Ergod. Th. & Dynam. Sys. (2003), **23**, 1323–1346
DOI: 10.1017/S0143385702001748
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Printed in the United Kingdom

Asymptotic invariance, amenability and generalized Cuntz algebras

WOLFGANG R. BERGMANN† and ROBERTO CONTI‡

† *Dipartimento di Matematica, Università della Basilicata, I-85100 Potenza, Italy*
(e-mail: bergmann@unibas.it)

‡ *Mathematisches Institut, FAU Erlangen-Nürnberg, D-91054 Erlangen, Germany*
(e-mail: conti@mi.uni-erlangen.de)

(*Received* 10 *October* 2002 *and accepted in final form* 17 *January* 2003)

Abstract. Given a unitary representation u of the locally compact group G on the Hilbert space H, we investigate the notion of infinite tensor power of u. Then we apply the results to study covariance of the associated canonical action of G on the generalized Cuntz algebra $\mathcal{O}_H$ in the GNS representation of (the gauge-invariant extension of) some pure quasi-free state. We examine in detail the case of a measure-preserving action of G on a measure space X. In this case, covariance is (almost) equivalent to the existence of a G-invariant state on $L^\infty(X)$.

0. *Introduction*

In this article we discuss the notion of infinite tensor power of a unitary representation of a locally compact group. In the familiar case of a group acting on a measure space, we identify the typical conditions ensuring this construction exists with well-known asymptotic invariance concepts scattered throughout the literature. Then we apply such a construction to solve a problem concerning certain C^*-dynamical systems and their covariant representations.

This introduction is divided into three parts. First we provide the reader with some background and motivations. This part also allows us to establish some useful notation. Then we describe the main ideas we have been dealing with and, finally, we examine in detail the actual content of the paper.

0.1. *Some background and motivations.*

A C^*-dynamical system is a triple $\{\mathcal{A}, \alpha, G\}$, where $\mathcal{A}$ is a C^*-algebra, G is a topological group and $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is an action of G by automorphisms of $\mathcal{A}$. A covariant representation of this system on the Hilbert space $\mathcal{K}$ is a pair (π, U), where $\pi : \mathcal{A} \to B(\mathcal{K})$ is a representation of $\mathcal{A}$ and $U : G \to U(\mathcal{K})$ is a (continuous) unitary representation of G such that

$$U(g)\pi(X)U(g)^* = \pi(\alpha_g(X)), \quad X \in \mathcal{A},\ g \in G.$$

https://doi.org/10.1017/S0143385702001748 Published online by Cambridge University Press

Let G be a locally compact group, and consider a unitary representation u on the (complex) Hilbert space H. We turn it into a C^*-dynamical system by considering the generalized Cuntz algebra $\mathcal{O}_H \supset H$ associated to H and the canonical action of G on $\mathcal{O}_H$ obtained by lifting u in the natural way. Our main purpose is to find 'product-type' covariant representations for it.

Here, H has to be considered as an object in a suitable tensor (or monoidal) category of Hilbert spaces [**26**], and the C^*-algebra $\mathcal{O}_H$ is the natural C^*-algebra generated by H in this categorical context. We refer the reader to [**8**] for a detailed account of the structure and the properties of $\mathcal{O}_H$, which is a special instance of $\mathcal{O}_\rho$, a $\mathbb{Z}$-graded C^*-algebra associated by Doplicher and Roberts to any object ρ in a tensor C^*-category [**13**].

By universality, there is a canonical action Λ of the unitary group $U(H)$ on $\mathcal{O}_H$, see [**1, 8**]. In particular $\mathcal{O}_H$ carries the natural 'gauge action' of the circle group $\mathbb{T} \simeq \mathbb{T} \cdot I$, compatible with the grading, see [**8**].

If H is finite-dimensional, $\mathcal{O}_H$ can be identified with the Cuntz algebra $\mathcal{O}_{\dim(H)}$ [**10**], a well-known C^*-algebra which has also been related in several guises to quantum field theory, abstract duality for compact groups, sector theory for subfactors, statistical mechanics, wavelet theory, groupoids and logic (see e.g. [**16**], the reports [**6, 12, 31**] and [**27**] for a small sample). The gauge-invariant subalgebra of $\mathcal{O}_d$ is the infinite tensor product of $d \times d$ complex matrices. If H is infinite dimensional and separable, $\mathcal{O}_H$ contains a canonical copy of the Cuntz algebra $\mathcal{O}_\infty$ as the C^*-subalgebra generated by H [**8**].

An important input for the present analysis is provided by considering infinite indexed families (sometimes nets, and in fact sequences) of unit vectors in the given Hilbert space H. From this point of view, our framework could be naturally related with ergodic theory and random walks [**33**], substitution dynamics, tiling problems, etc.

In fact, $\mathcal{O}_H$ admits a natural family of (locally normal) 'product-type' representations, hereafter called induced product representations (IPR) [**1**], which are parameterized by the space of all infinite sequences ϕ of unit vectors in H.

These representations are obtained by suitably combining the infinite tensor product construction of Hilbert spaces given by von Neumann [**28**], later revisited by Guichardet [**18**], with the Rieffel induction procedure [**32**]. (Recall that infinite products of Hilbert spaces are closely related, through the GNS construction, to infinite tensor products of states over (in general, non-commutative) operator algebras [**18**], a suitable counterpart for the infinite product construction of probability measures. This is the familiar situation met, e.g., when dealing with spin systems in quantum statistical mechanics [**7**]. Of course, an infinite family of vectors determines an infinite tensor product of (vector) states.)

Being the C^*-inductive limit of $B(H^{\otimes r})$, the zero-grade part $\mathcal{O}_H^0$ of $\mathcal{O}_H$ has a preferred family of representations on some inductive limit of $H^{\otimes r}$ as $r \to \infty$; compatibility requirements with the structure of $\mathcal{O}_H^0$ lead directly to a bijection between the class of such representations and the family of all sequences ϕ of unit vectors in H. Given a sequence ϕ, the infinite tensor power $\bigotimes_{i\in\mathbb{N}}^{\phi} H$ (we follow the notation in [**18**]) is the base space of a product representation $a^{\phi}_{(0)}$ of $\mathcal{O}_H^0$; the Rieffel induction step via the canonical conditional expectation $P : \mathcal{O}_H \to \mathcal{O}_H^0$ finally leads to the IPR $\langle H^\phi, a^\phi \rangle$ of $\mathcal{O}_H$. The carrier Hilbert space H^ϕ admits a $\mathbb{Z}$-direct sum description in which each summand H^{ϕ}_k is again an infinite

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tensor power along a sequence ϕ^k obtained by shifting (a bilateral extension of) ϕ by $k \in \mathbb{Z}$ (see [**1, 2**]).

By construction, H^ϕ is endowed with a canonical unitary representation V^ϕ of $\mathbb{T}$.

0.2. *Main ideas.* If we are given a unitary representation u acting on H, the composition $\Lambda \circ u$ is an action of G on $\mathcal{O}_H$. Now, the C^*-dynamical system $\{\mathcal{O}_H, \Lambda \circ u, G\}$ will have many covariant representations (π, U). The main problem addressed in this paper is to decide whether $\Lambda \circ u$ can be unitarily implemented in some IPR; in other words, we seek the existence of covariant representations of the given system of the form (a^ϕ, U^ϕ), with $U^\phi : G \to U(H^\phi)$ to be determined.

Note that G acts naturally on the family of all sequences of unit vectors.

As will become clear soon, our problem is closely related to the existence of the infinite tensor power (ITP) of u as a (strongly continuous) unitary representation on $\bigotimes_{i\in\mathbb{N}}^{\phi} H$ in the spirit of von Neumann–Guichardet theory, and in essence it amounts to exhibiting a sequence $\phi \subset H$ with certain specific asymptotic invariance properties under G. If such an infinite tensor power of u exists, then U^ϕ is easily obtained through a natural procedure related to Rieffel induction.

One of the main results of the paper is that, under suitable assumptions, a converse is also true. Exploiting the $\mathbb{Z}$-grading of $\mathcal{O}_H$, one of our arguments below consists in showing that, thanks to covariance, one may deduce the existence of a sequence ϕ of unit vectors on which G acts asymptotically like a shift, see §§7 and 8 for precise statements. (So that, after neglecting boundary terms, one gets a fully-fledged asymptotic invariant sequence.)

The construction of an ITP of unitary representations requires the existence of a suitable asymptotically G-invariant sequence of unit vectors. Quite surprisingly, this notion, which is already implicit in von Neumann–Guichardet theory, can hardly be found in the literature (although it is clearly related to other widely known concepts like, e.g., weak containment [**11**]), with few notable exceptions [**1, 20**]. Because of this lack of results, in order to discuss this notion satisfactorily (that is, providing some necessary or sufficient conditions for it) and show that it is available in some familiar cases we need to connect it to other asymptotic invariance concepts scattered in the literature, and related results.

For this reason we consider many other invariance properties of u as well, all of which are weakenings of 'u has non-zero invariant vectors'. Among them we quote the amenability of u in the sense of Bekka, namely the existence of an Adu-invariant mean on $B(H)$, or, equivalently, 'weak containment of the trivial representation after tensorizing with the conjugate representation' [**5**]; it plays the same role as the amenability of G in [**1**] for the u's arising from an action on a quasi-invariant measure space. (It is well known that the notion of amenability [**17, 29, 30**], which is important in operator algebras and harmonic analysis, is characterized in terms of the existence of a suitable G-invariant functional. Through some limiting process, it is often related to the existence of suitable asymptotically G-invariant nets of unit vectors.)

Along the way, in order to get rid of some subtleties, we need to show in a self-contained way on quite general grounds a technical result about uniform convergence over compact subsets of G of matrix coefficients which is, in our opinion, of independent interest (cf. [**21, 23**]).

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Additional properties for u, such as the existence of an 'absolute value' described below or the more specific requirement to arise from a measurable action, are sometimes required, as well as the σ-compactness of G.

For a unitary representation arising from an action on a measure space X, we get some useful existence results: its infinite tensor power exists provided that G is σ-compact and the representation is amenable as explained above or, equivalently, it weakly contains the trivial representation [**23**], §7 (cf. [**21**]). These conditions are, in turn, also equivalent to the existence of a G-invariant state on $L^\infty(X)$.

In fact, the origin of this paper goes back to [**1**]. There H was the space $L^2(G)$ of all Haar-measurable square-summable functions on G and u was the left regular representation λ; for a non-compact G, we showed the equivalence between σ-amenability (i.e. σ-compactness and amenability) of G and the existence of an irreducible $\Lambda \circ \lambda$-covariant IPR of $\mathcal{O}_H$. One of the early motivations of the present work was to extend the results found for λ to γ, the inner regular representation of G [**22**]. As a consequence of the results expounded in §§1–9, our analysis finally applies to γ, thus obtaining a characterization of inner-amenable (non-[IN]) groups which parallels the one for amenable (non-compact) groups. More precisely, for a σ-compact G we get equivalence between the amenability of γ (hence the inner amenability of G, i.e. the existence of a conjugation-invariant mean on $L^\infty(G)$ [**25, 34**]) and the existence of an $\Lambda \circ \gamma$-covariant IPR for which a certain obstruction vanishes. Similarly, one may relate the (strong) inner amenability of a countable discrete group Γ (see [**3, 15**]) to $\Lambda \circ \gamma_0$-covariance, where γ_0 is the canonical 'non-trivial part of γ'.

However our treatment goes far beyond these cases and applies to all the unitary representations satisfying a few simple properties.

0.3. *Getting into the organization and results of the paper.* The paper is divided in two parts. Sections 1–5 are devoted to clarify the group-representation results needed in the sequel. The existence of the covariant IPR of $\mathcal{O}_H$ is then discussed in the remaining sections.

In §1 we introduce some invariance properties of u, such as the existence of asymptotically invariant nets of unit vectors and its 'uniformly on compacta' version. We also recall the notion of amenability for u, along with some useful characterizations.

In §2 we discuss the notion of infinite tensor power of a unitary representation u and further refinements of this notion.

