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PROBLEMS AND SOLUTIONS

SOLUTIONS

00.1.1. *Determinant of a Skew Symmetric Matrix*—Solution. The statement of the problem is incorrect for n even, as noted by S. Puntanen, G.P.H. Styan, H.J. Werner, and G. Dhaene, because, e.g., the $n \times n$ zero matrix, which has 0 determinant for any n, is skew symmetric. The statement of the problem can be amended as follows:

00.1.1′. Let $S = \{s_{ij}\}$ be a skew symmetric matrix ($n \geq 2$). Show by induction that $|S| = 0$ for n odd and $|S| \geq 0$ for n even.

Puntanen, Styan, and Werner proposed a solution to the original problem for the case of n odd. Dhaene proposed a solution to the revised statement 00.1.1′. Both solutions are reported below. An excellent solution to 00.1.1′ was proposed independently by S. Zernov.

Solution 1, proposed by S. Puntanen, G.P.H. Styan, H.J. Werner, for n odd. If S is skew-symmetric, then $S = -S'$ by definition; here S' denotes the transpose of S. Therefore,

$$\det(S) = \det(-S') = \det(-S) = (-1)^n \det(S),$$

which for n odd becomes $\det(S) = -\det(S)$. Hence, necessarily $\det(S) = 0$, and so our proof is complete.

For further discussion of skew-symmetric matrices, see Gantmacher (1998, Vol. 1, Ch. 9, and Vol. 2, Ch. 11), Horn and Merino (1999), and Karro and Li (1997) who established the (nontrivial) result that the rank of a complex skew-symmetric matrix must be even (we note that a complex skew-symmetric matrix is not necessarily diagonalizable).

Solution 2, proposed by G. Dhaene. The induction starts by noting that

$$|0| = 0, \qquad \begin{vmatrix} 0 & -s_{21} \\ s_{21} & 0 \end{vmatrix} = s_{21}^2 \geq 0.$$

The property thus holds for $n = 1$ and $n = 2$. Suppose it holds up to some order $n = k - 1 \geq 2$. Now let $n = k$. We denote by S_{ij} the submatrix of S obtained by deleting the ith row and jth column and by $S_{ij,kl}$ the submatrix of S obtained by deleting the ith and kth rows and jth and lth columns. We can now write $|S|$ as

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$$
\begin{aligned}
|S| &= \sum_{i=2}^{k} (-1)^{i+1} s_{i1} |S_{i1}| \\
&= \sum_{i=2}^{k} (-1)^{i+1} s_{i1} \sum_{j=2}^{k} (-1)^{j} s_{1j} |S_{i1,1j}| \\
&= \sum_{i=2}^{k} \sum_{j=2}^{k} (-1)^{i+j} s_{i1} s_{j1} |S_{i1,1j}| \\
&= \sum_{i=2}^{k} s_{i1}^2 |S_{i1,1i}| + \sum_{i=2}^{k-1} \sum_{j=i+1}^{k} (-1)^{i+j} s_{i1} s_{j1} (|S_{i1,1j}| + |S_{j1,1i}|).
\end{aligned}
$$

It follows from the skew-symmetry of S that $(S_{j1,1i})' = -S_{i1,1j}$. Hence,

$$
|S| = \sum_{i=2}^{k} s_{i1}^2 |S_{i1,1i}|.
$$

The proof follows by noting that $S_{i1,1i}$, being a $(k-2) \times (k-2)$ principal submatrix of S, is skew-symmetric and thus has $|S_{i1,1i}| = 0$ for k odd and $|S_{i1,1i}| \geq 0$ for k even.

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Horn, R.A. & D.I. Merino (1999) The Jordan canonical forms of complex orthogonal and skew-symmetric matrices. *Linear Algebra and Its Applications* 302–303, 411–421.

Karro, J. & C.-K. Li (1997) A unified elementary approach to canonical forms of matrices. *SIAM Review* 39, 305–309.

