{U} : l(u) \leqslant n\}$. Finally write

$$\mathfrak{S}_u = \mathfrak{S}_{p_1}\mathfrak{S}_{p_2}\ldots\mathfrak{S}_{p_n},$$

where $u = (p_1, p_2, \ldots, p_n) \in \mathscr{U}$, for the usual class product.

LEMMA 2·6. (i) *For each $u \in \mathscr{U}$, $\mathfrak{S}_u$ is a primitive saturated formation of nilpotent length precisely $l(u)$.*

(ii) *If $\mathfrak{X}$ is a primitive saturated formation of nilpotent length precisely $l(u)$ contained in $\mathfrak{S}_u$, then $\mathfrak{X} = \mathfrak{S}_u$.*

Proof. Part (i) is an easy consequence of the definition of $\mathfrak{S}_u$. Part (ii) is proved by induction on $l(u)$ using the fact that $\mathfrak{X}$ is locally defined at p_1 by a primitive saturated formation $\mathfrak{X}_1$ contained in $\mathfrak{S}_{\hat{u}}$ (cf. theorem 3 of (1)).

If G is a group define $\mathscr{U}_n(G) = \{H : H \leqslant G, H \in \mathfrak{S}_u \text{ where } u \in \mathscr{U}_n\}$.

LEMMA 2·7. *Suppose that G is a group satisfying $G \in \{\mathfrak{s}, \mathfrak{N}_0\}\mathscr{U}_n(G)$. If M is a faithful module for G over $GF(p)$ then $MG \in \{\mathfrak{s}, \mathfrak{N}_0\}\mathscr{U}_{n+1}(MG)$.*

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Proof. Let $H \in \mathfrak{U}_n(G)$, say $H \in \mathfrak{S}_u$ where $u = (p_1, \ldots, p_k)$ with $k \leqslant n$. Then MH is an $\mathfrak{S}_{u'}$ subgroup of MG where $u' = u$ if $p = p_1$ and $u' = (p, p_1, \ldots, p_k)$ otherwise. In any case $l(u') \leqslant n+1$. Let $\mathfrak{X} = \{\mathrm{S}, \mathrm{N}_0\}\mathfrak{U}_{n+1}(MG)$. Then $MH \in \mathfrak{U}_{n+1}(MG) \subseteq \mathfrak{X}$. It follows that

$$\mathfrak{U}_n(G) \subseteq (p, \mathfrak{X})$$

and hence, by (2·5), that $G \in (p, \mathfrak{X})$. But by Lemma 2·4 $\mathscr{M}(G)$ contains all $GF(p)G$-modules and so $MG \in \mathfrak{X}$ as required.

LEMMA 2·8. *Suppose that subgroup closed Fitting classes of nilpotent length at most n are formations (and hence primitive saturated formations). If G is a group of nilpotent length at most $k \leqslant n$ then $G \in \{\mathrm{S}, \mathrm{N}_0\}\mathfrak{U}_k(G)$.*

Proof. We proceed by induction on $|G|$, the case $|G| = 1$ being obvious. Suppose that $|G| > 1$ and that for all groups H of order less than $|G|$, $H \in \{\mathrm{S}, \mathrm{N}_0\}\mathfrak{U}_k(H)$. Write $\mathfrak{X} = \{\mathrm{S}, \mathrm{N}_0\}\mathfrak{U}_k(G)$.

Let N be a minimal normal subgroup of G, and let $\mathfrak{U}_k(G/N) = \{T_i/N : 1 \leqslant i \leqslant r\}$. Choose a minimal supplement S_i for N in T_i. Since $S_i \cap N \leqslant \Phi(S_i)$ and

$$S_i/S_i \cap N \cong T_i/N \in \mathfrak{S}_u$$

for some $u \in \mathfrak{U}_k$, $S_i \in \mathfrak{S}_u$ by Lemma 2·6 (i). But then $S_i \in \mathfrak{X}$ and so $T_i/N \cong S_i/S_i \cap N \in \mathfrak{X}$ since $\mathfrak{X}$ is Q-closed. The induction hypothesis yields $G/N \in \{\mathrm{S}, \mathrm{N}_0\}\mathfrak{U}_k(G/N)$ and hence $G/N \in \mathfrak{X}$.

