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MONITORING STRUCTURAL CHANGES WITH THE GENERALIZED FLUCTUATION TEST

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In this paper we introduce the generalized fluctuation test for monitoring structural changes and establish a result characterizing the limiting behavior of this class of tests. As applications of the generalized fluctuation test, tests based on the maximum and range of the fluctuation of moving estimates are proposed. We also derive the boundary functions for the proposed tests and tabulate simulated critical values. Our simulations indicate that these tests compare favorably with the recursive-estimates-based test considered by Chu, Stinchcombe, and White (1996, *Econometrica* 64, 1045–1065) when a change occurs late.

1. INTRODUCTION

The structural change problem has long been an important research topic in the statistics and econometrics literature. Much research effort has been devoted to tests for structural changes and estimation of change points in a given sample; recent results include, e.g., Andrews (1993), Andrews and Ploberger (1994), Bai (1994, 1995, 1996), Kuan and Hornik (1995), and Bai and Perron (1998). These methods are all "retrospective" because they are designed to examine what happened in historical data sets. In practice, one may want to monitor new observations to see if a change occurs. Such forward-looking methods are closely related to the sequential test in the statistics literature but receive little attention in econometrics; Chu, Stinchcombe, and White (1996) is an exception.

Chu et al. (1996) propose two tests for monitoring potential structural changes. In their fluctuation test, when new observations arrive, estimates are computed sequentially from all available data (historical sample together with newly arriving data) and compared to the estimate based only on the historical sample. The hypothesis of no change is then rejected if the difference between these

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two estimates becomes too large. This testing procedure aims at detecting a change when new data arrive, whereas the (retrospective) fluctuation test of Ploberger, Krämer, and Kontrus (1989) checks whether a change exists in historical data. Moreover, the constant critical values for the latter cannot be used for monitoring. The law of iterated logarithm implies that, with probability one, the monitoring statistics would eventually exceed constant boundaries and hence signal a change even when there is none (see, e.g., Robbins, 1970). Thus, a challenging task is to find suitable boundary functions such that monitoring tests can maintain proper sizes.

In this paper we introduce a class of tests for monitoring structural changes. This class of tests is referred to as the generalized fluctuation test and includes the fluctuation test of Chu et al. (1996) as a special case. We extend Kuan and Hornik (1995) to establish a general result characterizing the limiting behavior of this test. As applications of the generalized fluctuation test, we propose to monitor changes using the maximum and range of the fluctuation of moving estimates. It is shown that the distributions of these tests are determined by the increments of the generalized Brownian bridge and that the corresponding boundary functions for the proposed tests grow approximately at the rate $\sqrt{\log t}$. By contrast, the boundary functions for the Chu et al. tests grow much faster (approximately at the rate t). As such, the Chu et al. tests are less sensitive to a change occurring late in the monitoring period, but the proposed tests are not. Our simulations confirm that the proposed test indeed detects a late change much more quickly.

The remainder of the paper is organized as follows. In Section 2 we extend the results of Kuan and Hornik (1995) and establish an asymptotic result for the generalized fluctuation test. In Section 3 we discuss recursive-estimates-based monitoring tests and show that the Chu et al. (1996) test is a special case of the generalized fluctuation test. In Section 4 we propose two moving-estimates-based monitoring tests and derive corresponding boundary functions. Simulation results are reported in Section 5. Section 6 concludes the paper. All mathematical proofs are deferred to the Appendix.

2. EXTENSION OF THE GENERALIZED FLUCTUATION TEST

Consider the multiple regression model

$$y_i = x_i' \beta_i + \epsilon_i, \qquad i = 1, \ldots, T, T+1, \ldots, \tag{1}$$

where x_i is the $n \times 1$ vector of explanatory variables. Suppose we are currently at time T and have observed historic data $(y_i, x_i')'$, $i = 1, \ldots, T$. We take as given that the parameter vector β_i was constant during the history period (i.e., $\beta_i \equiv \beta_0$ for $i = 1, \ldots, T$, where β_0 is unknown). We are interested in testing the null hypothesis that β_i remains constant (i.e., $\beta_i = \beta_0$ for all i) against the alternative that β_i changes at some unknown point in the future (i.e., $\beta_i \neq \beta_0$ for some $i > T$).

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Kuan and Hornik (1995) introduce the generalized fluctuation test, which provides a unified framework for analyzing tests for structural changes (or parameter constancy). They consider empirical processes that consist of two additive components: one satisfies a functional central limit theorem (FCLT), and one is roughly a "straight line" under the null hypothesis. For such a process Y_T, one can apply a suitable linear operator $\mathcal{L}_T$ to annihilate its straight line component so that the resulting process $\mathcal{L}_T Y_T$ is essentially governed by the FCLT under the null hypothesis. On the other hand, this process should exhibit excessive fluctuations when the null hypothesis is false. Tests for parameter constancy can then be constructed as $\lambda(\mathcal{L}_T Y_T)$, where λ is a functional "measuring" the amount of fluctuation of a process. This test includes the fluctuation (recursive-estimates) test of Ploberger et al. (1989), the moving-estimates test of Chu, Hornik, and Kuan (1995b), the OLS-CUSUM test of Ploberger and Krämer (1992), and the OLS-MOSUM test of Chu, Hornik, and Kuan (1995a) as special cases.

The aforementioned tests are all retrospective because they focus on testing parameter constancy in historical samples. We now want to extend the generalized fluctuation test so that it can be applied for monitoring future structural changes. In what follows, let $\to^{\mathbb{P}}$ denote convergence in probability, $\Rightarrow$ denote weak convergence of associated probability measures, $\mathbf{W}$ an n-dimensional, standard Wiener process, and $[a]$ the integer part of the real number a. For more details of the modes of convergence and related topics we refer to Billingsley (1968).

As in Krämer, Ploberger, and Alt (1988), we impose the following conditions on data.

(M1) $\{\epsilon_i\}$ is a martingale difference sequence with respect to $\{\mathcal{F}^i\}$, the σ-algebra generated by $\{(x_{t+1}, \epsilon_t), t \leq i\}$ such that $\mathbb{E}(\epsilon_i^2 | \mathcal{F}^{i-1}) = \sigma^2$.

(M2) $\{x_i\}$ is such that $\limsup_{T\to\infty} T^{-1} \sum_{i=1}^{T} \mathbb{E}|x_i|^{2+\delta} < \infty$, and

$$Q_{[Tt]} = \frac{1}{[Tt]} \sum_{i=1}^{[Tt]} x_i x_i' \to^{\mathbb{P}} Q, \tag{2}$$

uniformly in $0 < c < t$ for some c, where Q is a nonstochastic, positive definite matrix.