00.1.2. *A Necessary and Sufficient Condition of a Sequence of Random Variables Converging to a Normal Distribution*—Solution,[1] proposed by N. Christopeit and H.J. Werner. As it stands, the necessary part of the theorem is incorrect, as it is shown by any distribution with compact support or by the normal and the exponential laws. Actually, the condition

$$
\lim_{x \to \infty} \frac{P(|X_1| > \lambda x)}{P(|X_1| > x)} = \lambda^{-2} \quad \text{for} \quad \lambda > 0
$$

of the theorem in question just means that the function $U(x) = P(|X_1| > x)$ is regularly varying at ∞ with index $\rho = -2$. To formulate a valid version of Liang's conjecture, consider the class $\mathcal{U}_\rho$ of tail probability functions U that are regularly varying at ∞ with index ρ, i.e., for which we have

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$$\lim_{x\to\infty}\frac{U(\lambda x)}{U(x)}=\lambda^{\rho}\quad\text{for all}\quad 0<\lambda<\infty.$$

(Because U is decreasing, only $\rho\le 0$ need to be considered.) The link between the functions in the class $\mathcal{U}_\rho$ and the distribution functions F that are attracted by the normal law, i.e., which satisfy

$$\frac{1}{B_n}\left(\sum_{i=1}^{n}X_i-A_n\right)\xrightarrow{\mathcal{D}}N(0,1) \tag{A}$$

for suitably chosen constants A_n and B_n, is provided by the following theorem (see Cramér, 1970).

THEOREM 0. (A) *holds if and only if*

$$\frac{x^2U(x)}{\displaystyle\int_{\{|t|\le x\}}t^2F(dt)}\to 0\quad\text{as}\quad x\to\infty. \tag{B}$$

Note that, for given F, the corresponding tail probability function U is uniquely determined by $U(x)=1-F(x)+F(-x-)$. A correct version of the conjecture may then be stated as follows.

THEOREM CW1. *A distribution function F with regularly varying tail probability function U belongs to the domain of attraction of the normal law if and only if $U\in\mathcal{U}_\rho$ for some $\rho\le -2$.*

Proof.

(a) Consider first the case where F does not possess a finite second moment. Let $G(x)=1-U(x)$ denote the distribution function of $|X_1|$. Then

$$U(x)=\int_x^\infty G(dt)\quad\text{and}\quad U_2(x):=\int_0^x t^2G(dt)=\int_{\{|t|\le x\}}t^2F(dt).$$

Because $U_2(\infty)=\infty$ and $U\in\mathcal{U}_\rho$, Theorem 2 in VIII.9 of Feller (1971) applies (with $\eta=0$, $\zeta=2$, $V_0=U$, $\alpha=-\rho$) to show

$$\gamma:=\lim_{x\to\infty}\frac{x^2U(x)}{U_2(x)}=-\frac{2+\rho}{\rho} \tag{1}$$

(with $\gamma=\infty$ if $\rho=0$). Hence, the limit is 0 if and only if $\rho=-2$.

(b) If F has finite variance, (B) and a fortiori (A) is trivially satisfied. On the other hand, U must belong to $\mathcal{U}_\rho$ for some $\rho\le -2$ by the following lemma. ■

LEMMA CW1. *If U is in $\mathcal{U}_\rho$ for some $\rho\le 0$, then the second moment of F is*

(a) *finite for $\rho<-2$,*
(b) *infinite for $-2<\rho\le 0$.*

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Proof. The function $L(x) = x^{-\rho}U(x)$ is slowly varying. As is well known (see the lemma in Feller, 1971, VIII.9), the integral

$$\int_1^\infty t^p L(t)\,dt = \int_1^\infty t^{p-\rho}U(t)\,dt$$

converges for $p < -1$ and diverges for $p > -1$. Because

$$E(X_1^2) = 2\int_0^\infty tU(t)\,dt,$$

the second moment ($p = \rho + 1$) is finite for $\rho < -2$ and infinite for $\rho > -2$. ■

Remark 1. For $\rho = -2$, both cases may occur, as is shown by the following examples (both for $x \geq 1$):

(i) $U(x) = x^{-2}$ has infinite variance.
(ii) $U(x) = [x^2(1 + \log^2 x)]^{-1}$ has finite variance.

As an extra of the proof, we obtain the following version of the conjecture.