If G has distinct minimal normal subgroups N_1 and N_2 then G/N_1 and G/N_2 are both in $\mathfrak{X}$, whence $G \in \mathrm{R}_0\mathfrak{X} = \mathfrak{X}$. Suppose G has a unique minimal normal subgroup, N say. If $N \leqslant \Phi(G)$ then $G \in \mathfrak{X}$ since $\mathfrak{X}$ is saturated. If $N \nleqslant \Phi(G)$ then $G = NH$ with $N \cap H = 1$. But H has nilpotent length $k-1$ and, by the inductive hypothesis again,

$$H \in \{\mathrm{S}, \mathrm{N}_0\}\mathfrak{U}_{k-1}(H).$$

Hence $G = NH \in \{\mathrm{S}, \mathrm{N}_0\}\mathfrak{U}_k(G) = \mathfrak{X}$ by Lemma 2·7. The induction is complete.

LEMMA 2·9. *Let $u = (p_1, p_2, \ldots, p_n) \in \mathfrak{U}$. Suppose that $\mathfrak{S}_{\hat{u}}$ is the only subgroup closed Fitting class of nilpotent length precisely $l(\hat{u})$ contained in $\mathfrak{S}_{\hat{u}}$. If $G = MH \in \mathfrak{S}_u$, where M is a faithful module for H over $GF(p_1)$ and H has nilpotent length $l(\hat{u})$ exactly then $\{\mathrm{S}, \mathrm{N}_0\}(G) = \mathfrak{S}_u$.*

Proof. Write $\mathfrak{X} = \{\mathrm{S}, \mathrm{N}_0\}(G)$. Since $(p_1, \mathfrak{X})$ is an s-closed Fitting class and

$$H \in (p_1, \mathfrak{X}) \cap \mathfrak{S}_{\hat{u}}$$

has nilpotent length precisely $l(\hat{u})$, $\mathfrak{S}_{\hat{u}} \subseteq (p_1, \mathfrak{X})$.

We proceed by induction on the order of groups in $\mathfrak{S}_u$. Clearly $1 \in \mathfrak{X}$. Suppose that $A \in \mathfrak{S}_u$ has $|A| > 1$ and groups of order less than $|A|$ in $\mathfrak{S}_u$ are in $\mathfrak{X}$. If $A \in \mathfrak{S}_{\hat{u}}$ then $A \in \mathfrak{X}$ so we may suppose that A has a minimal normal subgroup N with $p_1 || N|$. Since $A/O_{p_1}(A) \in \mathfrak{S}_{\hat{u}} \subseteq (p_1, \mathfrak{X})$ there is a faithful module M for $A/O_{p_1}(A)$ over $GF(p_1)$ such that $M(A/O_{p_1}(A)) \in \mathfrak{X}$. If we regard M as an A/N-module by inflation, we have

$$M(A/N) \in \mathrm{R}_0\{M(A/O_{p_1}(A)),\ A/N\} \subseteq \mathfrak{X}$$

since $A/N \in \mathfrak{X}$ by induction. But lemmas 3·3 and 3·4 in (2) now yield that $\mathscr{M}(A/N)$ contains all $GF(p_1)(A/N)$-modules. In particular $N \operatorname{wr}(A/N) \in \mathfrak{X}$ and therefore, by (22·21) of (6), $A \in \mathrm{S}\mathfrak{X} = \mathfrak{X}$, completing the induction and the proof of (2·9).

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3. *The initial reduction.* Suppose that every subgroup closed Fitting class of nilpotent length at most $n-1$ is a formation, and that $\mathfrak{X}$ is a subgroup closed Fitting class of nilpotent length n which is not a formation. Let G in $\mathfrak{X}$ be minimal with respect to having a factor group not in $\mathfrak{X}$. As on page 290 of (2) we can show that if N is normal in G and maximal with $G/N \notin \mathfrak{X}$, then $N = \Phi(G)$, and $F(G)/\Phi(G)$ is the unique minimal normal subgroup of $G/\Phi(G)$. Consequently $F(G)/\Phi(G)$ is a p-group for some prime p. Note that G has nilpotent length precisely n, and $G/O_p(G)$ has nilpotent length $n-1$.