Under these conditions, a FCLT holds:

$$\left(\frac{1}{\sigma\sqrt{T}} Q^{-1/2} \sum_{i=1}^{[Tt]} x_i \epsilon_i, \qquad t \geq 0 \right) \Rightarrow \mathbf{W}. \tag{3}$$

The FCLT (3) also holds if σ^2 and Q are replaced by suitable estimators. For example, Q_T is consistent for Q by (2), and

$$\hat{\sigma}_T^2 = \frac{1}{T} \sum_{t=1}^{T} (y_t - x_t' \hat{\beta}_T)^2$$

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is consistent for σ^2, where $\hat{\beta}_T$ is the ordinary least squares (OLS) estimator of the model (1) computed from the historical sample.

Consider the piecewise constant process Y_T with jump points

$$Y_T\left(\frac{k}{T}\right) = \frac{1}{\hat{\sigma}_T\sqrt{T}} Q_T^{1/2} Q_k^{-1} \sum_{i=1}^{k} x_i y_i, \qquad k = 1,2,\ldots. \tag{4}$$

Hence, Y_T is in $D([0,\infty)^n)$, the space of functions that are right continuous with left hand limits on $[0,\infty)^n$; D is endowed with the Skorohod topology. Also observe that the key ingredients of Y_T are cumulative sums of data. In a monitoring process, data will arrive indefinitely so that the entire sample will be expanding. Using the number of observations T in the history period as the normalization factor, Y_T on $[0,1]$ thus corresponds to the historical sample, as in Kuan and Hornik (1995), whereas Y_T on $(1,\infty)$ corresponds to the monitoring period.

Under the null hypothesis, we have for $t \geq 0$

$$Y_T(t) = \sqrt{T}\,\frac{[Tt]}{\hat{\sigma}_T T} Q_T^{1/2}\beta_0 + \frac{1}{\hat{\sigma}_T\sqrt{T}} Q_T^{1/2} Q_{[Tt]}^{-1} \sum_{i=1}^{[Tt]} x_i \epsilon_i. \tag{5}$$

It is readily seen that the first term in (5) is roughly a "straight line" passing through the origin and that the second term satisfies the FCLT (3). Other empirical processes may also be employed to construct tests (see, e.g., Kuan, 1998). We consider the tests of the form $\lambda(\mathcal{L}_T Y_T)$, where Y_T is as defined in (4) and $\mathcal{L}_T$ and λ satisfy the conditions that follow.

(G1) $\mathcal{L}_T$ is a linear operator on $D([0,\infty)^n)$ such that $\mathcal{L}_T \iota_T = 0$, where $\iota_T(t) = [Tt]/T$.

(G2) λ is a positively homogeneous functional on $D([0,\infty)^n)$ that is continuous with respect to the Skorohod topology.

These conditions are virtually the same as those in Kuan and Hornik (1995). The difference is that we work on $D([0,\infty)^n)$, whereas Kuan and Hornik (1995) consider $D([0,1]^n)$.

The result that follows is an extension of Theorem 2.1 in Kuan and Hornik (1995) and gives a complete characterization of the limiting behavior of $\lambda(\mathcal{L}_T Y_T)$.

THEOREM 2.1 *Given the data generating process* (DGP) (1) *with*

$$\beta_i = \beta_0 + T^{-\delta} g(i/T), \tag{6}$$

where $\delta \leq 1/2$ *and* g *is a vector-valued function of bounded variation on* $[0,\infty)$ *that is identically zero on* $[0,1]$, *suppose that* (M1) *and* (M2) *hold. If* $\mathcal{L}$ *is a linear operator such that* $\mathcal{L}_T Y_T = \mathcal{L} Y_T + o_{\mathbb{P}}(1)$, *then for* $\delta = 1/2$,

$$\lambda(\mathcal{L}_T Y_T) \Rightarrow \lambda(\mathcal{L}(\mathbf{W} + \sigma^{-1} Q^{1/2} J g)),$$

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where Jg is the antiderivative of g: $Jg(t) = \int_0^t g(u)\,du$, *and for* $\delta < 1/2$,

$T^{\delta-1/2}\lambda(\mathcal{L}_T Y_T) \to^{\mathbb{P}} \lambda(\mathcal{L}(\sigma^{-1}Q^{1/2}Jg))$.

The first result of Theorem 2.1 indicates that under local alternatives of order $T^{-1/2}$, $\lambda(\mathcal{L}_T Y_T)$ has nontrivial local power, provided that $\mathcal{L}Jg \neq 0$; the second conclusion says that for nonlocal alternatives, the statistic diverges whenever $\lambda(\mathcal{L}Jg) > 0$ and hence is consistent against this class of alternatives. Under the null hypothesis, g is identically zero so that $\mathcal{L}Jg = 0$, and consequently,

$\lambda(\mathcal{L}_T Y_T) \Rightarrow \lambda(\mathcal{L}\mathbf{W})$.

The limiting distribution of $\lambda(\mathcal{L}_T Y_T)$ is therefore determined by the behavior of $\mathcal{L}\mathbf{W}$.

Note that we only consider structural breaks in the coefficients β of the model, whereas the noise variance σ is assumed to remain constant over time. Under the null, a change of σ affects only the last term in (5). Consider $\mathbb{E}(\epsilon_i^2|\mathcal{F}^{i-1}) = \sigma_i^2$ where $\sigma_i = \sigma s_i$, $s_i > 0$ such that the FCLT

$$\left(\frac{1}{\sigma\sqrt{T}}\,Q^{-1/2}\sum_{i=1}^{[Tt]} x_i\,\frac{\epsilon_i}{s_i},\quad t \geq 0\right) \Rightarrow \mathbf{W}$$

holds. Hence, if $s_i \equiv 1$ for $i < i_s$ and $s_i \equiv s$ for $i \geq i_s$ (shift of variance at $i_s > T$), then the limiting process of (3) becomes $s\mathbf{W}$ after i_s. Further we have $\lambda(\mathcal{L}s\mathbf{W}) = s\lambda(\mathcal{L}\mathbf{W})$, i.e., s directly modifies the test statistic. For $s < 1$ the probability of a type one error decreases; for $s > 1$ it increases.

3. MONITORING WITH RECURSIVE ESTIMATES

To illustrate the generalized fluctuation test $\lambda(\mathcal{L}_T Y_T)$, we now consider the tests based on recursive estimates and show that the Chu et al. (1996) fluctuation test is a special case of this class of tests.

Let $\hat{\beta}_k$ denote the recursive OLS estimator for (1) based on the information up to time k:

$$\hat{\beta}_k = \left(\sum_{i=1}^{k} x_i x_i'\right)^{-1}\sum_{i=1}^{k} x_i y_i, \qquad k = n, n+1, \ldots, T, T+1, \ldots.$$

We can see from (4) that the empirical process Y_T may also be expressed as

$$Y_T\left(\frac{k}{T}\right) = \frac{k}{\hat{\sigma}_T\sqrt{T}}\,Q_T^{1/2}\hat{\beta}_k. \tag{7}$$

This shows that monitoring tests can be constructed using recursive estimates.