A stronger ℓ^1-type property for u provides a natural tool for our purposes. It already appeared in [**1**]. On the contrary, we could have easily eliminated the ℓ^2-type version, also introduced; but it fits so nicely within our framework that it appeared in our work in the early stages and we did not like to throw it away.

All the properties for u considered here are implied by the existence of invariant vectors, and imply the amenability of u.

Then we establish some basic implications between this first set of conditions. In particular we may deduce the existence of infinite tensor powers of u provided that there are sequences of unit vectors which are asymptotically u-invariant uniformly on compacta. Further special equivalences can be recovered when G is σ-compact or u has C_0-matrix coefficients (i.e. vanishing at infinity).

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In the next few sections, we will be concerned with the construction of infinite tensor products in some concrete situations. However, this requires some care. The necessary background is covered in §§3–5.

In §3 we introduce and discuss a family of representations arising from measure-preserving reshuffling. These are the u's for which, up to unitary equivalence, an L^2-description is possible through a group $g \mapsto a_g$ of permutations of a measure space, namely $(u_g(f))(x) = J_g(x)^{1/2} f(a_{g^{-1}}(x))$.

We extend classical results obtained by Reiter (in connection with the existence of topological means on G, see [**29**]) as well as by Bekka. In this special case u is amenable iff there is an asymptotically invariant net of unit vectors in the 'uniformly on compacta' stronger form.

It is worth observing that the usual pointwise absolute value of L^2-functions is an example of the abstract notion of absolute value discussed in §5.

Section 4, dedicated to a general result about uniform convergence over compacta, is interesting in its own right but it is also the 'main engine' of §3. We started from material found in [**30**], Proposition 6.10. We have a Banach space X, a positive cone $P(X)$ in X satisfying simple assumptions and an isometric action T of G on X mapping $P(X)$ into itself: $\langle L^1(\mu), L^1(\mu)^+, \tau^1 \rangle$ is the example of $\langle X, P(X), T \rangle$ we are mainly interested in. We show that T has an asymptotically invariant net of unit vectors iff there is one for which asymptotic invariance holds uniformly on compacta.

In §5 we isolate a simple property for generic u called 'absolute value', namely a norm-preserving map $\xi \in H \mapsto |\xi| \in H$ commuting with u and having certain elementary positivity properties.

The reader can think of the vanishing of matrix coefficients, the existence of an absolute value, or the origin of a measurable action outlined above as some 'environment conditions' under which the properties of u considered in this paper bind more strictly.

In §6 we recall a few useful facts about $\mathcal{O}_H$, actions and representations.

Sections 7 and 8 are dedicated to the existence of covariant IPR representations. We also require further properties of the underlying covariant IPR a^ϕ, which express natural compatibility with the canonical 'gauge action' of $\mathbb{T}$; the stronger one is the irreducibility of a^ϕ, but we may relax it to some extent by requiring the Mackey's multiplicative commutator $[U, V^\phi]_m$ to be trivial or at least scalar-valued.

In §9 we collect our main achievements and draw some conclusion.

We end this introduction with a few remarks on possible developments. A second countable G is a-T-menable iff there exists a u with C_0-coefficients which admits a uniformly on compacta asymptotically invariant sequence of unit vectors, see [**9**]. By our results, such u admits an infinite tensor power, and thus it may be used to implement covariance in some IPR. However, at the present time, the converse is not at hand; for instance, we do not know if u admits an absolute value (or some asymptotic variant of this concept). Amenability, inner amenability and strong inner amenability are tied up with the behavior of a uniquely determined representation (namely $\lambda, \gamma, \gamma_0$ respectively). On the other hand the definition of a-T-menable groups does not seem to point out the choice of a more or less canonical representation.

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A more important issue is that there is no clear way to relate the C_0-hypothesis to the class IPR.

Hopefully our methods can be improved in order to treat a-T-menability, and also the other versions/variants of (weak) amenability scattered in the literature, e.g. K-amenability, compact or completely bounded approximation properties, DR or strong Feller properties. We hope to return to these problems elsewhere.

1. *Preliminaries*

We set up our notation to be used throughout the text.

- Let H, H' be Hilbert spaces. $B(H, H')$ is the Banach space of all bounded linear operators from H into H'; $B(H) = B(H, H)$; I is the identity in $B(H)$, and sometimes also the identity in other C^*-algebras. s, w, uw are used to specify the strong, weak, ultraweak operator topologies on $B(H)$.
- Let A be a C^*-algebra. $U(A)$ is the group of all unitary elements of A, $U(H) = U(B(H))$; $\mathrm{Aut}(A)$ is the group of all automorphisms of A.
- Throughout this paper G is a locally compact group with neutral element e and $g, h, \ldots$ stand for generic elements of G. μ_1 is a fixed left Haar measure on G. In integrals, dg, respectively dh stand for $d\mu_1(g)$, respectively $d\mu_1(h)$. Measure-theoretic structures on G, such as $L^p(G)$, $S^p(G)$ etc. will be understood with respect to μ_1. For instance, $L^\infty(G)$ is the W^*-algebra of all (equivalence classes of) essentially bounded complex functions G. $\mu_r(A) := \mu_1(A^{-1})$ is the corresponding right invariant Haar measure. The Radon–Nikodym derivative $\Delta = d\mu_1/d\mu_r : G \to (0, \infty)$ is the modular function on G. We have

$$\mu_1(Ag^{-1}) = \mu_1(gAg^{-1}) = \Delta(g^{-1})\mu_1(A) \quad (A \subset G \text{ Haar measurable})$$
$$\int_G f(hg)\,dh = \Delta(g^{-1}) \int_G f(h)\,dh \quad (f \in L^1(G)).$$

- $u : G \to U(H)$ denotes a (continuous) unitary representation of G on H.

1.1. *Some notions of asymptotical invariance.*

We need to recall some pertinent definitions.

A vector $\xi \in H$ is *u-invariant* if $u(g)\xi = \xi$ $(g \in G)$. If $\phi = \langle \phi_i \rangle$ is a *net* of unit vectors in H, ϕ is *asymptotically u-invariant* if $\|u(g)\phi_i - \phi_i\| \to 0$ $(g \in G)$; then the same holds for any subnet of ϕ. By standard arguments, u has an asymptotically invariant net iff for any finite $F \subset G$ and $\epsilon > 0$ there exists a unit vector $\xi \in H$ with $\max_{g \in F} \|u(g)\xi - \xi\| < \epsilon$. A net ϕ as above is *asymptotically u-invariant uniformly on compacta* if $\max_{g \in K} \|u(g)\phi_i - \phi_i\| \to 0$ for any compact set $K \subset G$. Such a ϕ exists iff for any compact $K \subset G$ and $\epsilon > 0$ there exists a unit vector $\xi \in H$ with $\max_{g \in K} \|u(g)\xi - \xi\| < \epsilon$; this means that u weakly contains the trivial one-dimensional representation ι of G (notation: $\iota \prec u$) [**11**, 18.1.4].

PROPOSITION 1.1. *Let ϕ be a sequence (i.e. $\mathcal{I} = \mathbb{N}$); ϕ is asymptotically u-invariant iff it is asymptotically u-invariant uniformly on compacta.*

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Proof. Let us write $\mathcal{E}$ for the scalar product $(u(g)\phi_i, \phi_i)$, and 'play the movie' $i \to \infty$: $\mathcal{E}$ becomes a sequence of positive definite functions in $L^\infty(G)$, converging pointwise to 1; by the Lebesgue dominated convergence theorem, $\mathcal{E} \to 1$ also under the $\sigma[L^\infty(G), L^1(G)]$-topology. $\mathcal{E}$ is continuous as an expression in the variable g and $\mathcal{E} = 1$ if $g = e$ thus, by [**11**, 13.5.2] and [**19**, 32.42], the convergence to 1 in g is uniform on compacta. □

The following lemma is certainly well known.

LEMMA 1.2. *Let ϕ be asymptotically u-invariant; then*

$$(\textit{weak cluster points of } \phi) \subset (u\textit{-invariant vectors}).$$

In particular, if u has no non-zero invariant vectors, then $\phi_i \to 0$ weakly.

Proof. Let ξ be a weak cluster point of ϕ; passing to a subnet, we may suppose that $\phi_i \to \xi$ weakly; then $(\phi_i, \xi) \to \|\xi\|^2$. From $\|u(g)\phi_i - \phi_i\| \to 0$ we get $(u(g)\phi_i, \xi) - (\phi_i, \xi) \to 0$, whence $(u(g)\phi_i, \xi) \to \|\xi\|^2$. Since u is weakly continuous, $(u(g)\phi_i, \xi) \to (u(g)\xi, \xi)$. Thus $(u(g)\xi, \xi) = \|\xi\|^2$, whence $u(g)\xi = \xi$.

Since the unit ball of H is weakly compact, for the second claim it is enough to show that (weak cluster points of ϕ) $= \{0\}$. Let ξ_0 be a weak cluster point of ϕ; then ξ_0 is u-invariant, whence it must be zero. On the other hand, again by compactness, ϕ must have some weak cluster point. □

Following [**5**], u is called *amenable* if there exists an Adu-invariant state ω on $B(H)$, that is $\omega(u_g T u_{g^{-1}}) = \omega(T)$ for any $g \in G$, $T \in B(H)$.

THEOREM 1.3. [**5**] *The following conditions are equivalent:*

(1) *u is amenable;*

(2) *there exists a net $S = \langle S_i | i \in \mathcal{I} \rangle$ with $S_i \in T(H)$, $S_i \geq 0$, $\|S_i\|_1 = 1$ such that $\lim_i \|\mathrm{Ad}u(g)S_i - S_i\|_1 = 0$;*

(3) *there exists a net $S = \langle S_i | i \in \mathcal{I} \rangle$ with $S_i \in T(H)$, $S_i \geq 0$, $\|S_i\|_1 = 1$ such that $\lim_i \max_{g \in K} \|\mathrm{Ad}u(g)S_i - S_i\|_1 = 0$ for any compact subset $K \subset G$;*

(4) *$\iota \prec u \otimes \overline{u}$.*

Here, $T(H)$ is the space of all trace class operators on H with trace class norm $\| \cdot \|_1$, Adu is the adjoint action of G on $B(H)$ associated to u (that is, $\mathrm{Ad}u(g)T := u(g)Tu(g^{-1})$, $T \in B(H)$) and $\overline{u}$ is the conjugate representation of u. Note that if ϕ is a unit vector and $S = t_\phi$ is the rank-one projection onto $\mathbb{C}\phi$, then $\|\mathrm{Ad}u(g)(S) - S\|_1 = \|u(g)Su(g^{-1}) - S\|_1 = 2[1 - |(u(g)\phi, \phi)|^2]^{1/2}$ (see Appendix B).

The notion of amenability for u may be related to the existence of hypertraces on $C^*(u(G))$ [**4**, Proposition 1.5].

We say that u is C_0 if $u(g)$ vanishes weakly as $g \to \infty$, i.e., for any $\xi \in H$, $g \mapsto (u(g)\xi, \xi)$ is a C_0 function on G. Note that G is compact iff 1 is a C_0 function, and that G is compact if some C_0 u has a non-zero invariant vector. C_0 u's occur in many situations; for instance, the left regular representation λ is always C_0; recall also the 'vanishing of matrix coefficients theorem' for simple connected Lie groups with finite center [**35**]. Now, let u be C_0. If it has an asymptotically invariant sequence, then G is σ-compact. Thus, for C_0 u's, the existence of such a sequence is equivalent to the existence

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of a net of unit vectors asymptotically u-invariant uniformly on compacta together with the σ-compactness of G.

2. *The infinite tensor power of u*

The main task of this section is to point out the properties of the (generally infinite) tensor powers of u that will be invoked more frequently in the sequel.

See [**18**] for a handy account of von Neumann notions of infinite tensor products of Hilbert spaces $\bigotimes^{\phi}_{i\in\mathcal{I}} H_i$, of vectors $\bigotimes^{\phi}_{i\in\mathcal{I}} \xi_i \in \bigotimes^{\phi}_{i\in\mathcal{I}} H_i$ and of operators $\bigotimes^{\phi}_{i\in\mathcal{I}} T_i \in B\big(\bigotimes^{\phi}_{i\in\mathcal{I}} H_i\big)$, cf. [**2**].