We show now that $G \in \mathfrak{S}_u$ for some $u \in \mathscr{U}$ with $l(u) = n$. This will follow if $G/\Phi(G)$ belongs to some $\mathfrak{S}_u$ with $l(u) = n$ because $\mathfrak{S}_u$ is saturated; and this in turn will follow by Lemma 2·7 if we can find $u = (p, p_2, \ldots, p_n)$ such that $G/F(G) \in \mathfrak{S}_{\hat{u}}$ or, equivalently $G/F(G) \in \mathscr{U}_{n-1}(G/F(G))$. If $G/F(G) \notin \mathscr{U}_{n-1}(G/F(G))$, then of course

$$G/\Phi(G) \notin \mathscr{U}_n(G/\Phi(G)).$$

But then, if $H/\Phi(G) \in \mathscr{U}_n(G/F(G))$, $H < G$, and by the minimality of G, $H/\Phi(G) \in \mathfrak{X}$: that is $\mathscr{U}_n(G/\Phi(G)) \subseteq \mathfrak{X}$. Since by Lemma 2·8

$$G/F(G) \in \{\mathfrak{s}, \mathfrak{N}_0\}\mathscr{U}_{n-1}(G/F(G)),$$

Lemma 2·7 gives

$$G/\Phi(G) \in \{\mathfrak{s}, \mathfrak{N}_0\}\mathscr{U}_n(G/\Phi(G)) \leqslant \mathfrak{X},$$

a contradiction. Hence $G/F(G) \in \mathfrak{S}_{u'}$ for some $u' = (p_2, \ldots, p_n)$, with $p_2 \neq p$ since G has nilpotent length n.

Thus we may suppose that the subgroup closed Fitting class $\mathfrak{X}$ under consideration is contained in some $\mathfrak{S}_u$, with $l(u) = n$; and that $\mathfrak{X}$ contains a group G with $G/\Phi(G) \notin \mathfrak{X}$. Among such groups let A be a minimal one: we claim that A is $\mathscr{L}(n-1)$-critical (that is, A has nilpotent length precisely n, and all the proper subgroups of A have nilpotent length at most $n-1$). For if A is not $\mathscr{L}(n-1)$-critical, choose an $\mathscr{L}(n-1)$-critical subgroup B of A. Then $B < A$, and by the minimality of A, $B/\Phi(B) \in \mathfrak{X}$. It follows from the structure theorems of Carter, Fischer and Hawkes (3) (see 4·1) that $F(B)/\Phi(B)$ is the unique minimal normal subgroup of $B/\Phi(B)$, and Lemma 2·9 then gives us $\mathfrak{X} = \mathfrak{S}_u$. But this means that $A/\Phi A \in \mathfrak{S}_u = \mathfrak{X}$, a contradiction.

To sum up: we may suppose that $\mathfrak{X} \subseteq \mathfrak{S}_u$ for some $u \in \mathscr{U}$ with $l(u) = n$, and that $\mathfrak{X}$ contains an $\mathscr{L}(n-1)$-critical group A such that $A/\Phi(A) \notin \mathfrak{X}$.

4. *Nilpotent length three.* A proof of Theorem 1·1 can now be given. To this end suppose that $\mathfrak{X}$ is a subgroup closed Fitting class of nilpotent length three which is not a formation. Since subgroup closed Fitting classes of nilpotent length two are formations (theorem 3·1 in (2)) the reduction of Section 3 allows us to suppose that $\mathfrak{X} \subseteq \mathfrak{S}_p\mathfrak{S}_q\mathfrak{S}_r$ where p, q and r are primes and $p \neq q \neq r$. Moreover there are groups in $\mathfrak{X}$ whose Frattini factor groups are not in $\mathfrak{X}$ and one of least order, G say, is $\mathscr{L}(2)$-critical. (Of course our aim is to show that no such group G exists.) The first step is to delineate the structure of G using results from (3). Some that we need are given now.