In the Chu et al. (1996) monitoring procedure, a recursive estimate is computed when new data arrive and then compared with $\hat{\beta}_T$, the estimate based on the historical sample. If their difference is too large, the null hypothesis is rejected and the monitoring procedure stops; otherwise, the monitoring proce-

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dure continues. Specifically, the Chu et al. statistic that follows must be computed sequentially:

$$\frac{k}{\hat{\sigma}_T\sqrt{T}}\|Q_T^{1/2}(\hat{\beta}_k - \hat{\beta}_T)\|, \qquad k = T+1, T+2, \ldots,$$

where $\|\cdot\|$ denotes the maximum norm. The null hypothesis would be rejected if there is at least one statistic exceeding proper boundaries. To characterize the limiting behavior of these monitoring statistics, we may write the Chu et al. fluctuation test as

$$\text{max-}RE_T(\tau) = \max_{k=T+1,\ldots,[T\tau]} \frac{k}{\hat{\sigma}_T\sqrt{T}}\|Q_T^{1/2}(\hat{\beta}_k - \hat{\beta}_T)\|, \tag{8}$$

where the period from time $T+1$ through $[T\tau]$, $\tau > 1$, is the expected monitoring period. When $\tau = \infty$, the monitoring procedure might last indefinitely.

For a function f, define the following maximal functional:

$$\max(f;[\tau_1,\tau_2]) := \max_{\tau_1 \le t \le \tau_2} \|f(t)\|.$$

It is not too difficult to see that (8) can also be written compactly as

$$\text{max-}RE_T(\tau) = \max(\mathcal{L}_T Y_T;[1,\tau]),$$

with $\mathcal{L}_T f(t) := f^0([Tt]/T)$ and $f^0(t) := f(t) - tf(1)$. Clearly, the operator $\mathcal{L}_T$ satisfies (G1), and the functional $\max(\cdot)$ satisfies (G2). Therefore, (8) is a special case of the generalized fluctuation test so that Theorem 2.1 applies. The result that follows is now immediate and complements the analysis of Chu et al. (1996).

COROLLARY 3.1. *Given the* DGP (1) *with* (6), *suppose that conditions* (M1) *and* (M2) *hold. Then for* $\delta = 1/2$,

$$\text{max-}RE_T(\tau) \Rightarrow \max(\mathcal{L}(\mathbf{W} + \sigma^{-1}Q^{1/2}Jg);[1,\tau]),$$

and for $\delta < 1/2$,

$$T^{\delta-1/2}[\text{max-}RE_T(\tau)] \to^{\mathbb{P}} \max(\mathcal{L}(\sigma^{-1}Q^{1/2}Jg);[1,\tau]),$$

where $\mathcal{L}f(t) = f^0(t)$.

Under the null hypothesis,

$$\text{max-}RE_T(\tau) \Rightarrow \max(\mathcal{L}\mathbf{W};[1,\tau]) = \max(\mathbf{W}^0;[1,\tau]),$$

where $\mathbf{W}^0$ is the generalized Brownian bridge on $[0,\infty)$, as shown by Chu et al. (1996). Note that the covariance function of $\mathbf{W}^0$ is $t(s-1)I_n$ for $t \ge s \ge 1$,

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where I_n is the $n \times n$ identity matrix, and that the variance of $\mathbf{W}^0$ is $t(t-1)I_n$. For a suitable boundary function q,

$$\lim_{T\to\infty} \mathbb{P}\left\{ \frac{k}{\hat{\sigma}_T\sqrt{T}} \|Q_T^{1/2}(\hat{\beta}_k - \hat{\beta}_T)\| < q(k/T), \quad \text{for all } T+1 \le k \le [T\tau] \right\}$$
$$= \mathbb{P}\{\|\mathbf{W}^0(t)\| < q(t), \quad \text{for all } 1 \le t \le \tau\}.$$

The limiting distribution of max-$RE_T(\tau)$ is thus determined by the boundary-crossing probability of $\mathbf{W}^0$ on $[1,\tau]$. Chu et al. considered $\tau = \infty$ and obtained the following result from Robbins and Siegmund (1970). For a one-dimensional Brownian bridge W^0,

$$\mathbb{P}\{|W^0(t)| < q(t), \quad \text{for all } t > 1\} = 2[\Phi(a) - a\phi(a)] - 1,$$

where Φ and ϕ are, respectively, the distribution and density functions of the standard normal random variable and

$$q(t) := \sqrt{t(t-1)\left[a^2 + \log\left(\frac{t}{t-1}\right)\right]}. \tag{9}$$

Choosing $a^2 = 7.78$ and $a^2 = 6.25$ gives 95% and 90% monitoring boundaries, respectively. See Chu et al. (1996) for more details.

From the preceding discussion we can see that numerous monitoring tests can be constructed by choosing suitable operators and/or functionals. The limiting behaviors of the tests thus constructed can be easily characterized by Theorem 2.1. As another example, consider the same operator as in the Chu et al. (1996) test and the range functional:

$$\text{range}(f;[\tau_1,\tau_2]) := \max_{i=1,\ldots,n}\left(\max_{\tau_1\le t\le\tau_2}[f(t)]_i - \min_{\tau_1\le s\le\tau_2}[f(s)]_i\right),$$

where $f(t)_i$ is the ith element of $f(t)$. Note that monitoring is plausible because the range of a function can be computed sequentially:

$$\text{range}(f;[\tau_1,\tau_2]) = \max_{\tau_1\le t\le\tau_2} \text{range}(f;[\tau_1,t]).$$

Employing this functional yields another monitoring test based on recursive estimates:

$$\text{range-}RE_T(\tau) = \max_{i=1,\ldots,n} \frac{1}{\hat{\sigma}_T\sqrt{T}}\Big(\max_{k=T+1,\ldots,[T\tau]}[Q_T^{1/2}(\hat{\beta}_k - \hat{\beta}_T)]_i$$
$$- \min_{l=T+1,\ldots,[T\tau]}[Q_T^{1/2}(\hat{\beta}_l - \hat{\beta}_T)]_i\Big)$$
$$= \text{range}(\mathcal{L}_T Y_T;[1,\tau]).$$

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In accordance with Corollary 3.1, the null distribution of range-RE_T is determined by the range of the generalized Brownian bridge; its boundary function is unknown, however.

4. MONITORING WITH MOVING ESTIMATES

From the simulation results Chu et al. (1996) found that their fluctuation test requires, on the average, a longer time to detect a change occurring late in the monitoring period. A major reason is that its boundary function grows too fast during the monitoring period. Chu et al. also noted that "The growing variance of W^0, which induces an increasing monitoring boundary, is a problem inherent to this type of monitoring" (p. 1061). In this section, we propose different monitoring schemes.

Define the moving OLS estimates computed from windows of a constant size $[Th]$, where $0 < h \leq 1$ and $[Th] > n$, as

$$\tilde{\beta}_T(k,[Th]) = \left(\sum_{i=k-[Th]+1}^{k} x_i x_i' \right)^{-1} \sum_{i=k-[Th]+1}^{k} x_i y_i, \qquad k = [Th], [Th]+1, \ldots.$$

Consider the simple example that data (without noise) shift from 2.0 to 2.8 at time $t = 100$. In Figure 1, we use the solid line to represent $\tilde{\beta}_T(k,[Th]) - \hat{\beta}_T$ with $h = 1$ and the dashed line for $\hat{\beta}_k - \hat{\beta}_T$. It can be seen that the former reacts more quickly to this change. This motivates us to consider monitoring tests based on moving estimates.