Let $\phi = \langle \phi_i | i \in \mathcal{I}\rangle$ be an indexed set (not necessarily a net) of unit vectors in H. We write $u^{\phi}_{(0)}(g)$ for the operator tensor power $\bigotimes^{\phi}_{i\in\mathcal{I}} u(g) \in B\big(\bigotimes^{\phi}_{i\in\mathcal{I}} H\big)$ of u whenever it exists, namely iff

$$\sum_{i\in\mathcal{I}} |1 - (u(g)\phi_i, \phi_i)| < \infty \quad (g \in G). \tag{2.1}$$

In the sequel we will use the stronger condition

$$\sum_{i\in\mathcal{I}} \max_{g\in K} |1 - (u(g)\phi_i, \phi_i)| < \infty \quad (K \subset G \text{ compact}). \tag{2.2}$$

Note that if (2.1) holds, then $u^{\phi}_{(0)}$ is a unitary representation of G on $\bigotimes^{\phi}_{i\in\mathcal{I}} H$. Of course these conditions are non-trivial iff $\mathcal{I}$ is infinite. If $\mathcal{I} = \emptyset$, then $\bigotimes^{\phi}_{i\in\mathcal{I}} H \simeq \mathbb{C}$ and $u^{\phi}_{(0)}$ is the trivial representation of G.

If there exists a ϕ satisfying (2.1) with $\mathcal{I}$ infinite then, simply by extracting any countable subset of $\mathcal{I}$, we can have another instance of ϕ with (2.1) and $\mathcal{I} = \mathbb{N}$. The same holds for (2.2).

Remark 2.1. If (2.1) holds and $\mathcal{I}$ is countable, then $u^{\phi}_{(0)}$ is weakly measurable, being an s- (hence w-) limit of a sequence of (objects equivalent to) finite tensor powers of u; if in addition H is separable, then also $\bigotimes^{\phi}_{i\in\mathcal{I}} H$ is, and thus, by the von Neumann theorem on weakly measurable representations, $u^{\phi}_{(0)}$ is strongly continuous.

PROPOSITION 2.2. *If formula (2.2) holds for ϕ then $u^{\phi}_{(0)}$ is strongly continuous.*

Proof. The w-continuity is sufficient. We can limit ourselves to a total subset $\big\{\bigotimes^{\phi}_{i\in\mathcal{I}} \xi_i\big\}$ of $\bigotimes^{\phi}_{i\in\mathcal{I}} H$. Let us check that, if $\langle \xi_i | i \in \mathcal{I}\rangle$ is a subset of H indexed by $\mathcal{I}$ and $\xi_i = \phi_i$ for all but finitely many i's, then the complex function F on G defined by $F(g) = \big(u^{\phi}_{(0)}(g) \bigotimes^{\phi}_{i\in\mathcal{I}} \xi_i, \bigotimes^{\phi}_{i\in\mathcal{I}} \xi_i\big)$ is continuous on G. F continuous on any K-compact subset of G will suffice. By (2.2), the product $\prod_{i\in\mathcal{I}} (u(g)\xi_i, \xi_i)$ converges uniformly for $g \in K$ (see [**1**, Appendix 2]). Thus, since the functions $g \mapsto (u(g)\xi_i, \xi_i)$ are continuous on G, F is continuous on K. □

If G is σ-compact, it follows readily from Proposition 1.1 that if a sequence ϕ satisfies (2.1) there is a (possibly different) sequence satisfying (2.2).

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2.1. *Some ℓ^p-type conditions.* Let $\phi = \langle \phi_i | i \in \mathcal{I} \rangle$ be an indexed set of unit vectors in H. With $p \in \{1, 2\}$, we will consider the following condition on ϕ:

$$\sum_{i \in \mathcal{I}} \|u(g)\phi_i - \phi_i\|^p = \sum_{i \in \mathcal{I}} \{2[1 - \Re(u(g)\phi_i, \phi_i)]\}^{p/2} < \infty \quad (g \in G) \tag{2.3}$$

and the stronger version (with the same p)

$$\sum_{i \in \mathcal{I}} \max_{g \in K} \|u(g)\phi_i - \phi_i\|^p < \infty \quad (K \subset G \text{ compact}). \tag{2.4}$$

We stress the simple but useful fact that for each $g \in G$ it holds that $\|u(g)\phi_i - \phi_i\|^2 = 2[1 - \Re(u(g)\phi_i, \phi_i)] \leq 2|1 - (u(g)\phi_i, \phi_i)| \leq 2\|u(g)\phi_i - \phi_i\|$. Therefore (2.3) with $p = 1$ implies (2.1), which in turn implies (2.3) with $p = 2$; similarly (2.4) with $p = 1$ implies (2.2) which implies (2.4) with $p = 2$.

Now, let $\mathcal{I}$ be infinite; let also be defined a direction on $\mathcal{I}$, i.e. let ϕ be a *net*. Clearly, (2.3) (respectively (2.4)) with $p = 2$ implies that ϕ is asymptotically u-invariant (respectively uniformly on compacta) and hence, by Proposition 1.1, also that there exists a sequence which is asymptotically u-invariant uniformly on compacta. In the following we have a kind of converse.

PROPOSITION 2.3. *If there is an asymptotically invariant net of unit vectors $\phi = \langle \phi_i \rangle$ for u, then u is amenable. Furthermore, if G is σ-compact and the net ϕ is asymptotically u-invariant uniformly on compacta, then (2.4) with $p = 1$, and thus (2.2), holds for some sequence ξ of unit vectors.*

Proof. Define a net of operators by $t_\phi = \langle t_{\phi_i} \rangle$, where $t_{\phi_i}(\eta) = (\eta, \phi_i)\phi_i$, $\eta \in H$. Clearly, $t_{\phi_i} \geq 0$, $\operatorname{tr}(t_{\phi_i}) = 1$, $u(g)t_{\phi_i}u(g^{-1}) = t_{u(g)\phi_i}$, and moreover $\|t_{u(g)\phi_i} - t_{\phi_i}\|_1 = 2[1 - |(u(g)\phi_i, \phi_i)|^2]^{1/2}$. The conclusion follows by Theorem 1.3.

Let $H_1, H_2, \ldots$ be compact subsets of G with $H_n \subset \operatorname{int}(H_{n+1})$ $(n = 1, 2, \ldots)$ and $\bigcup_n H_n = G$. Then we can choose a sequence of unit vectors $\xi = \langle \xi_n \rangle$ in H with $\max_{g \in H_n} \|u(g)\xi_n - \xi_n\| < 1/2^n$ and we are done. □

Notice that an amenable representation might not have asymptotically invariant nets at all; see Remark 7.5 with $\eta \not\equiv 1$. Also, it is not so clear (and perhaps not true, for general u) that from the existence of an asymptotically invariant net of unit vectors one can deduce the existence of a net for which the convergence is uniform over compact sets; in §3 we discuss a situation for which this is possible.

We leave to the reader the task of examining the content of this section under additional conditions such as σ-compactness for G, or C_0 for u.

3. *Actions of G on measure spaces*

In this section we set up the framework for the most important class of unitary representations considered in this paper. Loosely speaking, this is the class of all u's for which a description in terms of translation of measurable sets is available. Let us get into the details.

Let $\langle \Omega, \mathcal{S}, \mu \rangle$ be a (positive) measure space in the sense of Chapter II of [**14**]; $\mathcal{S}$ is the ring of all measurable subsets of Ω; let x stand for the generic element of Ω, and dx for $d\mu(x)$. In the sequel, p, q range over $[1, \infty]$, and $p^{-1} + q^{-1} = 1$.

https://doi.org/10.1017/S0143385702001748 Published online by Cambridge University Press

By classical Hölder-type reasons, the duality

$$\langle \xi, \eta \rangle = \int_\Omega \xi(x)\eta(x)\,dx \quad (f \in L^p(\mu), \eta \in L^q(\mu)) \tag{3.1}$$

is available, by which $L^q(\mu)$ is embedded in the dual space $L^p(\mu)'$, and this happens isometrically; even for $q = \infty$, by the semifiniteness assumptions intrinsic in Chapter II of [**14**]; moreover, if $q \in (0, \infty)$, the embedding is *onto*. Throughout this §3, the important property

$$L^\infty(\mu) \text{ exhausts the dual space } L^1(\mu)' \tag{3.2}$$

will be explicitly invoked when needed, and not assumed from the start; our aim is to make clear the logical structure of our treatment, rather than extending parts of our results to exotic, non-localizable measure-theoretical counterexamples. After all, in the next subsection we use the embedding of $L^1(\mu)$ in the dual space $L^\infty(\mu)'$, not only the opposite embedding.

Remark 3.1. (See [**14**], ch. II) $\langle \Omega, \mu \rangle = \langle G, \mu_1 \rangle \Rightarrow \mu$ regular Borel measure $\Rightarrow \langle \Omega, \mu \rangle$ parabounded $\Rightarrow$ property (3.2) holds; moreover $\langle \Omega, \mu \rangle$ bounded $\Rightarrow \langle \Omega, \mu \rangle$ σ-bounded $\Rightarrow \langle \Omega, \mu \rangle$ parabounded.

3.1. *Polar facts.* We set $S^p(\mu) = \{\xi \in L^p(\mu) \mid \|\xi\|_p = 1\}$ (unit shell), $PB^p(\mu) = \{\xi \in L^p(\mu) \mid \xi \geq 0, \|\xi\|_p \leq 1\}$ (positive ball) and $PS^p(\mu) = \{\xi \in L^p(\mu) \mid \xi \geq 0, \|\xi\|_p = 1\}$ (positive shell).

Let $w^*[\infty]$ be the weak*-topology of the dual space $L^\infty(\mu)'$.

Let $B^\infty = \{\varphi \in L^\infty(\mu)' \mid \varphi \geq 0, \|\varphi\| \leq 1\}$, and $E^\infty = \{\varphi \in L^\infty(\mu)' \mid \varphi \geq 0, \|\varphi\| = 1\}$. B^∞ and E^∞ are both convex and $w^*[\infty]$-closed.

Let $wk[1]$ be the weak topology of $L^1(\mu)$ and $wk[1, \infty]$ the weak dual topology $\sigma[L^1, L^\infty]$ on $L^1(\mu)$ associated to the family of all functionals of the dual space $L^1(\mu)'$ which 'belong to $L^\infty(\mu)$'. Of course, $wk[1, \infty] \leq wk[1]$, where we have equality iff (3.2) holds.

Now let us think of $L^1(\mu)$ as a subspace of the dual space $L^\infty(\mu)'$ in the usual way; then we can also think that $PB^1(\mu) = B^\infty \cap L^1(\mu)$, that $PS^1(\mu) = E^\infty \cap L^1(\mu)$ and that $wk[1, \infty]$ is the $w^*[\infty]$-relative topology of $L^1(\mu) \subset L^\infty(\mu)'$.

In the sequel, polars $(\cdot)^\circ$ are taken with respect to the duality $\langle \cdot, \cdot \rangle$ between $L^\infty(\mu)$ and the dual space $L^\infty(\mu)'$.

LEMMA 3.2. *With the above notation* $(PB^1(\mu))^{-w^*[\infty]} = B^\infty$.

Proof. $PB^1(\mu)$ is convex and contains 0, hence $(PB^1(\mu))^{-w^*[\infty]} = PB^1(\mu)^{\circ\circ}$. $B^\infty = (B^\infty)^{\circ\circ}$, thus $PB^1(\mu)^\circ \subset (B^\infty)^\circ$ is sufficient. We have $(B^\infty)^\circ = \{f \in L^\infty(\mu) \mid \Re\langle f, \varphi \rangle \leq 1 \ \forall \varphi \in B^\infty\} = \{f \in L^\infty(\mu) \mid \langle \Re f, \varphi \rangle \leq 1 \ \forall \varphi \in B^\infty\} = \{f \in L^\infty(\mu) \mid \Re f \leq 1\} = \{f \in L^\infty(\mu) \mid \langle \Re f, \varphi \rangle \leq 1 \ \forall \varphi \in PB^1(\mu)\} = \{f \in L^\infty(\mu) \mid \Re\langle f, \varphi \rangle \leq 1 \ \forall \varphi \in PB^1(\mu)\} = PB^1(\mu)^\circ$. □

LEMMA 3.3. $(PS^1(\mu))^{-w^*[\infty]} = E^\infty$.