(4·1) (theorems 5·1 and 2·9 (iv) in (3)). *Each factor group of G has at most one complemented minimal normal subgroup.*

(4·2) (theorem 5·1 in (3)). *If H/K is a chief factor in G with $1 \leqslant K < H \leqslant \Phi(G)$ then $G/C_G(H/K)$ is nilpotent.*

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(4·3) (theorem 5·1 in (3)). *$G/G_{\mathfrak{N}^2}$ has trivial Frattini subgroup.*

For ease of quotation we also record an easy consequence of lemma 3·5 in (2).

(4·4) *The centre of G is trivial.*

The first part of the reduction of G is summed up in the following result.

(4·5) LEMMA. *Let $P = O_p(G)$ and let Q be a Sylow q-subgroup of G. Then*

(i) *P is a minimal normal subgroup of G.*

There exists a cyclic r-subgroup R of G normalizing Q such that

(ii) *QR complements P in G;*

(iii) *$Q/\Phi(Q)$ is irreducible and non-trivial for the action of R;*

(iv) *if $r = p$ then $|R| = p$;*

(v) *if $r \neq p$ R acts faithfully on P and $[\Phi(R), Q] = 1$.*

(vi) $C_{QR}(P) \leqslant \Phi(Q)$.

Proof. Since $PQ/P\Phi(Q)$ is normal and complemented in $G/P\Phi(Q)$ it is, by (4·1), a chief factor of G. It is moreover non-central since G is not metanilpotent. It follows that PQ/P is the nilpotent residual of G/P.

Now let H/K be a chief factor of G with $H \leqslant \Phi(G)$. Since P centralizes H/K it follows from (4·2) that $Q \leqslant C_G(H/K)$. Therefore, since $q \neq p$,

$$[Q, \Phi(G) \cap P] = 1.$$

The case $r = p$ can now be done quickly. In this case $\Phi(G) \cap P$ is hypercentral in G and so by (4·4) is trivial. But $\Phi(G) \cap P = 1$ means that P is elementary abelian and that every minimal normal subgroup of G contained in P is complemented. Hence P is a complemented minimal normal subgroup of G. Some complement of P contains Q. Write R for a Sylow p-subgroup of this complement; then, of course, R normalizes Q. Now $C_R(Q) = O_p(QR)$ which centralizes P. Hence $C_R(Q) = 1$ or else G has non-trivial centre, contradicting (4·4). It now follows that $PQ = G_{\mathfrak{N}^2}$ and then, from (4·3) and (4·1), that $R \cong G/PQ$ has order p.

We now take up the case $r \neq p$. In this case P is complemented. Again choose R to be a Sylow r-subgroup of a complement containing Q. Now $G_{\mathfrak{N}^2} = PQC_R(Q)$; for certainly $PQC_R(Q) \leqslant G_{\mathfrak{N}^2}$, and no larger subgroup of G is still metanilpotent. By (4·3) $R/C_R(Q)$ ($\cong G/G_{\mathfrak{N}^2}$) is elementary and so $\Phi(R) \leqslant C_R(Q)$. But then $R/\Phi(R)$ ($\cong G/PQ\Phi(R)$) is, by (4·1), of order r. The upshot is that R is cyclic, and $[\Phi(R), Q] = 1$. Also $C_R(P/\Phi(P)) = 1$. For R does not centralize $P/\Phi(P)$, or else G would be metanilpotent, and hence $C_R(P/\Phi(P)) \leqslant \Phi(R)$ so is central in G and therefore trivial by (4·4).

From theorem 3·1 in (2) and theorem 1 in (4) it follows that

$$\mathfrak{S}_p\mathfrak{S}_r \subseteq \mathfrak{X} \quad \text{and} \quad \mathfrak{S}_q\mathfrak{S}_r \subseteq \mathfrak{X}. \tag{4·6}$$

Since $P/\Phi(P)$ is complemented in $G/\Phi(P)$ it is a chief factor of G by (4·1). The proof of (4·5) will therefore be complete if we show that $\Phi(P) = 1$. To this end we need one last quote from (3): by its theorem 5·3, P is a special group: in particular $N = \Phi(P)$ is central in P.