Analogous to the recursive-estimates-based tests discussed earlier, we can base tests on the maximum and range of the fluctuation of moving estimates:

$$\text{max-}ME_{T,h}(\tau) = \max_{k=T+1,\ldots,[T\tau]} \frac{[Th]}{\hat{\sigma}_T \sqrt{T}} \| Q_T^{1/2} (\tilde{\beta}_T(k,[Th]) - \hat{\beta}_T) \|, \tag{10}$$

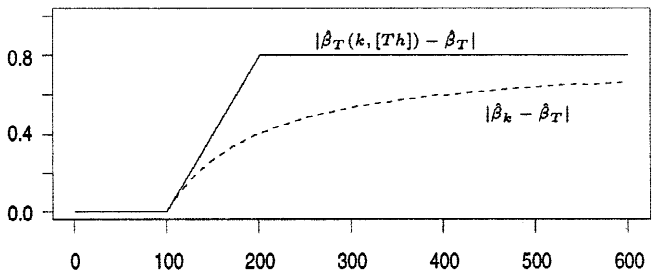


FIGURE 1. Comparison of test statistics in a simple location model without noise.

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$$\text{range-}ME_{T,h}(\tau) = \max_{i=1,\ldots,n} \frac{[Th]}{\hat{\sigma}_T\sqrt{T}} \Big(\max_{k=T+1,\ldots,[T\tau]} [Q_T^{1/2}(\tilde{\beta}_T(k,[Th]) - \hat{\beta}_T)]_i - \min_{k=T+1,\ldots,[T\tau]} [Q_T^{1/2}(\tilde{\beta}_T(k,[Th]) - \hat{\beta}_T)]_i \Big); \tag{11}$$

(cf. Chu et al., 1995b; Kuan and Hornik, 1995).

We first show that (10) and (11) are special cases of the generalized fluctuation test. For $t > 1$, write $Q_{[Tt],[Th]} := [Th]^{-1} \sum_{i=[Tt]-[Th]+1}^{[Tt]} x_i x_i'$. Then,

$$\begin{aligned}
&\frac{[Th]}{\hat{\sigma}_T\sqrt{T}} Q_T^{1/2} \tilde{\beta}_T([Tt],[Th]) \\
&\quad = \frac{1}{\hat{\sigma}_T\sqrt{T}} Q_T^{1/2} Q_{[Tt],[Th]}^{-1} \left(\sum_{i=1}^{[Tt]} x_i y_i - \sum_{i=1}^{[Tt]-[Th]} x_i y_i \right) \\
&\quad = \frac{1}{\hat{\sigma}_T\sqrt{T}} Q_T^{1/2} \Bigg[Q_{[Tt]}^{-1} \sum_{i=1}^{[Tt]} x_i y_i - Q_{[Tt]-[Th]}^{-1} \sum_{i=1}^{[Tt]-[Th]} x_i y_i \\
&\qquad\qquad + (Q_{[Tt],[Th]}^{-1} - Q_{[Tt]}^{-1}) \sum_{i=1}^{[Tt]} x_i y_i \\
&\qquad\qquad - (Q_{[Tt],[Th]}^{-1} - Q_{[Tt]-[Th]}^{-1}) \sum_{i=1}^{[Tt]-[Th]} x_i y_i \Bigg] \\
&\quad = Y_T\left(\frac{[Tt]}{T}\right) - Y_T\left(\frac{[Tt]-[Th]}{T}\right) + o_{\mathbb{P}}(1),
\end{aligned}$$

where the last equality follows from the definition of Y_T and (2). It follows that

$$\begin{aligned}
&\frac{[Th]}{\hat{\sigma}_T\sqrt{T}} Q_T^{1/2}(\tilde{\beta}_T(k,[Th]) - \hat{\beta}_T) \\
&\quad = Y_T\left(\frac{k}{T}\right) - Y_T\left(\frac{k-[Th]}{T}\right) - \frac{[Th]}{T} Y_T(1) + o_{\mathbb{P}}(1) \\
&\quad = Y_T^0\left(\frac{k}{T}\right) - Y_T^0\left(\frac{k-[Th]}{T}\right) + o_{\mathbb{P}}(1).
\end{aligned}$$

Thus, (10) and (11) can be expressed as

$$\begin{aligned}
\text{max-}ME_{T,h}(\tau) &= \max(\mathcal{L}_{T,h} Y_T; [1,\tau]), \\
\text{range-}ME_{T,h}(\tau) &= \text{range}(\mathcal{L}_{T,h} Y_T; [1,\tau]),
\end{aligned}$$

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where

$$\mathcal{L}_{T,h} f(t) = f^0\left(\frac{[Tt]}{T}\right) - f^0\left(\frac{[Tt]-[Th]}{T}\right) + o_{\mathbb{P}}(1).$$

Applying Theorem 2.1 we have the following corollary.

COROLLARY 4.1. *Given the* DGP (1) *with* (6), *suppose that conditions* (M1) *and* (M2) *hold. Then for* $\delta = 1/2$ *we have*

$$\text{max-}ME_{T,h}(\tau) \Rightarrow \max(\mathcal{L}_h(\mathbf{W} + \sigma^{-1}Q^{1/2}Jg);[1,\tau]),$$

$$\text{range-}ME_{T,h}(\tau) \Rightarrow \text{range}(\mathcal{L}_h(\mathbf{W} + \sigma^{-1}Q^{1/2}Jg);[1,\tau]),$$

and for $\delta < 1/2$,

$$T^{\delta-1/2}[\text{max-}ME_{T,h}(\tau)] \to^{\mathbb{P}} \max(\|\mathcal{L}_h(\sigma^{-1}Q^{1/2}Jg)\|;[1,\tau]),$$

$$T^{\delta-1/2}[\text{range-}ME_{T,h}(\tau)] \to^{\mathbb{P}} \text{range}(\mathcal{L}_h(\sigma^{-1}Q^{1/2}Jg);[1,\tau]),$$

where $\mathcal{L}_h f(t) = f^0(t) - f^0(t-h)$.

Corollary 4.1 implies that under the null hypothesis

$$\begin{aligned}&\text{max-}ME_{T,h}(\tau) \Rightarrow \max(\mathcal{L}_h\mathbf{W};[1,\tau]),\\ &\text{range-}ME_{T,h}(\tau) \Rightarrow \text{range}(\mathcal{L}_h\mathbf{W};[1,\tau]),\end{aligned} \tag{12}$$

where $\mathcal{L}_h\mathbf{W}(t) = \mathbf{W}^0(t) - \mathbf{W}^0(t-h)$, the increments (with step size h) of the Brownian bridge. It can be verified that the covariance function of $\mathcal{L}_h\mathbf{W}$ is, for $t \geq s \geq 1$,

$$\mathbb{E}[\mathcal{L}_h\mathbf{W}(t)\mathcal{L}_h\mathbf{W}(s)'] = \begin{cases} h^2 I_n, & \text{if } t-s \geq h,\\ (h^2 + h + s - t)I_n, & \text{if } t-s < h,\end{cases}$$

which depends on $t - s$, so that the variance is a constant $(h^2 + h)I_n$. Thus, $\mathcal{L}_h\mathbf{W}$ is a stationary Gaussian process. This is in sharp contrast with $\mathbf{W}^0$, which has a growing variance.