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Proof. Let $\varphi \in E^\infty$. By Lemma 3.2, there exists a net ξ_α in $PB^1(\mu)$ $w^*[\infty]$-converging to φ. Since $\|\xi_\alpha\|_1 = \xi_\alpha(I) \to 1$, we can assume $\|\xi_\alpha\|_1 \neq 0$ for each α. Put $\varphi_\alpha := \xi_\alpha/\|\xi_\alpha\|_1$. We have $\|\varphi_\alpha\|_1 = 1$, $\varphi_\alpha \to_{w^*[\infty]} \varphi$. □

3.2. *Representations associated to quasi-invariant measures.* Let $a : G \to$ Permutations(Ω) be a measure class-preserving action of G on Ω, in the sense: a density $J_g(x) \in (0, \infty)$ is given, with

$$d(a_{g^{-1}}(x)) = J_g(x)\,dx; \tag{3.3}$$

by (3.3) we mean: for any complex-valued expression $f(x)$ in the variable x and for any g, $f(x)$ is in $L^1(\Omega, dx)$ iff $f(a_{g^{-1}}(x))J_g(x)$ is, and in this case we have $\int_\Omega f(x)\,dx = \int_\Omega f(a_{g^{-1}}(x))J_g(x)\,dx$. By $(\tau_g^p(f))(x) = J_g(x)^{1/p} f(a_{g^{-1}}(x))$ we define a norm-preserving linear operator τ_g^p on $L^p(\mu)$, compatible with the duality (3.1): $\langle \tau_g^p(\xi), \tau_g^q(\eta)\rangle = \langle \xi, \eta\rangle$ $(\xi \in L^p(\mu), \eta \in L^q(\mu))$. We assume

$$J_e(x) = 1 \quad \text{and} \quad J_g(x)J_h(a_{g^{-1}}(x)) = J_{gh}(x). \tag{3.4}$$

By (3.4), τ_g^p is a representation of G as a group of norm-preserving linear operators on $L^p(\mu)$. Of course, by (3.4), we have $J_g(x)^{-1} = J_{g^{-1}}(a_{g^{-1}}(x))$. Let us now assume that

$$g \mapsto \tau_g^p(\xi) \in L^p(\mu) \text{ is continuous whenever } p < \infty \text{ and } \xi \in L^p(\mu). \tag{3.5}$$

In this case, $g \mapsto \tau_g^\infty(\xi) \in L^\infty(\mu)$ is continuous in the dual $\sigma[L^\infty, L^1]$ topology. Note that if in the proof of Lemma 3.3, φ is a τ^1-invariant state then, for any g, $\tau_g^1(\xi_\alpha) - \xi_\alpha \to 0$ in the topology $wk[1, \infty]$.

Let us write τ for τ^2.

Let $|f|(x) := |f(x)|$ for any $f : \Omega \to \mathbb{C}$. If $f \in L^p(\mu)$, $|\tau_g^p(f)| = \tau_g^p(|f|)$. If $\xi, \eta \in L^2(\mu)$, then $|(\xi, \eta)| \leq (|\xi|, |\eta|)$; if $\|\xi\| = 1$, it holds that

$$\begin{aligned}\|\tau_g(|\xi|) - |\xi|\|^2 &= 2[1 - (\tau_g(|\xi|), |\xi|)] \leq 2[1 - |(\tau_g(\xi), \xi)|] \\ &\leq 2[1 - \Re(\tau_g(\xi), \xi)] = \|\tau_g(\xi) - \xi\|^2.\end{aligned}$$

Let $M : L^\infty(\mu) \to B(L^2(\mu))$ be the usual $*$-isomorphism obtained by multiplication: $(M_f(\xi))(x) := f(x)\xi(x)$ $(f \in L^\infty(\mu), \xi \in L^2(\mu))$. We have $\tau_g M_f \tau_{g^{-1}} = M_{\tau_g^\infty f}$ $(f \in L^\infty(\mu))$.

3.3. *Invariance notions for τ^p.* We show that many reasonable asymptotic invariance (AI) notions for τ^p are equivalent to each other.

THEOREM 3.4. *The following properties are equivalent:*

(1) $AI[PS^2]$*: there exists in* $PS^2(\mu)$ *an asymptotically* τ*-invariant net* $\langle \phi_i | i \in \mathcal{I}\rangle$*;*
(2) $AI[PS^2, \kappa]$*: there exists in* $PS^2(\mu)$ *a net* $\langle \phi_i | i \in \mathcal{I}\rangle$ *which is asymptotically* τ*-invariant uniformly on compacta;*
(3) $AI[S^2]$*: the same as* $AI[PS^2]$*, but with* $S^2(\mu)$ *instead of* $PS^2(\mu)$*;*
(4) $AI[S^2, \kappa]$*: the same as* $AI[PS^2, \kappa]$*, but with* $S^2(\mu)$ *instead of* $PS^2(\mu)$*;*
(5) $AI[2s]$*: there exists a net* $\phi = \langle \phi_i | i \in \mathcal{I}\rangle$ *in* $S^2(\mu)$ *such that, for any* g*,* $\lim_i |(\tau_g \phi_i, \phi_i)| = 1$*;*

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(6) $AI[PS^1]$: *there exists in* $PS^1(\mu)$ *an asymptotically* τ^1*-invariant net* $\langle \psi_j | j \in \mathcal{J} \rangle$;
(7) $AI[PS^1, \kappa]$: *there exists in* $PS^1(\mu)$ *a net* $\langle \psi_j | j \in \mathcal{J} \rangle$ *which is asymptotically* τ^1*-invariant uniformly on compacta;*
(8) $AI[PS^1, wk[1]]$: *there exists in* $PS^1(\mu)$ *a* $wk[1]$*-asymptotically* τ^1*-invariant net* $\langle \psi_j | j \in \mathcal{J} \rangle$.
Moreover, with:
(9) $AI[Bk]$: τ *is amenable;*
(10) $AI[\infty]$: *there exists a* τ^∞*-invariant state on* $L^\infty(\mu)$;
(11) $AI[PS^1, wk[1, \infty]]$: *there exists in* $PS^1(\mu)$ *a* $wk[1, \infty]$*-asymptotically* τ^1*-invariant net* $\langle \psi_j | j \in \mathcal{J} \rangle$,
we have (1) $\Rightarrow$ (9) $\Rightarrow$ (10) $\Rightarrow$ (11).

Remark 3.5. If (3.2) holds, then $wk[1, \infty] = wk[1]$, thus all claims (1) through (11) are equivalent. If $\mu \equiv 0$, then all claims (1) through (11) are false.

Proof. (1) $\Rightarrow$ (9): this is direct; let ω_i be the vector state on $B(L^2(\mu))$ associated to ϕ_i. Let $w^*[B]$ be the weak*-topology of the (norm-)dual space of $B(L^2(\mu))$. Any $w^*[B]$-cluster point ω of the net $\langle \omega_i \rangle$ in the dual space of $B(L^2(\mu))$ is G-invariant.
(9) $\Rightarrow$ (10): if ω is a G-invariant state on $B(L^2(\mu))$, then $\nu := \omega \circ M$ is a τ^∞-invariant state on $L^\infty(\mu)$.
(10) $\Rightarrow$ (11): let ν be a τ^∞-invariant state on $L^\infty(\mu)$. By Lemma 3.3 there exists a net ϕ_i in $PS^1(\mu)$ $w^*[\infty]$-converging to ν. For any g, $\tau^1_g \phi_i$ $wk[1, \infty]$-converges to $\nu \circ \tau^\infty_{g^{-1}} = \nu$, hence $\tau^1_g \phi_i - \phi_i \to 0$ in the $wk[1, \infty]$-topology.
(8) $\Rightarrow$ (6): clearly, it is sufficient: for any finite $F \subset G$ and $\epsilon > 0$, there is a ψ in $PS^1(\mu)$ such that $\|\tau^1_g(\psi) - \psi\|_1 < \epsilon$ for any $g \in F$. Let us use the Day's trick: put $E := \{\langle \tau^1_g(\xi) - \xi \mid g \in F \rangle \mid \xi \in PS^1(\mu)\} \subset X_F := \bigoplus_{g \in F} L^1(\mu)$. Let $nt[F]$, $wt[F]$ be the norm and the weak topologies on X_F, respectively; by (8), there is a net ϕ_i in $PS^1(\mu)$ with $\tau^1_g(\phi_i) - \phi_i \to 0$ weakly for any $g \in F$, thus $(0, \ldots, 0) \in \overline{E}^{wt[F]}$. Since E is convex, $\overline{E}^{wt[F]} = \overline{E}^{nt[F]}$, whence $0_{X_F} \in \overline{E}^{nt[F]}$, thus ψ exists.
(6) $\Rightarrow$ (1), (7) $\Rightarrow$ (2): take the 'p-bus', i.e. $\mathcal{I} = \mathcal{J}$, $\phi_i := \psi_i^{1/2}$.
(1) $\Rightarrow$ (6), (2) $\Rightarrow$ (7): $\mathcal{J} = \mathcal{I}$, $\psi_i := \phi_i^2$.
(1) $\Leftrightarrow$ (3), (2) $\Leftrightarrow$ (4), (5) $\Leftrightarrow$ (1): see the last paragraph in §3.2.
(7) $\Rightarrow$ (6) $\Rightarrow$ (8), (2) $\Rightarrow$ (1): obvious.
(6) $\Rightarrow$ (7): not obvious; use §5 with $X = L^1(\mu)$, $\mathcal{P} = L^1(\mu)^+$, $T_g = \tau^1_g$. □

If τ is C_0, then $AI[2s] \wedge (G\ \sigma$-compact) holds iff we can satisfy $AI[2s]$ with a *sequence* ϕ (i.e. $\mathcal{I} = \mathbb{N}$).

Remark 3.6. Let us assume: (1) $a_g(A)$ is measurable whenever $A \subset \Omega$ is; (2) $D : G \to (0, \infty)$ is a *continuous* group homomorphism, with $\mu(a_g(A)) = D(g^{-1})\mu(A)$ ($A \subset \Omega$ measurable); (3) for any measurable $A \subset \Omega$, $\mu(a_g(A) \triangle A) \to 0$ as $g \to e$ in G ($\triangle$ denotes the symmetric difference between sets); (4) $J_g(x) = D(g)$. Then (3.3), (3.4) and (3.5) (whence also Theorem 3.4) hold; τ^p is given by $(\tau^p_g(f))(x) = D(g)^{1/p} f(a_{g^{-1}}(x))$.

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Maybe the reader has noticed that, apart from a brief introductory comment, the symbol u did not occur in this §3 until now. We are now ready to specify what we mean by 'u arises from a measurable action of G'.

Definition 3.7. We say that u arises from a measurable action of G if there exists a representation of u as the result of an action of G as a group of permutations of an underlying measure space, in the sense: it is possible to arrange this §3 (more precisely $\langle \Omega, \mathcal{S}, \mu, a, J \rangle$) so that (A) there exists a unitary equivalence between u and τ, (B) (3.2) holds, i.e. $L^\infty(\mu)$ exhausts the dual of $L^1(\mu)$.

By the Remarks 3.6 and 3.1, λ, γ and γ_0 are of this kind. Of course, all such u's have an absolute value in the sense of §5.

4. *More about uniform convergence on compacta*

In this section we pass from $\langle L^1(\mu), \tau^1 \rangle$ to $\langle X, T \rangle$, where X is a Banach space endowed with a positive cone $P(X)$ interacting with the norm in the L^1-like manner (4.1) below and T is an isometric action of G on X mapping $P(X)$ into itself.