Now let $D \cong R$ and form the twisted wreath product

$$W = P \operatorname{twr}_{QR}(QR \times D).$$

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As in the proof of Lemma 2·3, $W \in \mathfrak{X}$. Let Π be the base group of W. Then, as $(QR \times D)$-module over $GF(p)$,

$$\Pi/N^D \cong (P/N) \# GF(p) D.$$

Choose M, a faithful and irreducible direct summand of $GF(p) D$. Now $(P/N)_{\Phi(R)}$ and $M_{\Phi(D)}$ are direct sums of faithful submodules. It follows that $((P/N) \# M)_{\Phi(R) \times \Phi(D)}$ is a direct sum of modules of the form $U \# V$ where U is a faithful and irreducible $\Phi(R)$-module and V is a faithful and irreducible $\Phi(D)$-module. The kernels of the irreducible direct summands of $U \# V$ are diagonal subgroups of $\Phi(R) \times \Phi(D)$ different from $\Phi(R)$ and $\Phi(D)$. Hence we may write

$$(P/N) \# M = \bigoplus_{i=1}^{t} M_i,$$

where M_i is precisely the set of fixed points of some diagonal subgroup D_i^* of

$$\Phi(R) \times \Phi(D)$$

different from $\Phi(R)$ and $\Phi(D)$. Moreover since D_i^* is normal in $QR \times D$ each M_i admits $QR \times D$.

Choose a cyclic subgroup D_i of $R \times D$ so that $\Phi(D_i) = D_i^*$. We have

$$QD_i \leqslant QR \times D \quad \text{and} \quad (QD_i) D = QR \times D.$$

Now let S_i be a subgroup of W minimal with respect to satisfying

$$S_i/S_i \cap N^D = M_i QD_i.$$

Since $S_i \cap N^D \leqslant \Phi(S_i)$ and since $p \nmid |QD_i|$,

$$S_i \cap N^D \leqslant \Phi(S_i \cap \Pi).$$

Hence since $\Phi(D_i)$ centralizes $M_i = (S_i \cap \Pi) N^D/N^D \cong S_i \cap \Pi/S_i \cap N^D$, it also centralizes $S_i \cap \Pi$; and therefore

$$|S_i : C_{S_i}(S_i \cap N^D)| \leqslant r.$$

Lemma 2·3 now allows us to factor out of S_i one minimal normal subgroup contained in $S_i \cap N^D$ at a time to conclude that

$$S_i/S_i \cap N^D = M_i QD_i \in \mathfrak{X}.$$

Then $M_i(QR \times D) \in \mathfrak{N}_0\{M_i QD_i, M_i D\} \subseteq \mathfrak{X}$ (because $M_i D \in \mathfrak{X}$ by (4·6)) and so

$$((P/N) \# M)(QR \times D) \in \mathfrak{R}_0 \mathfrak{X} = \mathfrak{X}.$$

Finally $G/N \cong (P/N) QR \in \mathfrak{S}\mathfrak{X} = \mathfrak{X}$. But $N \leqslant \Phi(G)$ and the minimality of G is contradicted unless $N = 1$ in which case P is a minimal normal subgroup as required.

The proof of (vi) is easy: $C_{QR}(P)$ can have no Sylow r-subgroup because, being normal in QR it would be in R, contradicting (v) in the case $r \neq p$, and making G metanilpotent in the case $r = p$. Hence $C_{QR}(P) \leqslant Q$. Since $Q/\Phi(Q)$ is irreducible for R and $Q \nleq C_{QR}(P)$, $C_{QR}(P) \leqslant \Phi(Q)$. This completes the proof of Lemma 4·5.

We now embark on the last stage of the proof, the outcome of which will be that $G/\Phi(G) \in \mathfrak{X}$, contradicting our supposition that $\mathfrak{X}$ is not a formation.

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Let F_1 and F_2 be free groups in the variety $\mathfrak{V}$ generated by Q, both of rank d, where d is the minimum number of generators of Q. As in lemmas 6·1 and 6·3 of (2) we can define actions of R on F_1 and F_2 in such a way that there exist R-homomorphisms $F_1 \twoheadrightarrow Q$ and $F_2 \twoheadrightarrow Q$. In particular $F_1/\Phi(F_1)$ and $F_2/\Phi(F_2)$ are, as R-modules over $GF(q)$, isomorphic to $Q/\Phi(Q)$.