It remains to find proper boundary functions for the monitoring tests (10) and (11). When the expected monitoring period is finite ($\tau < \infty$), any (strictly positive) function $zq(t)$ can serve as a boundary, where z is a scaling factor. By choosing a proper function $q(t)$, a monitoring test can be made more powerful in the beginning or at the end of the monitoring phase. For example, to make monitoring more powerful at the beginning, one may want to have $q(t)$ that is decreasing for small t and eventually increasing for large t. For a given q, the scaling factor z can be used to determine the size of the test. For an infinite monitoring period ($\tau = \infty$), we need an analytic result on the growth rate of the limiting process, which enables us to determine the correct functional form of the boundary functions. With a boundary function whose growth rate is too slow, the monitoring test will commit the type one error almost surely, regard-

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less of any scaling factor z. For a boundary function whose growth rate is too large, the test will lose power for large t.

The most flexible monitoring scheme would allow for the specification of a prior distribution on the time of the structural break t. A prior with more mass on the beginning of the monitoring period would give more power against early breaks; this type of prior is (indirectly) induced by Chu et al. (1996) monitoring. Power against late breaks could be achieved by shifting the mass of the prior toward the end of the monitoring period. However, we currently see no general way of solving this task and restrict ourselves to uninformative priors with almost constant power over the complete monitoring period.

From Révész (1982) we get that the asymptotic growth rate of the supremum of the increments of the Wiener process is $\sqrt{\log t}$ almost surely. Hence proper boundary functions with constant hitting probability over time for the increments of the Brownian bridge should grow approximately at this rate (at least for large t). This growth rate is much slower than (9), which grows approximately at the rate t. In Theorem 4.2, which follows, we show that one can indeed construct boundary functions using the asymptotic growth rate.

THEOREM 4.2. *Let $W^0(t)$ be a one-dimensional Brownian bridge on $[0,\infty)$. Then for $0 < h \leq 1$ and $\tau > 0$ (including $\tau = \infty$),*

$$\mathbb{P}\{|W^0(t) - W^0(t-h)| < z(h)\sqrt{2\log_+ t},\ \forall 1 < t < \tau\} = F_1(z(h),\tau),$$

$$\mathbb{P}\{\text{range}(W^0(s) - W^0(s-h);[1,t]) < z(h)\sqrt{2\log_+ t},\ \forall 1 < t < \tau\} = F_2(z(h),\tau),$$

where

$$\log_+ t := \begin{cases} 1, & \text{if} \quad t \leq e, \\ \log t, & \text{if} \quad t > e, \end{cases}$$

and $F_i(z(h),\infty)$, $i = 1,2$ are zero for $z \leq \sqrt{h}$ and positive otherwise.

The preceding boundaries $z(h)\sqrt{2\log t}$ are asymptotically optimal for uninformative priors on the time of the structural break. Given a boundary that grows faster, it is less likely that this boundary will be crossed for large t. On the other hand, there can be no boundary function growing more slowly than $\sqrt{\log t}$; any such function will eventually become smaller than $z(h)\sqrt{2\log t}$ such that by Theorem 4.2 the boundary will be hit with probability one. Thus, any boundary for $\tau = \infty$ must have a limiting growth rate of at least $\sqrt{\log t}$. However, for small t we can use any functional form of the boundary. Using $\log_+$ is just one way of ensuring that the boundary is strictly positive on $[0,\infty]$. Any other strictly positive function can be used for $0 < t < t_1 < \tau$ with t_1 arbitrary (but finite).

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In contrast with the boundary-crossing probability (9) of Chu et al. (1996) the F_i do not have analytic forms. Nevertheless, the critical values $z(h)$ can be obtained via simulations. Consider F_1, which gives the boundary-crossing probability of the h increments of the Brownian bridge. For a fixed τ, we simulate the Wiener process using 1,000 independent and identically distributed (i.i.d.) standard normal random variables on a unit interval (e.g., 10,000 values for $\tau = 10$) and transform it into a Brownian bridge. Then for selected values of h, we record

$$c_i(h) = \max_{1<t<\tau} \left\{ \frac{|W^0(t) - W^0(t-h)|}{\sqrt{2 \log_+ t}} \right\}, \qquad i = 1, \ldots, R,$$

where $R = 100{,}000$ is the number of replications in our simulations. The critical values $z(h)$ are just the respective empirical quantiles of the $c_i(h)$ values. Table 1 contains some critical values. The S code for simulating the critical values can be obtained from the authors upon request.

The result given in the following corollary now follows easily from (12) and Theorem 4.2.

COROLLARY 4.3. *Given the* DGP (1) *with* (6), *suppose that conditions* (M1) *and* (M2) *hold. Then,*

$$\lim_{T\to\infty} \mathbb{P}\left\{ \frac{[Th]}{\hat{\sigma}_T \sqrt{T}} \| Q_T^{1/2}(\tilde{\beta}_T(k, [Th]) - \hat{\beta}_T) \| < z(h)\sqrt{2 \log_+(k/T)}, \right.$$
$$\left. \text{for all } T + 1 \le k < [T\tau] \right\}$$
$$= [F_1(z(h), \tau)]^n,$$

TABLE 1. Simulated critical values z for the increments of the Brownian bridge

	$\tau = 4$		$\tau = 6$		$\tau = 8$		$\tau = 10$	
h	10%	5%	10%	5%	10%	5%	10%	5%
				Maximum of the increments				
0.25	1.2437	1.3347	1.2510	1.3399	1.2523	1.3408	1.2525	1.3409
0.5	1.7324	1.8861	1.7495	1.8996	1.7526	1.9016	1.7541	1.9021
1	2.4318	2.7051	2.4783	2.7361	2.4900	2.7419	2.4937	2.7446
				Range of the increments				
0.25	1.9724	2.0838	2.0043	2.1110	2.0143	2.1197	2.0190	2.1237
0.5	2.5106	2.6809	2.5820	2.7388	2.6075	2.7578	2.6205	2.7683
1	3.0744	3.3283	3.2260	3.4563	3.2829	3.5084	3.3165	3.5381