Let X be a Banach space, with ξ, η generic elements in X. Let T be a linear norm-preserving action of G on X such that, for any ξ, $g \in G \mapsto T_g(\xi) \in X$ is continuous. Let $P(X)$ be a norm-closed positive cone in X: $P(X) + cP(X) \subset P(X)$ $(c \geq 0)$. We assume

$$\|\xi + \eta\| = \|\xi\| + \|\eta\| \quad (\xi, \eta \in P(X)) \tag{4.1}$$

$$T_g(P(X)) \subset P(X) \quad (g \in G). \tag{4.2}$$

An example satisfying these assumptions is the complexification of a Riesz L-space, in particular an L^1-space from measure theory; but there are other situations in which the linear partial ordering obtained from $P(X)$ does not satisfy all lattice axioms, e.g. $X = T(H)$, $P(X) = T(H)^+$, $T_g = \mathrm{Ad}u(g)|T(H)$.

Let us write $PS(X)$ for $\{\xi \in P(X) \mid \|\xi\| = 1\}$.

THEOREM 4.1. *The following conditions are equivalent:*

(1) *for every $\epsilon > 0$ and finite $F \subset G$, there exists $\phi \in PS(X)$ such that $\|T_g(\phi) - \phi\| < \epsilon$ $(g \in F)$;*

(2) *for every $\epsilon > 0$ and compact $K \subset G$, there exists $\phi \in PS(X)$ such that $\|T_g(\phi) - \phi\| < \epsilon$ $(g \in K)$.*

The proof requires some preparation. Since this would lead us too far astray, we include it in Appendix C.

5. *Absolute values for u*

The discussion in the previous sections and notably the role played by a positive cone there provides some motivation for this section, where we turn back to generic unitary representations u for which much less structure is present.

The abstract notion of positivity for a generic u introduced below fits nicely with our analysis and will be useful later to suppress undesirable phase factors.

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Definition 5.1. An absolute value for u is a map $\xi \in H \mapsto |\xi| \in H$ with the following properties: (A) $|(\xi, \eta)| \leq (|\xi|, |\eta|)$ $(\xi, \eta \in H)$; (B) $\||\xi|\| = \|\xi\|$ $(\xi \in H)$; and $(*)$ $|u(g)\xi| = u(g)|\xi|$ $(\xi \in H, g \in G)$.

If u arises from a measurable action of G, as in the previous sections, then u has an absolute value by taking the ordinary absolute value of complex functions in L^2. It is worthwhile to stress that the class of representations having an absolute value is closed under direct sums (by $|\bigoplus_k \xi_k| := \bigoplus_k |\xi_k|$) and Mackey's induction, see Appendix A. For an example of a u without any absolute value, see also Remark 7.5 with $\eta \not\equiv 1$.

PROPOSITION 5.2. *If G is compact, then u has an absolute value iff it has a non-zero invariant vector.*

Proof. '$\Leftarrow$': see the examples below. For '$\Rightarrow$': take $0 \neq \eta \in H$, set $\xi = |\eta|$ and let $\xi_0 = (1/\mu_1(G)) \int_G u(g)\xi \, dg$. Then ξ_0 is u-invariant, and it is direct to check that $(\xi_0, \xi) > 0$. □

Now let there be given a function $\xi \mapsto |\xi|$ from H into H; we consider the following additional properties: (c) $||\xi|| = |\xi|$ $(\xi \in H)$; (d) $||\xi| + |\eta|| = ||\xi|| + ||\eta||$ $(\xi, \eta \in H)$.

For instance, if u arises from a measurable action of G, the usual absolute value has (A)∧(B)∧(c)∧(∗); in particular, if $H = L^2(G)$ and $u = \lambda$, then (A)∧(B)∧(c)∧(d)∧(∗) holds. As an example of quite different nature, let $|\xi| := \|\xi\|\psi_0$, with ψ_0 a u-invariant unit vector; it looks naive, but (A)∧(B)∧(c)∧(∗) holds.

More generally, if $|\cdot|$ has (A)∧(B)∧(c) and $u(g)|\xi| = |\xi|$ $(\xi \in H, g \in G)$, then $(*)$ holds iff $|u(g)\xi| = |\xi|$ $(\xi \in H, g \in G)$, and (d) holds iff $|H| + |H| \subset |H|$.

We set $P := \{|\xi| \mid \xi \in H\}$, and $PP := \{\xi \in H \mid \xi = |\xi|\} \subset P$.

We have $(\xi, \eta) \geq 0$ whenever $\xi, \eta \in P$. In general, (A)∧(B)∧(∗) does not imply (c); in fact, even in the case where (A)∧(B)∧(c)∧(∗) holds, for any isometry $s \in B(H)$ commuting with $u(G)$, the new absolute value $|\cdot|' := s(|\cdot|)$ still has the property (A)∧(B)∧(∗), but not necessarily (c). Take for example $s = c$ with $1 \neq c \in \mathbb{T}$; or, in the case where $u = \lambda$, take as s the *right* translation $\rho(g_0)$ by a fixed element $g_0 \neq e$ of G.

Note that u does not appear in the next result.

PROPOSITION 5.3. *Assume that (A)∧(B) holds. With $\xi, \eta \in H$, $c \in \mathbb{C}$, we have: (0) $||\xi|| = |\xi|$ $(\xi \in PP)$; (1) $\| |\xi| - |\eta| \| \leq \|\xi - \eta\|$; (2) PP is norm-closed in H; (3) $|c\xi| = |c||\xi|$; in particular, $|-\xi| = |\xi|$; (4) $\xi \in PP \Leftrightarrow \xi = |\xi| \Leftrightarrow (\xi, |\xi|) = (\xi, \xi)$; (5) $P \cap (-P) = \{0\}$; (6) $PP + PP \subset PP$; (7) PP is a positive cone in H; (8) $P = PP \Leftrightarrow$ also (c) holds; (9) if also (c) holds, then P is a positive cone in H; and (10) (c) $\Rightarrow$ (d).*

Proof. (0): obvious. (1): $\| |\xi| - |\eta| \|^2 = \|\xi\|^2 + \|\eta\|^2 - 2(|\xi|, |\eta|) \leq \|\xi\|^2 + \|\eta\|^2 - 2\Re(\xi, \eta) = \|\xi - \eta\|^2$. (2): by (1). (3): $0 \leq \| |c| |\xi| - |c\xi| \|^2 = |c|^2 \| |\xi| \|^2 + \| |c\xi| \|^2 - 2|c|(|\xi|, |c\xi|) \leq |c|^2\|\xi\|^2 + \|c\xi\|^2 - 2|c| \, |(\xi, c\xi)| = 2|c|^2\|\xi\|^2 - 2|c|^2|(\xi, \xi)| = 0$. (4): obvious. (5): if $\xi \in P \cap (-P)$, then $\xi, -\xi \in P$, thus $(\xi, \xi) \geq 0$, $(\xi, -\xi) \geq 0$. (6): let $\xi, \eta \in PP$; by (4), $(\xi + \eta, |\xi + \eta|) = \|\xi + \eta\|^2$ is sufficient: but $(\xi + \eta, |\xi + \eta|) = (\xi, |\xi + \eta|) + (\eta, |\xi + \eta|) = (|\xi|, |\xi + \eta|) + (|\eta|, |\xi + \eta|) \geq |(\xi, \xi + \eta)| + |(\eta, \xi + \eta)| = | \|\xi\|^2 + (\xi, \eta) | + | \|\eta\|^2 + (\eta, \xi) | = \|\xi + \eta\|^2$. (7): obvious from (5) and (6). (8): direct

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to check. (9): it follows from $P = PP$. (10): by (c), $||\xi|| + ||\eta|| = |\xi| + |\eta|$; since $|\xi|, |\eta| \in P = PP$, $|\xi| + |\eta| \in PP$, whence $||\xi| + |\eta|| = |\xi| + |\eta|$. □

6. *Induced product representations of $\mathcal{O}_H$*

Let $\mathcal{O}_H$ be the extended Cuntz algebra over H with its grades $\mathcal{O}_H^k$ ($k \in \mathbb{Z}$), see [**8**]. We write $w \mapsto \Lambda_w$ for the canonical action of $U(H)$ on $\mathcal{O}_H$, leaving each $\mathcal{O}_H^k$ stable, see [**1, 8**]; we write Λ_0 for the ITP-type action of $U(H)$ obtained by restriction of Λ to $\mathcal{O}_H^0$. For any $z \in \mathbb{T}$, Λ_{zI} reduces on $\mathcal{O}_H^k$ to scalar multiplication by z^k; in particular, $\mathcal{O}_H^0$ is the fixed point subalgebra under the gauge action $z \mapsto \Lambda_{zI}$. $P : \mathcal{O}_H \to \mathcal{O}_H^0$ is the canonical conditional expectation defined by $P(x) := 1/2\pi \int_{\mathbb{T}} \Lambda_{zI}(x)\, dz$ ($x \in \mathcal{O}_H$).

Let $\phi = (\phi_1, \phi_2, \dots)$ be a sequence of unit vectors of H. In the sequel we use the theory of infinite tensor products of Hilbert spaces [**18, 28**]. Let us write $H_{(0)}^\phi$ for $\bigotimes_{i\in\mathbb{N}}^\phi H$. $\langle H_{(0)}^\phi, a_{(0)}^\phi \rangle$ denotes the product representation (PR) of $\mathcal{O}_H^0$ associated to ϕ, defined by $a_{(0)}^\phi(T) = T \otimes I_{K_r}$ ($T \in B(H^{\otimes r})$); here T has been identified with its copy in $\mathcal{O}_H^0$, K_r is the 'trailing' product $\bigotimes_{i\geq r+1}^{\phi_i} H$, and the r.h.s. operator tensor power refers to the natural identification between $H_{(0)}^\phi$ and $H^{\otimes r} \otimes K_r$, see [**1, 2**] (in [**1**] the notation π_0^ϕ is used instead of $a_{(0)}^\phi$, and H is $L^2(G)$).

If π, π' are two representations of the C^*-algebra $\mathcal{A}$, we write $T : \pi \to \pi'$ if T intertwines π and π', i.e. $T \in B(H, H')$ and, for any $x \in A$, $T \circ \pi(x) = \pi'(x) \circ T$; $T : \pi \simeq \pi'$ if $T : \pi \to \pi'$ *and* T is unitary; $\pi \simeq \pi'$ if $T : \pi \simeq \pi'$ for some T.

In the sequel we will use repeatedly the following fact (see [**1**], and also [**24**, p. 395]):

LEMMA 6.1. *Given two sequences ϕ, ϕ' of unit vectors of H, $a_{(0)}^\phi \simeq a_{(0)}^{\phi'}$ if and only if $\sum_i [1 - |(\phi_i, \phi_i')|] < \infty$.*

The induced product representation (IPR) of $\mathcal{O}_H$ associated to ϕ is the representation $\langle H^\phi, a^\phi \rangle$ obtained by Rieffel induction procedure from the PR $\langle H_{(0)}^\phi, a_{(0)}^\phi \rangle$ via P. See [**2**] for more details. The reader can also see [**1**], p. 56 (we wrote there π^ϕ instead of a^ϕ). The proof of the following result is straightforward (cf. Lemma 1.2 and [**2**]); we omit the details.

PROPOSITION 6.2. *If the sequence $\phi = (\phi_1, \phi_2, \dots)$ of unit vectors is asymptotically u-invariant (e.g., if $u_{(0)}^\phi$ exists) for some u on H without non-zero invariant vectors, then there exists a subsequence ϕ' of ϕ with $a^{\phi'}$ irreducible.*

7. *On the existence of covariant IPRs*

Let there be given actions $\beta_0 : G \to \mathrm{Aut}(\mathcal{O}_H^0)$ and $\beta : G \to \mathrm{Aut}(\mathcal{O}_H)$. We will consider eventually the case $\beta_0 = \Lambda_0 \circ u$ and $\beta = \Lambda \circ u$.