COROLLARY CW1. *If U is regularly varying and U does not possess a finite variance, then F is attracted by the normal distribution if and only if the index of regular variation is $\rho = -2$.*

Remark 2. If U is *any* tail probability function (i.e., not necessarily regularly varying) such that (1) holds with $\gamma > 0$ ($\Leftrightarrow \rho > -2$), then it may be inferred (theorem cited in the proof of Theorem CW1) that U is actually regularly varying with index ρ. For $-2 < \rho < 0$, this in turn is equivalent to F belonging to the domain of attraction of a stable law with exponent $\alpha = -\rho$, provided the tails of F are balanced (Feller, 1971, XVII.5). In particular, for such distributions, regular variations of U turns out to be a necessary condition for F being attracted. In case $\gamma = 0$, however, it can only be concluded that U_2 varies slowly, but no inference about regular variation of U itself can be made. For an example of an atomic distribution with infinite variance and satisfying (B), but for which U is not regularly varying, see Problem 32 in VIII.10 of Feller (1971). On the other hand, by the same theorem cited in the proof of Theorem CW1, slow variation of U_2 is sufficient for (1) to hold with $\gamma = 0$. Therefore, a different valid variant of the conjecture could be the following: A non-degenerate distribution function F is attracted by the normal law if and only if U_2 varies slowly. This can already be found in Feller (1971, XVII.5, Corollary 1).

NOTE

1. The observation that the given condition is only sufficient but not necessary has been made also by C. Kleiber, referring to Ibragimov and Linnik (1971, Theorem 2.6.2). Independent solutions of the sufficiency part have been proposed by J. Hill and H. Liang, the poser of the problem.

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00.1.3. *A Relationship Satisfied by Two Representations of a Positive Semidefinite Matrix.*—Solution,[1] proposed by H. Neudecker and M. van de Velden. In the last line of the statement of the problem, the phrase "rank r and k" should be corrected to "rank k and k." Consider the singular-value decompositions

$$\mathbf{F} = \mathbf{S}_1\mathbf{\Lambda}_1\mathbf{T}_1', \quad \text{and} \quad \mathbf{G} = \mathbf{S}_1\mathbf{\Lambda}_1\mathbf{R}_1',$$

with $\mathbf{S}_1'\mathbf{S}_1 = \mathbf{T}_1'\mathbf{T}_1 = \mathbf{R}_1'\mathbf{R}_1 = \mathbf{I}_r$, and diagonal positive definite $\mathbf{\Lambda}_1$.

For $r \leq k \leq n$, define

$$\mathbf{S} \equiv (\mathbf{S}_1, \mathbf{S}_2), \quad \text{where } \mathbf{S} \text{ is of order } m \times k,$$

$$\mathbf{T} \equiv (\mathbf{T}_1, \mathbf{T}_2), \quad \text{where } \mathbf{T} \text{ is of order } n \times k, \text{ and } \mathbf{T}'\mathbf{T} = \mathbf{I}_k,$$

$$\mathbf{R} \equiv (\mathbf{R}_1, \mathbf{R}_2), \quad \text{where } \mathbf{R} \text{ is of order } n \times k, \text{ and } \mathbf{R}'\mathbf{R} = \mathbf{I}_k,$$

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{\Lambda}_1 & \\ & \mathbf{0} \end{pmatrix}, \quad \text{where } \mathbf{\Lambda} \text{ is of order } k \times k.$$

It follows that

$$\mathbf{F} = \mathbf{S}\mathbf{\Lambda}\mathbf{T}' = \mathbf{S}\mathbf{\Lambda}\mathbf{R}'\mathbf{R}\mathbf{T}' = \mathbf{S}_1\mathbf{\Lambda}_1\mathbf{R}_1'\mathbf{R}\mathbf{T}' = \mathbf{G}\mathbf{Q}_k,$$

where

$$(\mathbf{Q}_k\mathbf{Q}_k')^2 = (\mathbf{R}\mathbf{T}'\mathbf{T}\mathbf{R}')^2 = \mathbf{R}\mathbf{R}' = \mathbf{Q}_k\mathbf{Q}_k',$$

and

$$\operatorname{rank}(\mathbf{Q}_k) = \operatorname{rank}(\mathbf{Q}_k\mathbf{Q}_k') = \operatorname{rank}(\mathbf{R}\mathbf{R}') = k.$$

Note that, for $r = n$ or $k = n$,

$$\mathbf{Q}_k\mathbf{Q}_k' = \mathbf{Q}_k'\mathbf{Q}_k = \mathbf{I}_n.$$

Hence, one can always find an orthogonal matrix $\mathbf{Q}$.

The solution for $k = n$ is known and proofs can be found in Srivastava and Khatri (1979, p. 20) and Muirhead (1982, pp. 589–591). Young et al. (1999) presented the result without proof.

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NOTE

1. An excellent solution has been proposed independently by G. Dhaene.

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