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$$\lim_{T\to\infty} \mathbb{P}\left\{ \max_{i=1,\ldots,n} \frac{[Th]}{\hat{\sigma}_T\sqrt{T}} \left(\max_{k=T+1,\ldots,J} [Q_T^{1/2}(\tilde{\beta}_T(k,[Th]) - \hat{\beta}_T)]_i \right.\right.$$

$$\left. - \min_{k=T+1,\ldots,J} [Q_T^{1/2}(\tilde{\beta}_T(k,[Th]) - \hat{\beta}_T)]_i \right)$$

$$\left. < z(h)\sqrt{2\log_+(J/T)}, \qquad \text{for all } T+1 \le J < [T\tau] \right\}$$

$$= [F_2(z(h),\tau)]^n.$$

5. SIMULATION RESULTS

In this section we present some simulation results on the proposed tests. For easy comparison we follow Chu et al. (1996) and generate data from i.i.d. $N(2,1)$ under the null hypothesis. We consider historical samples of sizes $T = 25$, 50, 100, 200, and 300, moving window sizes $h = 0.25, 0.5$, and 1, and $\tau = 10$ for the expected monitoring period $[T\tau]$. Under the alternative, the mean changes from 2.0 to 2.8 at $1.1T$ or $3T$. We used the monitoring boundaries for the significance levels 5% and 10%; as the results are similar, we report only the results for the 10% level. All experiments were repeated 1,000 times. These simulations were performed using the R software package (an implementation of the S language distributed under the GNU general public license; see Ihaka and Gentleman (1996)).

Table 2 shows the empirical sizes of the Chu et al. (1996) test and the proposed tests. For the max-*RE* test we use $a^2 = 6.25$ in boundary (9), which gives a 10% monitoring procedure for $\tau = \infty$. As our data are only simulated for $\tau = 10$, the empirical size of the max-*RE* test should be smaller than the nominal size of 10%. For the moving-estimates tests we simulate the critical values for $\tau = \infty$ by choosing $\tau = 1000$. For the max-*ME* test this choice seems to result in boundaries close to the nominal value (but still lower); for the range-*ME* test the boundaries are a bit more conservative. Comparing to the max-*RE* test,

TABLE 2. Empirical sizes for monitoring tests

		max-*ME*			range-*ME*		
T	max-*RE*	$h = 1/4$	$h = 1/2$	$h = 1$	$h = 1/4$	$h = 1/2$	$h = 1$
25	.086	.101	.112	.132	.064	.073	.088
50	.070	.082	.092	.092	.059	.061	.063
100	.079	.087	.083	.081	.078	.075	.076
200	.074	.090	.084	.079	.064	.066	.067
300	.084	.087	.088	.089	.075	.077	.076

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the empirical sizes of the max-*ME* test are all closer to the nominal size, except when $T = 25$ and $h = 1$.

When the mean shifts, these monitoring tests would eventually signal a change by consistency. It is therefore more interesting to know how soon a change can be detected. Table 3 shows the mean and standard deviation of the detection delay. For example, when $T = 100$ and a structural change occurs at time $1.1T = 110$ ($3T = 300$), the max-*RE* test can detect this change, on average, after a delay of 27 (128), with the standard deviation of 16 (74); i.e., the change is detected at time $110 + 27 = 137$ ($300 + 128 = 428$).

Given an early change at time $1.1T$, the average detection delay of the max-*RE* test is almost independent from T, and the standard deviation decreases for increasing T and stabilizes for $T \geq 100$. On the other hand, the max-*RE* test performs quite poorly for a late change at $3T$. In this case, the average detection delay increases with T. Moreover, both the mean and standard deviation of detection delay are much larger than those for an early change. This is precisely the difficulty of the Chu et al. (1996) test discussed in the beginning of Section 4.

For the moving-estimates-based tests, it is easy to see that their performance depends on the window sizes. A smaller window usually results in quicker detection. In contrast with the max-*RE* test, the average detection delays of the max-*ME* test (for a given window size) do not vary with the location of the change, although they may be different across T. In fact, the average detection delays of the max-*ME* test (with $h = 1/4$) are close to that of the max-*RE* test for an early change at $1.1T$ but are much shorter for a late change at $3T$. For an early change, the range-*ME* test yields longer detection delay than does the max-*ME* test; for a late change, they perform quite similarly, at least for $T \geq 100$. The average detection delays of the range-*ME* test (with $h = 1/4$) are longer than that of the max-*RE* test for an early change but are also much shorter for a late change. For example, when $T = 200$, the average delays of the max-*RE*, max-*ME*, and range-*ME* tests are, respectively, 29, 31, and 52 for an early change and 152, 33, and 37 for a late change. That is, a late change can be detected much more quickly by the proposed tests.

6. CONCLUSIONS

In this paper, we introduce a unified framework for constructing monitoring tests and establish a general asymptotic result. As applications, two new monitoring tests based on moving estimates are proposed. We also show that the proper boundary function of the proposed tests can grow at a much slower rate of $\sqrt{\log t}$, whereas the boundary function for the Chu et al. (1996) test has to grow rather fast (approximately at rate t). As such, the proposed tests have roughly equal sensitivity to a change occurring early or late in the monitoring period, whereas the Chu et al. test is insensitive to a late change. This result is confirmed by our simulations. As the proposed tests avoid the difficulty arising from the Chu et al. test, they are of more practical use.

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TABLE 3. The mean and standard deviation of detection delay

			max-*ME*			range-*ME*		
T	Change point	max-*RE*	$h = 1/4$	$h = 1/2$	$h = 1$	$h = 1/4$	$h = 1/2$	$h = 1$
25	28	29 (32)	24 (26)	24 (24)	25 (18)	25 (19)	27 (23)	31 (25)
50	55	30 (32)	32 (37)	27 (21)	34 (17)	43 (31)	41 (29)	42 (24)
100	110	27 (16)	26 (16)	32 (12)	46 (13)	62 (58)	47 (26)	57 (14)
200	220	29 (14)	31 (10)	44 (13)	64 (16)	52 (32)	59 (13)	78 (17)
300	330	32 (15)	38 (12)	54 (16)	78 (18)	55 (17)	73 (16)	96 (18)
25	75	72 (42)	29 (32)	31 (32)	32 (28)	20 (23)	26 (26)	31 (26)
50	150	109 (69)	40 (48)	34 (33)	25 (25)	35 (34)	37 (37)	36 (17)
100	300	128 (74)	34 (38)	36 (21)	49 (20)	41 (49)	37 (20)	47 (16)
200	600	152 (71)	33 (12)	47 (16)	68 (25)	37 (20)	49 (16)	66 (21)
300	900	174 (76)	41 (12)	58 (18)	82 (30)	44 (13)	59 (18)	79 (26)

Note: The numbers in each cell are the mean and standard deviation (in parentheses).

Although we observed from our simulations that the moving-estimates-based tests with a smaller window size h can detect a change more quickly, it remains unknown how small h should be. In fact, it would be of great interest if one could find a window size that is optimal in the sense of shortest detection delay. Another open question is monitoring schemes for nonlinear models. The current approach cannot be used directly because there is no closed form solution for the parameter estimates (which we use for the definition of the empirical process Y_T in (4)). These topics are currently under investigation.