In this section we study the existence of a unitary representation U_0 (respectively U) of G on $H_{(0)}^\phi$ (respectively H^ϕ), for some sequence ϕ, such that $\langle H_{(0)}^\phi, a_{(0)}^\phi, U_0 \rangle$ (respectively $\langle H^\phi, a^\phi, U \rangle$) is a covariant representation of $(\mathcal{O}_H^0, \beta_0)$ (respectively $(\mathcal{O}_H, \beta)$).

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Let $\langle \alpha_0, \alpha \rangle$ be an instance of the $\langle \beta_0, \beta \rangle$ as above with α not too wild and giving rise by restriction to α_0, in the sense: (1) for any $g \in G$, α_g commutes with P, thus $\mathcal{O}_H^0$ is α_g-invariant; (2) $(\alpha_0)_g = \alpha_g | \mathcal{O}_H^0$.

A condition on α stronger than (1) is the following (1+): for any $g \in G, z \in \mathbb{T}$, α_g commutes with Λ_{zI}. Moreover, in the sequel we will sometimes consider the following continuity hypothesis (3) on α: for any $x, x' \in \mathcal{O}_H$, the map $g \in G \mapsto a^\phi_{(0)}(P(x'^* \alpha_g(x))) \in B(H^\phi_{(0)})$ is w-continuous. For $\langle \alpha_0, \alpha \rangle = \langle \Lambda_0 \circ u, \Lambda \circ u \rangle$, (1+), (2) and (3) hold.

Let there be given a unitary representation U_0 of G on $H^\phi_{(0)}$ which makes $a^\phi_{(0)}$ α_0-covariant. Then it is possible to construct a unitary representation $U_0^\sim$ of G on H^ϕ by Rieffel induction using the above assumptions (1) and (2) about the pair $\langle \alpha_0, \alpha \rangle$: $U_0^\sim$ is the unique unitary action which extends U_0 and makes the IPR a^ϕ α-covariant, (well) defined by $U_0^\sim(g)(a^\phi(x)(\xi)) := a^\phi(\alpha_g(x))(U_0(g)(\xi))$ $(x \in \mathcal{O}_H, \xi \in H^\phi_{(0)})$, see [**1**]. It is possible to show that if U_0 is s-continuous, and if α has the continuity property (3) then $U_0^\sim$ is also s-continuous.

Arguing in the same way with $G = \mathbb{T}$, $\alpha_z = \Lambda(zI)$, $\alpha_0 =$ the trivial action made covariant by I, one defines an s-continuous representation V^ϕ of $\mathbb{T}$ on H^ϕ, simply by $V^\phi(z)(a^\phi(x)\xi) := a^\phi(\Lambda_{zI}(x))(\xi)$ $(z \in \mathbb{T}, x \in \mathcal{O}_H, \xi \in H^\phi_{(0)})$.

Note that if (1+) holds, then $U_0^\sim$ and V^ϕ commute on H^ϕ. For an example of a U making a^ϕ covariant but not commuting with V^ϕ, see the comment preceding Proposition 8.2.

7.1. *Infinite tensor powers of u and $\Lambda \circ u$ covariance.* Recall that u is a given unitary representation of G on H. In this section we consider the actions $\alpha_0 = \Lambda_0 \circ u$ and $\alpha = \Lambda \circ u$. The proof of the following result is left to the reader (cf. [**1**], Theorem 3.0.2).

PROPOSITION 7.1. *Let ϕ be a sequence of unit vectors in H. If $u^\phi_{(0)}$ exists then it makes $a^\phi_{(0)}$ $\Lambda_0 \circ u$-covariant.*

Definition 7.2. If $u^\phi_{(0)}$ exists, we define the induced infinite tensor power u^ϕ of u as the unitary representation of G on H^ϕ given by $u^\phi := (u^\phi_{(0)})^\sim$.

By the discussion above, u^ϕ makes a^ϕ $\Lambda \circ u$-covariant, moreover it is s-continuous if $u^\phi_{(0)}$ is. Also, $V^\phi(z)$ and $u^\phi(g)$ commute for all $z \in \mathbb{T}$, $g \in G$.

COROLLARY 7.3. *Let u have no non-zero invariant vectors. If (2.2) holds, then there exists a subsequence ϕ' of ϕ such that $a^{\phi'}$ is $\Lambda \circ u$-covariant and irreducible, and both $u^{\phi'}_{(0)}$, $u^{\phi'}$ are s-continuous.*

Proof. By Proposition 6.2 we can take a subsequence ϕ' with $a^{\phi'}$ irreducible; then (2.2) still holds with ϕ' instead of ϕ, thus $u^{\phi'}_{(0)}$ (whence also $u^{\phi'}$) is s-continuous. □

Notice that a converse to Proposition 7.1 is not possible in general, see Remark 7.5. However, under the assumption that u has an absolute value, we will show that the existence of $u^\phi_{(0)}$ for some ϕ is equivalent to that of a $\Lambda_0 \circ u$-covariant PR $a^{\phi'}_{(0)}$ for some

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(possibly different) ϕ'. (See [**1**], p. 56 for an example of a $\Lambda_0 \circ \lambda$-covariant $a^{\phi}_{(0)}$ for which $u^{\phi}_{(0)}$ does not exist.)

PROPOSITION 7.4. *If $a^{\phi}_{(0)}$ is $\Lambda_0 \circ u$-covariant, then $\sum_i [1 - |(u(g)\phi_i, \phi_i)|] < \infty$ $(g \in G)$ and u is amenable. If in addition u has an absolute value $|\cdot|$, then $u^{|\phi|}_{(0)}$ exists.*

Proof. Let U_0 implement the asserted covariance, then for any $g \in G$ we have $U_0(g) : a^{\phi}_{(0)} \circ \Lambda_0(u(g^{-1})) \simeq a^{\phi}_{(0)}$; but we also have $a^{\phi}_{(0)} \circ \Lambda_0(u(g^{-1})) \simeq a^{(u(g)\phi_1, u(g)\phi_2, \dots)}_{(0)}$ [**1**, p. 59], thus $a^{\phi}_{(0)} \simeq a^{(u(g)\phi_1, u(g)\phi_2, \dots)}_{(0)}$ and the first claim follows by Lemma 6.1. Let $S_i = t_{\phi_i}$, then by Lemma B.2 we have $\|u(g)S_i u(g^{-1}) - S_i\|_1 = 2[1 - |(u(g)\phi_i, \phi_i)|^2]^{1/2} \to 0$ $(g \in G)$. The last statement follows since $\sum_i [1 - (u(g)|\phi_i|, |\phi_i|)] \leq \sum_i (1 - |(u(g)\phi_i, \phi_i)|) < \infty$. □

Remark 7.5. Let η be a character of G, i.e. a *continuous* group homomorphism from G into $\mathbb{T}$. Let $u(g) = \eta(g) I_H$. Then: (1) u is amenable: pick any state ω on $B(H)$; (2) for any ξ unit vector in H, $\|u(g)\xi - \xi\| = |1 - \eta(g)|$ $(g \in G)$; (3) u has an asymptotically invariant net of unit vectors iff $u^{\phi}_{(0)}$ exists for some sequence ϕ of unit vectors iff u has a non-zero invariant vector iff $\eta \equiv 1$; this follows from point (2); (4) $\Lambda_0 \circ u$ is trivial; (5) $U_0(g) \equiv I$ is s-continuous and makes any PR $\Lambda_0 \circ u$-covariant; (6) $U(g) = V^{\phi}(\eta(g))$ is s-continuous and makes any IPR a^{ϕ} $\Lambda \circ u$-covariant; (7) u has an absolute value iff $\eta \equiv 1$; this follows from Proposition 7.4, and points (3) and (5).

7.2. *'Wildness' of covariance and its measure χ.* Let us assume that U makes the IPR a^{ϕ} $\Lambda \circ u$-covariant. We set $\chi(g, z) \equiv \chi_{\phi,U}(g, z) := U(g)V^{\phi}(z)U(g)^{-1}V^{\phi}(z)^{-1}$. Since $\Lambda_{u(g)}$ and Λ_{zI} commute on $\mathcal{O}_H$, $\chi(g, z) \in a^{\phi}(\mathcal{O}_H)'$. Clearly, $U(g)V^{\phi}(z) = \chi(g, z)V^{\phi}(z)U(g)$.

If $[V^{\phi}(z), \chi(g, w)] = 0$ for any $g \in G, z, w \in \mathbb{T}$, it is direct to check that $z \mapsto \chi(g, z)$ defines a unitary representation of $\mathbb{T}$, s-continuous since V^{ϕ} is; similarly, if $[\chi(h, z), U(g)] = 0$ for any $h, g \in G, z \in \mathbb{T}$, then, for every z, $g \mapsto \chi(g, z)$ is a unitary *anti*representation of G, s-continuous if U is. Thus we have obtained the following.

LEMMA 7.6. *With the above notation, if $\chi(g, z) \in \mathbb{C}$ then, for any g, $z \mapsto \chi(g, z)$ is a unitary representation of $\mathbb{T}$ and, for any z, $g \mapsto \chi(g, z)$ is a unitary representation of G.*

Next we wonder whether the $\Lambda \circ u$-covariance of a^{ϕ} implies the $\Lambda_0 \circ u$-covariance of $a^{\phi'}_{(0)}$ for some sequence ϕ' of unit vectors. In the light of the results established so far, we provide the missing step.

PROPOSITION 7.7. *Let u have an absolute value. If the IPR a^{ϕ} is $\Lambda \circ u$-covariant through a unitary representation U of G on H^{ϕ} such that $\chi_{\phi,U}(g, z)$ is scalar-valued, then there exists a sequence $\Phi = (\Phi_1, \Phi_2, \dots)$ of unit vectors which is asymptotically u-invariant uniformly on compacta.*

Proof. By Proposition 8.2 we have (8.1), thus the conclusion follows taking Φ as in Proposition 8.3. □

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8. *Structure and covariance of the IPRs*

The material contained in this §8 should appear interesting in its own right, although it could make sense to skip it in a first reading.

Let $k \in \mathbb{Z}$, and set $\Omega^\phi := \bigotimes_{i\in\mathbb{N}}^\phi \phi_i \in H^\phi_{(0)}$, a distinguished unit vector of H^ϕ; $H^\phi_k := \overline{a^\phi(\mathcal{O}^k_H)\Omega^\phi}$, a closed subspace of H^ϕ; $V_k :=$ the isometric inclusion of H^ϕ_k into H^ϕ; $E_k := V_k \circ V_k^* \in B(H^\phi)$, the orthogonal projection from H^ϕ onto H^ϕ_k. H^ϕ_k is $a^\phi(\mathcal{O}^0_H)$-invariant; let us write a^ϕ_k for the subrepresentation of $a^\phi|\mathcal{O}^0_H$ corresponding to H^ϕ_k, i.e. $a^\phi_k(x) := V_k^* \circ a^\phi(x) \circ V_k$ $(x \in \mathcal{O}^0_H)$.

$\langle H^\phi_k | k \in \mathbb{Z}\rangle$ is the spectral decomposition of V^ϕ, i.e. $V^\phi(z) = \sum_{k\in\mathbb{Z}} z^k E_k$ [**1**, p. 58].

In order to give a closer look at the a^ϕ_k's, we fix additional, arbitrary unit vectors $\langle \phi_k | k \in \{0, -1, -2, \dots\}\rangle$ of H, introduce the shifted sequences $\phi^k = (\phi_{1-k}, \phi_{2-k}, \dots)$ and extend the definition of Ω^ϕ by setting $\Omega^\phi_k = a^\phi(\phi_{1-k} \dots \phi_{-1}\phi_0)$ if $k > 0$, $\Omega^\phi_0 = \Omega^\phi$ and $\Omega^\phi_k = a^\phi(\phi^*_k \dots \phi^*_1)\Omega^\phi$ if $k < 0$. Then we have $H^\phi_k = \overline{a^\phi(\mathcal{O}^0_H)\Omega^\phi_k}$, $a^\phi_k \simeq a^{\phi^k}_{(0)}$ (in particular, a^ϕ_k is irreducible), $a^\phi(\phi_{-k})\Omega^\phi_k = \Omega^\phi_{k+1}$ and $a^\phi(\phi^*_{-k+1})\Omega^\phi_k = \Omega^\phi_{k-1}$ [**1**, §4].