Now let F be the free $\mathfrak{V}$-product of F_1 and F_2. There is an action of R on F whose restrictions to F_1 and F_2 are those already defined (theorem 18·42 of (6)). Note that F is the free group of rank $2d$ in $\mathfrak{V}$. We select an R-subgroup F_0 in F as follows. Since $F_i \cap \Phi(F) = \Phi(F_i)$, $i = 1, 2$, there is an R-isomorphism

$$F/\Phi(F) \cong F_1/\Phi(F_1) \times F_2/\Phi(F_2);$$

and since $F_1/\Phi(F_1) \cong F_2/\Phi(F_2)$ as R-modules over $GF(q)$, there is an irreducible submodule M_0 of $F/\Phi(F)$ different from $F_1\Phi(F)/\Phi(F)$ and from $F_2\Phi(F)/\Phi(F)$. Let F_0 be a subgroup of F such that F_0R is minimal with respect to

$$F_0R\Phi(F)/\Phi(F) = M_0R.$$

Then $F_0 \cap \Phi(F) \leqslant \Phi(F_0R)$ and, since $r \neq q$, $F_0 \cap \Phi(F) \leqslant \Phi(F_0)$. But M_0 is a d-generator group and therefore so is F_0. In fact d generators for F_0 and d generators for F_1 together generate F and therefore do so freely ((6), 41·31 and 41·33). Consequently F_0 is free in $\mathfrak{V}$ of rank d and F is the free $\mathfrak{V}$-product of F_0 and F_1. Note also that $F_0/\Phi(F_0)$ as R-module is isomorphic to $Q/\Phi(Q)$.

Let P_0 be a faithful and irreducible module for $QR/\Phi(Q)$ over $GF(p)$, regarded as a QR-module by inflation. Lemma 2·4 asserts that $P_0QR \in \mathfrak{X}$ because

$$C_{QR}(P) \leqslant \Phi(Q) = C_{QR}(P_0).$$

We are going to inflate P_0 further as follows.

Let K be the normal closure of F_0 in F. Then $K \cap F_1 = 1$ by 18·36 in (6) and hence there is a homomorphism from $FR/\Phi(K)$ onto F_1R and hence one onto QR as well: suppose that its kernel is $M/\Phi(K)$. Then we may regard P_0 as an $FR/\Phi(K)$ module by inflation; and

$$P_0(FR/\Phi(K)) \in \mathfrak{X},$$

because $P_0(FR/\Phi(K))/M/\Phi(K) \cong P_0QR \in \mathfrak{X}$ and $P_0(FR/\Phi(K))/P_0 \in \mathfrak{S}_q\mathfrak{S}_r \subseteq \mathfrak{X}$ by (4·6) so $P_0(FR/\Phi(K)) \in \mathfrak{R}_0\mathfrak{X} = \mathfrak{X}$.

Finally,

$$P_0(F_0R\Phi(K)/\Phi(K)) \in \mathfrak{S}\mathfrak{X} = \mathfrak{X}.$$

Let L be the normal closure of F_1 in F. Then $L \cap F_0 = 1$, $LF_0 = F$, and hence $K = (L \cap K)F_0$ and $(L \cap K) \cap F_0 = 1$. Now

$$\Phi(K/L \cap K) = (L \cap K)\Phi(F_0)/(L \cap K),$$

whence $\Phi(K) \cap F_0 \leqslant \Phi(F_0)$. On the other hand the Frattini subgroup of a q-group is generated by qth powers and commutators, and so $\Phi(F_0) \leqslant \Phi(K)$. It follows that $F_0 \cap \Phi(K) = \Phi(F_0)$, and therefore that $P(F_0/\Phi(F_0))R \in \mathfrak{X}$. Lemma 2·9 now yields $\mathfrak{S}_p\mathfrak{S}_q\mathfrak{S}_r \subseteq \mathfrak{X}$ which means $G/\Phi(G) \in \mathfrak{X}$, the contradiction we sought. The proof of Theorem 1·1 is now complete.

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