REFERENCES

Andrews, D.W.K. (1993) Tests for parameter instability and structural change with unknown change points. *Econometrica* 61, 821–856.

Andrews, D.W.K. & W. Ploberger (1994) Optimal tests when a nuisance parameter is present only under the alternative. *Econometrica* 62, 1383–1414.

Bai, J. (1994) Least square estimation of a shift in linear processes. *Journal of Time Series Analysis* 15, 453–472.

Bai, J. (1995) Least absolute deviation estimation of a shift. *Econometric Theory* 11, 403–436.

Bai, J. (1996) Testing for parameter constancy in linear regressions: An empirical distribution function approach. *Econometrica* 64, 597–622.

Bai, J. & P. Perron (1998) Estimating and testing linear models with multiple structural changes. *Econometrica* 66, 47–78.

Billingsley, P. (1968) *Weak Convergence of Probability Measures*. New York: Wiley.

Chu, C.-S.J., K. Hornik, & C.-M. Kuan (1995a) MOSUM tests for parameter constancy. *Biometrika* 82, 603–617.

Chu, C.-S.J., K. Hornik, & C.-M. Kuan (1995b) The moving-estimates test for parameter stability. *Econometric Theory* 11, 699–720.

Chu, C.-S.J., M. Stinchcombe, & H. White (1996) Monitoring structural change. *Econometrica* 64, 1045–1065.

Deheuvels, P. & P. Révész (1987) Weak laws for the increments of Wiener processes, Brownian bridges, empirical processes and partial sums of iid random variables. *Mathematical Statistics and Probability Theory* A, 69–88.

Feller, W. (1968) *An Introduction to Probability Theory and Its Applications*. New York: Wiley.

Ihaka, R. & R. Gentleman (1996) R: A language for data analysis and graphics. *Journal of Computational and Graphical Statistics* 5, 290–314.

Krämer, W., W. Ploberger, & R. Alt (1988) Testing for structural change in dynamic models. *Econometrica* 56, 1355–1369.

Kuan, C.-M. (1998) Tests for changes in models with a polynomial trend. *Journal of Econometrics* 84, 75–91.

Kuan, C.-M. & K. Hornik (1995) The generalized fluctuation test: A unifying view. *Econometric Reviews* 14, 135–161.

Ploberger, W. & W. Krämer (1992) The CUSUM test with OLS residuals. *Econometrica* 60, 271–285.

Ploberger, W., W. Krämer, & K. Kontrus (1989) A new test for structural stability in the linear regression model. *Journal of Econometrics* 40, 307–318.

Révész, P. (1982) On the increments of Wiener and related processes. *Annals of Probability* 10, 613–622.

Robbins, H. (1970) Statistical methods related to the law of the iterated logarithm. *Annals of Mathematical Statistics* 41, 1397–1409.

Robbins, H. & D. Siegmund (1970) Boundary crossing probabilities for the Wiener process and sample sums. *Annals of Mathematical Statistics* 41, 1410–1429.

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APPENDIX

Proof of Theorem 2.1. The proof is essentially the same as in Kuan and Hornik (1995). Let V_g be the variation of g and M_g the maximum of $\|g\|$, respectively. Under (6),

$$Y_T(t) = \frac{Q_T^{1/2}}{\hat{\sigma}_T\sqrt{T}}\left([Tt]\beta_0 + T^{1-\delta}Q_{[Tt]}^{-1}\frac{1}{T}\sum_{i=1}^{[Tt]} x_i x_i' g(t_i) + Q_{[Tt]}^{-1}\sum_{i=1}^{[Tt]} x_i \epsilon_i\right),$$

where $t_i = i/T$. For arbitrary but fixed t,

$$\begin{aligned}
&\left\|\frac{1}{T}\sum_{i=1}^{[Tt]} x_i x_i' g(t_i) - Q\int_0^t g(s)\,ds\right\| \\
&\quad\le \left\|\frac{1}{T}\sum_{i=1}^{[Tt]}(x_i x_i' - Q)g(t_i)\right\| + \left\|Q\left(\frac{1}{T}\sum_{i=1}^{[Tt]} g(t_i) - \int_0^t g(s)\,ds\right)\right\| \\
&\quad\le M_g\left|\frac{[Tt]}{T}\right|\left\|\frac{1}{[Tt]}\sum_{i=1}^{[Tt]} x_i x_i' - Q\right\| + \|Q\|\left\|\frac{1}{T}\sum_{i=1}^{[Tt]} g(t_i) - \int_0^t g(s)\,ds\right\|,
\end{aligned}$$

where the first term on the right hand side is $o_{\mathbb{P}}(1)$ by (2) and the second term is

$$\begin{aligned}
&\|Q\|\left\|\sum_{i=1}^{[Tt]}\int_{t_{i-1}}^{t_i}(g(t_i) - g(s))\,ds - \int_{[Tt]/T}^t g(s)\,ds\right\| \\
&\quad= \|Q\|\left\|\int_0^{1/T}\sum_{i=1}^{[Tt]}(g(t_i) - g(t_{i-1} + s))\,ds - \int_{[Tt]/T}^t g(s)\,ds\right\| \\
&\quad\le \|Q\|(V_g + M_g)/T \\
&\quad= o_{\mathbb{P}}(1).
\end{aligned}$$

We can then write

$$Y_T = \frac{1}{\hat{\sigma}_T}\left(\sqrt{T}\,\frac{[Tt]}{T}\,Q^{1/2}\beta_0 + T^{1/2-\delta}Q^{1/2}Jg + Q^{-1/2}\sum_{i=1}^{[Tt]} x_i\epsilon_i + o_{\mathbb{P}}(1)\right).$$

It follows from (M1), (M2), and the continuous mapping theorem that, for $\delta = 1/2$,

$$\lambda(\mathcal{L}_T Y_T) \Rightarrow \lambda(\mathcal{L}\mathbf{W} + \sigma^{-1}\mathcal{L}Q^{1/2}Jg)$$

and for $\delta < 1/2$,

$$T^{\delta-1/2}\lambda(\mathcal{L}_T Y_T) \to^{\mathbb{P}} \lambda(\mathcal{L}\sigma^{-1}Q^{1/2}Jg). \qquad \blacksquare$$