Next let us assume that U makes the IPR a^ϕ $\Lambda \circ u$-covariant. The following result is related to [**1**, Lemma 4.1.2]; we omit the details.

PROPOSITION 8.1. *$\chi_{\phi,U}(g, z) \in \mathbb{C}I$ for any g, z iff there exists an 'offset' map $p : G \to \mathbb{Z}$ with $U_g(H^\phi_k) \subset H^\phi_{k+p(g)}$ $(k \in \mathbb{Z})$. In that case, $U_g(H^\phi_k) = H^\phi_{k+p(g)}$ holds for each k; p is a group homomorphism, continuous if U is s-continuous, and $\chi_{\phi,U}(g, z) = z^{-p(g)}I$.*

As a consequence, it is easy to see that for all g, z, $[U(g), V^\phi(z)] = 0$ iff for all g, k, $U(g)H^\phi_k \subset H^\phi_k$ iff for all g, $U(g)H^\phi_0 \subset H^\phi_0$ iff for all g, $U(g)\Omega^\phi \in H^\phi_0$ iff for all g, z, $V^\phi(z)U(g)\Omega^\phi = U(g)\Omega^\phi$.

Note that in general $p(g) \not\equiv 0$, whence $U(g)$ and $V^\phi(z)$ do not commute, see [**1**, §4.2].

PROPOSITION 8.2. *If U makes a^ϕ $\Lambda \circ u$-covariant and $\chi_{\phi,U} \in \mathbb{C}$, then*

$$\sum_{i\in\mathbb{N}} [1 - |(u(g)\phi_i, \phi_{i-p(g)})|] < \infty \quad (g \in G), \tag{8.1}$$

where $p : G \to \mathbb{Z}$ is the group homomorphism implicitly defined by $U(g)H^\phi_0 = H^\phi_{p(g)}$ $(g \in G)$ by Proposition 8.1.

Proof. The covariance relation $U(g)a^\phi(\cdot)U(g)^{-1} = a^\phi(\Lambda(u(g))(\cdot))$ restricted to $\mathcal{O}^0_H$ and then to H^ϕ_0 may be understood as a unitary equivalence between $a^\phi_{(0)} \circ \Lambda_0(u(g^{-1}))$ and $a^\phi_{p(g)}$. On the other hand we have $a^\phi_{(0)} \circ \Lambda_0(u(g^{-1})) \simeq a^{(u(g)\phi_1, u(g)\phi_2, \dots)}_{(0)}$, and $a^\phi_{p(g)} \simeq a^{(\phi_{1-p(g)}, \phi_{2-p(g)}, \dots)}$ and hence the conclusion follows by Lemma 6.1. □

PROPOSITION 8.3. *Let u have an absolute value and suppose that (8.1) holds. Then there exists a sequence Φ asymptotically u-invariant uniformly on compacta.*

Proof. By Proposition 1.1, it is sufficient to show that, for every g, $\|u(g)\Phi_N - \Phi_N\| \to 0$ as $N \to \infty$. Putting $\psi_i := |\phi_i|$, (8.1) remains true under the replacement $\phi_i \to \psi_i$. Now we can construct Φ from the ψ_i's using the same averaging technique of [**1**, Theorem 4.1.7]. □

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9. *Summing up*

Collecting all the results established so far, we can now state the main result of this paper.

It is convenient to enumerate the relevant properties of a generic u:

(1) u is amenable;
(2) there exists a net of unit vectors which is asymptotically u-invariant;
(3) there exists a net of unit vectors which is asymptotically u-invariant uniformly on compacta;
(4) there exists a sequence of unit vectors which is asymptotically u-invariant;
(5) there exists an indexed set of unit vectors satisfying the ℓ^1-type property expressed by equation (2.3) with $p = 1$;
(6) there exists an indexed set of unit vectors satisfying the ℓ^1-type property expressed by equation (2.4) with $p = 1$;
(7) there exists some infinite tensor power $\bigotimes_{\mathbb{N}}^{\phi} u$ in the sense of equation (2.1);
(8) there exists some infinite tensor power $\bigotimes_{\mathbb{N}}^{\phi} u$ in the sense of equation (2.2);
(9) there exists some infinite tensor power $\bigotimes_{\mathbb{N}}^{\phi} u$ in the sense of equation (2.1) which is s-continuous;
(10) there exists an indexed set of unit vectors satisfying the ℓ^2-type property expressed by equation (2.3) with $p = 2$;
(11) there exists an indexed set of unit vectors satisfying the ℓ^2-type property expressed by equation (2.4) with $p = 2$;
(12) there exists a PR of $\mathcal{O}_H^0$ which makes $\Lambda_0 \circ u$ covariant;
(13) there exists a covariant representation (a^ϕ, U^ϕ) of $\{\mathcal{O}_H, \Lambda \circ u, G\}$ such that U^ϕ and V^ϕ commute;
(14) there exists an IPR a^ϕ of $\mathcal{O}_H$ which makes the action $\Lambda \circ u$ covariant with $\chi \in \mathbb{C}I$;
(15) there exists an irreducible IPR a^ϕ which makes the action $\Lambda \circ u$ covariant.

THEOREM 9.1. *Let G be a locally compact, σ-compact group and u a unitary representation of G which admits an absolute value. Then the properties (3) through (14) are equivalent. If in addition u has no non-zero fixed vector then the properties (3) to (15) are equivalent.*

COROLLARY 9.2. *If G is σ-compact and u comes from a measurable action of G on (X, μ), then the properties (1) to (14) are equivalent to the existence of a G-invariant state on $L^\infty(X, \mu)$.*

COROLLARY 9.3. *If G is a locally compact group and u is a C_0 unitary representation of G with an absolute value, then the properties (4) through (14) are equivalent and also to (3) $\vee$ G σ-compact. If, in addition, u has no non-zero fixed vector all these properties are also equivalent to (15).*

Let us define a new property (xσ) if property (x) holds and G is σ-compact.

COROLLARY 9.4. *If G is a locally compact group and u is a C_0 unitary representation coming from a measurable action of G, then the properties (1σ) through (3σ) and (4) through (14) are equivalent. If in addition u has no non-zero fixed vector all these properties are also equivalent to (15).*

https://doi.org/10.1017/S0143385702001748 Published online by Cambridge University Press

Acknowledgements. We are very grateful to Professor Sergio Doplicher for his helpful advice. We are also grateful to Professors E. Bédos, R. V. Kadison, A. L. T. Paterson, G. Skandalis and L. Zsido for useful hints. Part of this work has been done while the second named author was visiting the Équipe de Algèbre d'Opérateurs of the University P & M Curie (Paris VI), the Mathematics Department of the University of California at Berkeley and the Mathematics Department of the University of Oslo. He thanks these institutions for their hospitality, and CNR and EU for financial support.

A. *Appendix. Absolute value and Frobenius–Mackey induction*

We show that the class of representations with an absolute value is closed under Frobenius–Mackey induction. The notations used here are taken, with a few slight changes, from [**14**], especially XI.8.2. Now our G is a closed subgroup of a locally compact group G' whose left Haar measure and modular function are denoted by μ_1' and Δ' respectively. Let g' and g'' stand for generic elements of G', let dg' stand for $d\mu_1'(g')$. A G-ρ function on G' is a locally μ_1'-summable, complex-valued function ρ on G' such that $\rho(g'g) = \rho(g')\Delta(g)/\Delta'(g)$, see [**14**, III.13.2]. We fix a continuous positive G-ρ function ρ on G', [**14**, III.14.5] (sometimes called a 'Bruhat function'), and we normalize it, i.e. $\rho(e) = 1$. We denote by $\rho^\sharp$ the regular Borel measure on G'/G constructed from ρ as in [**14**, III.13.10]; we have $\int_{G'/G} f^G(x)\, d\rho^\sharp(x) = \int_{G'} f(g')\rho(g')\, dg'$ for any f continuous function on G' with compact support and with $f^G(g'G) = \int_G f(g'g)\, dg$, [**14**, III.13.10]. By our choice of ρ, $\rho^\sharp$ is quasi-invariant, see [**14**, III.14.7].

Let us pass to $\langle H', u'\rangle$, the induced representation of $\langle H, u\rangle$. H' can be realized as the linear space of all functions $\xi' : G' \to H$ such that (i) ξ' is locally μ_1'-measurable; (ii) $\xi'(g'g) = u(g^{-1})\xi'(g')$; (iii) $g'G \mapsto \|\xi'(g')\|_H^2$ (well defined by (ii)) is $\rho^\sharp$-summable on G'/G. Identifying the functions on G' which differ only on a μ_1'-null set, H' becomes a Hilbert space with the inner product $(\xi', \eta') = \int_{G'}(\xi'(g'), \eta'(g'))_H \rho(g')\, dg'$. Let $\sigma(g', g''G) = \rho(g'')^{-1}\rho(g'^{-1}g'')$. Defining $(u'(g')\xi')(g'') = (\sigma(g', g''G))^{1/2}\xi'(g'^{-1}g'')$ ($\xi' \in H'$), we have $u' \simeq \mathrm{Ind}_{G\uparrow G'}(u)$ (with the notations of [**14**, XI.10.9]).

Now let there be given an absolute value $|\cdot| : H \to H$ for u. We show that $|\xi'|^\sim(g') := |\xi'(g')|$ defines an absolute value $|\cdot|^\sim$ for u'. First we observe that $|\cdot|^\sim$ maps H' into H': in fact, property (i) for $|\xi'|^\sim$ follows by local approximation with simple functions; for (ii): $|\xi'|^\sim(g'g) = |\xi'(g'g)| = |u(g^{-1})\xi'(g')| = u(g^{-1})|\xi'(g')| = u(g^{-1})(|\xi'|^\sim(g'))$; for (iii): $g'G \mapsto \||\xi'|^\sim(g')\|_H = \||\xi'(g')|\|_H = \|\xi'(g')\|_H$.

Now we check the conditions (A), (B), (∗) of Definition 5.1:

(A) $|(\xi', \eta')| \le \int_{G'} |(\xi'(g'), \eta'(g'))_H| \rho(g')\, dg' \le \int_{G'} (|\xi'(g')|, |\eta'(g')|)_H \rho(g')\, dg' = \int_{G'} (|\xi'|^\sim(g'), |\eta'|^\sim(g'))_H \rho(g')\, dg' = (|\xi'|^\sim, |\eta'|^\sim)_{H'}$;

(B) $\||\xi'|^\sim\|_{H'}^2 = (|\xi'|^\sim, |\xi'|^\sim)_{H'} = \int_{G'} (|\xi'|^\sim(g'), |\xi'|^\sim(g'))_H \rho(g')\, dg' = \int_{G'} \||\xi'(g')|\|_H^2 \rho(g')\, dg' = \int_{G'} \|\xi'(g')\|_H^2 \rho(g')\, dg' = (\xi', \xi') = \|\xi'\|_H^2$;

(∗) $(u'(g')|\xi'|^\sim)(g'') = (\sigma(g', g''G))^{1/2}|\xi'|^\sim(g'^{-1}g'') = (\sigma(g', g''G))^{1/2}|\xi'(g'^{-1}g'')|$, whilst $|u'(g')\xi'|^\sim(g'') = |u'(g')\xi'(g'')| = |(\sigma(g', g''G))^{1/2}\xi'(g'^{-1}g'')|$; this yields the conclusion since $(\sigma(g', g''G))^{1/2} > 0$.

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B. *Appendix. Some inequalities on norms and traces*

Let $t_\xi \in T(H)$ be the rank (zero or) one operator $\eta \mapsto (\eta, \xi)\xi$ from H into H; if ξ has norm one, t_ξ is the rank-one projection onto $\mathbb{C}\xi$.

LEMMA B.1. *Let ξ, η be unital vectors in H, then $\|t_\xi - t_\eta\|_2 = [2(1 - |(\xi, \eta)|^2)]^{1/2}$.*

Proof. $\|t_\xi - t_\eta\|_2^2 = \mathrm{tr}((t_\xi - t_\eta)(t_\xi - t_\eta)) = \mathrm{tr}(t_\xi - (\eta, \xi)t_{\xi,\eta} - (\xi, \eta)t_{\eta,\xi} + t_\eta) = 1 - 2|(\eta, \xi)|^2 + 1$. □

LEMMA B.2. *If $\xi, \eta \in H$, $\|\xi\| = \|\eta\| = 1$, then $\|t_\xi - t_\eta\|_1 = 2[1 - |(\xi, \eta)|^2]^{1/2}$.*

Proof. The claim is obvious if ξ, η are linearly dependent; otherwise, it is sufficient to consider the case where H is equal to the linear span of ξ, η. Since we are dealing with a hermitian operator, the trace is the sum of the absolute values of the eigenvalues. We have (relaxing the condition $\|\eta\| = 1$) $(t_\xi - t_\eta)\xi = \xi - (\xi, \eta)\eta$ and $(t_\xi - t_\eta)\eta = (\eta, \xi)\xi - \|\eta\|^2\eta$, thus the matrix $A = (a_{ij})$ associated to $t_\xi - t_\eta - \lambda I$ is given by $a_{11} = 1 - \lambda$, $a_{12} = (\eta, \xi)$, $a_{21} = -(\xi, \eta)$, $a_{22} = -\|\eta\|^2 - \lambda$, with characteristic polynomial $(1 - \lambda)(-\|\eta\|^2 - \lambda) + |(\eta, \xi)|^2 = \lambda^2 + (\|\eta\|^2 - 1)\lambda - \|\eta\|^2 + |(\eta, \xi)|^2$; the discriminant is $d = (\|\eta\|^2 - 1)^2 - 4[-\|\eta\|^2 + |(\eta, \xi)|^2] = \|\eta\|^4 + 2\|\eta\|^2 + 1 - 4|(\eta, \xi)|^2 = (\|\eta\|^2 + 1)^2 - [2|(\eta, \xi)|]^2 = [\|\eta\|^2 + 2|(\eta, \xi)| + 1][\|\eta\|^2 - 2|(\eta, \xi)| + 1]$. Now, if $\|\eta\|^2 = 1$, we have $d = 4[1 - |(\eta, \xi)|^2]$; the eigenvalues of A are $\lambda_\pm = \pm 2[1 - |(\eta, \xi)|^2]^{1/2}/2$, hence $|\lambda_+| + |\lambda_-| = 2[1 - |(\eta, \xi)|^2]^{1/2}$. □

C. *Appendix. Proof of Theorem 4.1*

Clearly (1) ⇒ (2) is sufficient. We need to collect some preliminaries. Let φ, φ' be generic elements of $L^1(G)$; let us write $(g\varphi)$ for $\lambda_g^1(\varphi)$, (φg) for $\rho_g^1(\varphi)$. We refer to the integration theory for vector-valued functions contained in Chapter II of [**14**]. By setting $\hat{T}_\varphi(\xi) := \int_G \varphi(g)T_g(\xi)\,dg$ we get a representation of the convolution algebra $L^1(G)$ on X: we have $\hat{T}_{(g\varphi)}(\xi) = T_g(\hat{T}_\varphi(\xi))$ and $\hat{T}_{(\varphi g)}(\xi) = \hat{T}_\varphi(T_g\xi)$; we also have $\hat{T}_\varphi(\hat{T}_{\varphi'}(\xi)) = \hat{T}_{\varphi * \varphi'}(\xi)$ and $\|\hat{T}_\varphi(\xi)\| \le \|\varphi\|\|\xi\|$. Moreover, if $f_n, F \in L^1(G)$, if $|f_n(g)| \le F(g)$ for each n and, as $n \to \infty$, $f_n(g) \to f(g)$ for almost all $g \in G$, then we have $\hat{T}_{f_n}(\xi) \to \hat{T}_f(\xi)$ in X. Finally, we have always $\hat{T}_\varphi(\xi) \in P(X)$, $\|\hat{T}_\varphi(\xi)\| = \|\varphi\|\|\xi\|$ $(\varphi \ge 0, \xi \in P(X))$.

Proof. Fix $\varphi \in PS^1(G)$. Let $K_1 = K \cup K^{-1} \cup \{e\}$ and $\delta > 0$. We will construct functions $Y \in PS^1(G)$, $\psi \in PS(X)$ such that

$$\|Y * (a\varphi) - (a\varphi)\| < \delta \quad (a \in K_1) \tag{C.1}$$

$$\|\hat{T}_{Y*(a\varphi)}(\psi) - \hat{T}_Y(\psi)\| < 4\delta \quad (a \in K_1). \tag{C.2}$$

Then we put $\phi := \hat{T}_\varphi(\psi) \in PS(X)$. We have, for any $a \in K$

$$\begin{aligned}\|T_a(\phi) - \phi\| &= \|T_a(\hat{T}_\varphi(\psi)) - \hat{T}_\varphi(\psi)\| \\ &\le \|T_a(\hat{T}_\varphi(\psi)) - \hat{T}_Y(\hat{T}_{(a\varphi)}(\psi))\| + \|\hat{T}_Y(\hat{T}_{(a\varphi)}(\psi)) - \hat{T}_Y(\psi)\| \\ &\quad + \|\hat{T}_Y(\psi) - \hat{T}_{Y*\varphi}(\psi)\| + \|\hat{T}_{Y*\varphi}(\psi) - \hat{T}_\varphi(\psi)\|\end{aligned}$$

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$$
\begin{aligned}
&= \|\hat{T}_{(a\varphi)-Y*(a\varphi)}(\psi)\| + \|\hat{T}_{Y*(a\varphi)}(\psi) - \hat{T}_Y(\psi)\| + \|\hat{T}_Y(\psi) - \hat{T}_{Y*\varphi}(\psi)\| \\
&\quad + \|\hat{T}_{Y*\varphi}(\psi) - \hat{T}_\varphi(\psi)\| \\
&\le \delta + 4\delta + 4\delta + \delta = 10\delta.
\end{aligned}
$$

To complete the proof, we have 'only' to construct ψ and Y. There exists a neighborhood U of e such that $\|(g\varphi) - \varphi\| < \delta$ whenever $g \in U$. Since K_1 is compact, there is a neighborhood V_0 of e whose closure V is compact and such that $a^{-1}Va \subset U$ for each $a \in K_1$. Hence, for every $a \in K_1$ and $z \in V$, we have $\|(a^{-1}za\varphi) - \varphi\| < \delta$. Putting $\eta_V := \mathrm{Ch}_V/\mu_1(V) \in PS^1(G)$ (Ch_V is the characteristic function of V), for any $a \in K_1$ we have (cf. [**30**], p. 52)

$$
\begin{aligned}
\|\eta_V * (a\varphi) - a\varphi\| &= \left\| \int_V \eta_V(g)(ga\varphi)\,dg - a\varphi \right\| = \left\| \int_V \eta_V(g)((ga\varphi) - a\varphi)\,dg \right\| \\
&\le \int_V \eta_V(g)\|(ga\varphi) - a\varphi\|\,dg = \int_V \eta_V(g)\|(a^{-1}ga\varphi) - \varphi\|\,dg < \delta.
\end{aligned}
$$

Let $C \subset G$ be compact with $\int_{G\setminus C} \varphi(g)\,dg < \delta$. For any $a \in K_1$ we have $a^{-1}K_1^{-1}C \supset C$ then, with $C_1 := K_1^{-1}C = K_1C \supset C$ (compact)

$$
\begin{aligned}
\int_{G\setminus C_1} \varphi(a^{-1}g)\,dg &= \int_{G\setminus K_1^{-1}C} \varphi(a^{-1}g)\,dg \\
&= \int_{G\setminus a^{-1}K_1^{-1}C} \varphi(h)\,dh \le \int_{G\setminus C} \varphi(h)\,dh < \delta.
\end{aligned}
$$

There is a neighborhood W_0 of e whose closure W is compact and such that

$$
\|(\eta_V g) - \eta_V\| < \delta \quad (g \in W). \tag{C.3}
$$

Let W' be a neighborhood of e such that $W'W'^{-1} \subset W$. Since C_1 is compact, there are $c_1, \ldots, c_m \in C_1$ such that $C_1 \subset \bigcup_{i=1}^m W'c_i$. With $A_i = W'c_i$, we have $A_iA_i^{-1} \subset W$. Thus there is a finite partition $B_1, \ldots, B_n$ of C_1 formed by Borel subsets such that

$$
B_jB_j^{-1} \subset W. \tag{C.4}
$$

Fix, for each j, $a_j \in B_j$. If $a \in B_j$ we have, by (C.3) and (C.4), $\|\eta_V a - \eta_V a_j\| = \|\eta_V aa_j^{-1} - \eta_V\| < \delta$.

Let $Y := \eta_V$. Let $\psi \in PS(X)$ be such that $\|T_{a_j}(\psi) - \psi\| < \delta (j = 1, \ldots, n)$ (use (1) in Theorem 4.1). Then, for each $a \in K_1$, we have

$$
\begin{aligned}
&\|\hat{T}_{Y*(a\varphi)}(\psi) - \hat{T}_Y(\psi)\| \\
&\quad = \left\| \int_G (\eta_V * (a\varphi))(g) \cdot T_g(\psi)\,dg - \hat{T}_{\eta_V}(\psi) \right\| \\
&\quad = \left\| \int_G \int_G \eta_V(h) \cdot (a\varphi)(h^{-1}g) \cdot T_g(\psi)\,dh\,dg - \hat{T}_{\eta_V}(\psi) \right\| \\
&\quad = \left\| \int_G \int_G \eta_V(gh) \cdot (a\varphi)(h^{-1}) \cdot T_g(\psi)\,dh\,dg - \hat{T}_{\eta_V}(\psi) \right\| \\
&\quad = \left\| \int_G \int_G \eta_V(gh^{-1}) \cdot \Delta(h^{-1}) \cdot (a\varphi)(h) \cdot T_g(\psi)\,dh\,dg - \hat{T}_{\eta_V}(\psi) \right\|
\end{aligned}
$$

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$$
\begin{aligned}
&= \left\| \int_G \int_G (\eta_V h)(g) \cdot (a\varphi)(h) \cdot T_g(\psi)\, dh\, dg - \hat{T}_{\eta_V}(\psi) \right\| \\
&= \left\| \int_G \left(\int_G (\eta_V h)(g) \cdot T_g(\psi)\, dg \right) \cdot (a\varphi)(h)\, dh - \hat{T}_{\eta_V}(\psi) \right\| \\
&= \left\| \int_G \hat{T}_{(\eta_V h)}(\psi) \cdot (a\varphi)(h)\, dh - \hat{T}_{\eta_V}(\psi) \right\| \\
&= \left\| \int_G \hat{T}_{(\eta_V h)}(\psi) \cdot (a\varphi)(h)\, dh - \left(\int_G (a\varphi)(h)\, dh \right) \cdot \hat{T}_{\eta_V}(\psi) \right\| \\
&\leq \int_G (a\varphi)(h) \cdot \|\hat{T}_{(\eta_V h)}(\psi) - \hat{T}_{\eta_V}(\psi)\|\, dh \\
&= \int_{G \setminus C_1} (a\varphi)(h) \cdot \|\hat{T}_{(\eta_V h)}(\psi) - \hat{T}_{\eta_V}(\psi)\|\, dh \\
&\quad + \sum_j \int_{B_j} (a\varphi)(h) \cdot \|\hat{T}_{(\eta_V h)}(\psi) - \hat{T}_{\eta_V}(\psi)\|\, dh \\
&\leq 2\delta + \sum_j \int_{B_j} (a\varphi)(h) \cdot \|\hat{T}_{(\eta_V h)}(\psi) - \hat{T}_{(\eta_V a_j)}(\psi)\|\, dh \\
&\quad + \sum_j \int_{B_j} (a\varphi)(h) \cdot \|\hat{T}_{(\eta_V a_j)}(\psi) - \hat{T}_{\eta_V}(\psi)\|\, dh \\
&< 2\delta + \delta + \delta = 4\delta. \qquad \square
\end{aligned}
$$

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