Proof of Corollary 3.1. Straightforward application of Theorem 2.1. ■

Proof of Corollary 4.1. Straightforward application of Theorem 2.1. ■

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Proof of Theorem 4.2 Let N_i, $i = 1,2,\ldots$, be i.i.d. $N(0,1)$ random variables. Then

$$\mathbb{P}\{N_i < f(i), \forall i\} = \prod_{i=1}^{\infty} \Phi(f(i)) = \begin{cases} 0, & \text{if} \quad \sum_{i=1}^{\infty} (1 - \Phi(f(i))) = \infty, \\ a > 0, & \text{if} \quad \sum_{i=1}^{\infty} (1 - \Phi(f(i))) < \infty, \end{cases}$$

by the Borel–Cantelli lemma (e.g., Feller, 1968, p. 201). Using the inequality in Feller (1968, p. 175),

$$\frac{1}{\sqrt{2\pi}} (x^{-1} - x^{-3}) e^{-x^2/2} \leq 1 - \Phi(x) \leq \frac{1}{\sqrt{2\pi}} x^{-1} e^{-x^2/2},$$

and setting $f(i) = z\sqrt{2 \log i}$, we get

$$\sum_{i=3}^{\infty} (1 - \Phi(f(i))) = \begin{cases} \infty, & \text{if} \quad z \leq 1, \\ c(z) < \infty, & \text{if} \quad z > 1, \end{cases}$$

where $c(z)$ is some real number. In what follows we also let $F(z)$ denote a positive number whose value varies from equation to equation. Hence,

$$\mathbb{P}\{N_i < z\sqrt{2 \log i}, \forall i \geq 3\} = \begin{cases} 0, & \text{if} \quad z \leq 1, \\ F(z) > 0, & \text{if} \quad z > 1. \end{cases}$$

Because

$$\mathbb{P}\{|N_i| < f(i), \forall i \geq 3\} = \prod_{i=1}^{\infty} (2\Phi(f(i)) - 1)$$

$$= \begin{cases} 0, & \text{if} \quad 2\sum_{i=1}^{\infty} (1 - \Phi(f(i))) = \infty, \\ F(z) > 0, & \text{if} \quad 2\sum_{i=1}^{\infty} (1 - \Phi(f(i))) < \infty, \end{cases}$$

we have

$$\mathbb{P}\{|N_i| < z\sqrt{2 \log i}, \forall i \geq 3\} = \begin{cases} 0, & \text{if} \quad z \leq 1, \\ F(z) > 0, & \text{if} \quad z > 1. \end{cases}$$

Clearly,

$$\mathbb{P}\{W(t+1) - W(t) < z\sqrt{2 \log t}, \forall t > e\}$$
$$\leq \mathbb{P}\{W(i+1) - W(i) < z\sqrt{2 \log i}, \forall i \geq 3, i \in \mathbb{N}\},$$

where W is the standard Wiener process. When $z \leq 1$, the probability on the right hand side is zero, and so is the probability on the left hand side. The same relation is still true if we take the absolute value of the increment of the Wiener process.

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From Deheuvels and Révész (1987, Lemma 11), we get

$$\alpha(z) \le \mathbb{P}\Big\{ \sup_{0\le s\le 1} (W(s+1) - W(s)) > z \Big\} \le \beta(z),$$

with

$$\alpha(z) := \frac{(1-\epsilon)}{\sqrt{2\pi}}\, z e^{-z^2/2},$$

$$\beta(z) := \frac{(1+\epsilon)}{\sqrt{2\pi}}\, z e^{-z^2/2},$$

for $z \ge z_0(\epsilon)$. The technical constant z_0 is needed to bound z away from zero (in dependence on ϵ, which defines the "spread" between the bounds α and β). In the following we only need the upper bound β for some $\epsilon > 0$; hence we do not care about the spread between α and β.

Let

$$\mathcal{A} := \Big\{ \sup_{0\le s\le 1} (W(2i+s+1) - W(2i+s)) < z\sqrt{2\log i},\ \forall i \ge 2 \Big\},$$

$$\mathcal{B} := \Big\{ \sup_{0\le s\le 1} (W(2i+s+2) - W(2i+s+1)) < z\sqrt{2\log i},\ \forall i \ge 2 \Big\},$$

where $\mathcal{A}$ is the event that the increments of W are smaller than $z\sqrt{2\log i}$ on "odd" intervals, i.e., intervals starting at odd points $2i+1$. Event $\mathcal{B}$ marks processes with increments smaller than the boundary on intervals starting at even points $2i+2$. As W is independent on disjunct intervals, the probability of event $\mathcal{A}$ is the product of the probabilities that the increments are bounded on all odd intervals. The increments on odd intervals originate from disjunct intervals of the Wiener process and therefore are independent (the same is true for the even intervals in $\mathcal{B}$). Hence, for $z > 1$,

$$\mathbb{P}(\mathcal{A}) = \mathbb{P}(\mathcal{B}) \ge \prod_{i=1}^{\infty} (1 - \beta(z\sqrt{2\log i})) > 0.$$

We easily get $\mathbb{P}(\mathcal{AB}) > 0$ by piecewise combination of processes from $\mathcal{A}$ and $\mathcal{B}$: As W is a Markov process, we can restart it at any point t in time at the current value $W(t)$ and again get a Wiener process. For any process in $\mathcal{A}$ that is not in $\mathcal{B}$, we can construct a new process by stopping it before the boundary is hit for the first time (which must occur in an even interval) and appending a process from $\mathcal{B}$. This new process might hit the boundary in an odd interval, but then we can stop again and continue with the first process.

Combining this and the fact that $\sqrt{h}W(t/h)$ is again a Wiener process, we get with $s = t/h$

$$\begin{aligned}
&\mathbb{P}\{W(t+h) - W(t) < z\sqrt{2h\log_+ t/h},\ \forall t > 0\} \\
&\quad = \mathbb{P}\{W(s+1) - W(s) < z\sqrt{2\log_+ s},\ \forall s > 0\} \\
&\quad = \begin{cases} 0, & \text{if} \quad z \le 1, \\ F(z) > 0, & \text{if} \quad z > 1. \end{cases}
\end{aligned}$$

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Because $W^0(t+h) - W^0(t) = W(t+h) - W(t) - hW(1)$, the increments of the Brownian bridge and the Wiener process differ only by the term $hW(1)$, which is independent from $W(t+h) - W(t)$ for $t > 1$. Boundaries on the increments of the Brownian bridge can basically be the same as boundaries on the increments of the Wiener process; they only have to account for the additional uncertainty added by $W(1)$. Hence, there exist $z_1 < z_2$ such that

$$\mathbb{P}\{|W^0(t+h) - W^0(t)| < z_1\sqrt{2\log_+ t}\} \leq \mathbb{P}\{|W(t+h) - W(t)| < z_2\sqrt{2\log_+ t}\}$$

and simultaneously $z_3 < z_4$ such that

$$\mathbb{P}\{|W(t+h) - W(t)| < z_3\sqrt{2\log_+ t}\} \leq \mathbb{P}\{|W^0(t+h) - W^0(t)| < z_4\sqrt{2\log_+ t}\}.$$

Using this we easily get

$$\mathbb{P}\{|W^0(t+h) - W^0(t)| < z(h)\sqrt{2\log_+ t}, \forall t \geq 0\} = \begin{cases} 0, & \text{if} \quad z(h) \leq 1, \\ F(z) > 0, & \text{if} \quad z(h) > 1. \end{cases}$$

This result extends straightforwardly to the range of the increments of the Brownian bridge. ■

Proof of Corollary 4.3. From (12), the limiting process under the null hypothesis is the increments of the n-dimensional Brownian bridge $\mathbf{W}^0$. As the elements of $\mathbf{W}^0$ are independent, the assertions follow directly from Theorem 4.2. ■

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