ON THE STABILITY PROBLEM IN HYDRODYNAMICS

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ABSTRACT. The paper deals with the problem of the effect of two-dimensional first order disturbances on the linear and parabolic flows of a viscous fluid. If the laminar flow is in the direction x and is bounded by the planes $y = \pm h$, and if the stream function of the disturbance is assumed to be of the form $f(y) \exp i\alpha \left(\frac{CU}{h} t - \frac{x}{h}\right)$, it is found that, for small Reynolds numbers $R\ (= Uh/\nu)$, C can be developed in the power series

$$C = \frac{C_{-1}}{(i\alpha R)} + C_0 + (i\alpha R)\, C_1 + (i\alpha R)^2\, C_2 + \ldots . \qquad (A)$$

The first term in (1) corresponds to a disturbance in still water and is known from Rayleigh's investigation of that problem; the succeeding C's are real and can be obtained by an application of Schrödinger's perturbation theory of wave mechanics. In the case of the parabolic flow, expressions for C_0 and C_1 were thus derived for both the symmetrical and anti-symmetrical types of disturbance. The imaginary part of C, as represented by $-\frac{C_{-1}}{\alpha R} + \alpha R C_1$ in this approximation, is found to be positive. When α is of the order of unity, the value of R which makes the second term $\frac{1}{10}$ of the first is about 50 for the symmetrical disturbance and 100 for the antisymmetrical disturbance. C_0, the first term in the expression for the phase velocity, is less than 1 and increases with α, which implies a group velocity greater than the phase velocity.

In the case of the linear flow it is found that the C's with even subscripts vanish, implying a vanishing phase velocity. This verifies the assumption to that effect made by Southwell and Chitty, and it also suggests that the region of convergence of (A) is limited to the range of values of R covered by these authors. Expressions are derived for C_1 and C_3 and are evaluated for $\alpha = 1$. The formulae thus obtained for the damping constant give an initial decrease with R, which is not shown in the corresponding curve of Southwell and Chitty.

An alternative method of determining C is outlined in the last section. Use is made of the orthogonal system of the characteristic functions for the vorticity of a disturbance in still water. The equation determining C is reduced to an infinite determinant in which C occurs only in the diagonal elements. When $R \to 0$, the determinant can be evaluated and the roots C are found to coincide with the first two terms in (A).

Finally it is shown that the two-dimensional laminar flow is stable for values of R less than the smaller of the quantities 2α and α^3.

Introduction. The theoretical investigations of the problem of the breakdown of the laminar flow of a viscous fluid, which have extended over a period of fifty years, present a record of only partial success*. Three cases have been studied:

* For a review of the literature on the subject see Lamb, *Hydrodynamics* (1932), 670, also ref. §, p. 2.

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(1) the rotatory motion of a fluid between concentric cylinders; (2) the linear laminar flow of a fluid between parallel planes in relative motion; (3) the parabolic laminar flow of a fluid between fixed parallel planes. The first has been successfully treated by G. I. Taylor*, who also verified his conclusions by experiment. From the work of L. Hopf† and von Mises‡ it appears that the linear flow is stable for first order perturbations. Such is also the opinion of R. V. Southwell and L. Chitty§, who have recently made a detailed study of the nature of the disturbed flow for small Reynolds numbers. Very little information of a definite nature is, however, available on the stability of the parabolic flow.

We know from the investigation of Rayleigh‖ that the parabolic flow, as well as all other types of laminar flow, is stable in the limit of vanishing R (Reynolds number). As R is increased from the value zero, the character of the possible small oscillations presumably changes continuously from that of a disturbance in a still fluid, and stability presumably persists in some finite range of small values of R. Thus the idea suggests itself of building up a solution for finite values of R from the known solution for $R=0$. This can be accomplished by the well-known procedure of Schrödinger's perturbation theory of wave mechanics¶, as will be shown in the following.

1. Let the motion take place in the xy plane, and let the boundaries be at $y=\pm h$. If ζ denotes the z-component of the vorticity,

$$\zeta=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}, \tag{1}$$

the equations of motion and of continuity are

$$\frac{D\zeta}{Dt}=\nu\nabla^2\zeta, \tag{2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{3}$$

Equation (3) is solved by putting $u=\dfrac{\partial\psi}{\partial y}$, $v=-\dfrac{\partial\psi}{\partial x}$, whereby (2) becomes

$$\frac{\partial\nabla^2\psi}{\partial t}+\frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y}=\nu\nabla^2\nabla^2\psi. \tag{4}$$

We now introduce non-dimensional coordinates $x'=x/h$, $y'=y/h$, $t'=tU/h$, where U stands for the velocity at the centre of the canal in the parabolic case, or

* G. I. Taylor, *Phil. Trans.* A, 223 (1923), 289.
† L. Hopf, *Ann. d. Physik*, 44 (1914), 1.
‡ v. Mises, *Jahresb. d.D. Math. Ver.* 21 (1912), 241.
§ R. V. Southwell and L. Chitty, *Phil. Trans.* A, 229 (1930), 205.
‖ Rayleigh, *Scientific Papers*, Vol. 3, p. 582.
¶ E. Schrödinger, *Ann. d. Physik*, 80 (1926), 437. See also A. Sommerfeld, *Ergänzungsband* (1929), 172.

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for the absolute value of the velocity of the boundaries in the linear case*. On substituting in (4) and dropping the accents, we obtain

$$\frac{\partial \nabla^2\psi}{\partial t}+\frac{\partial \psi}{\partial y}\frac{\partial \nabla^2\psi}{\partial x}-\frac{\partial \psi}{\partial x}\frac{\partial \nabla^2\psi}{\partial y}=\frac{1}{R}\nabla^2\nabla^2\psi, \quad R\equiv\frac{Uh}{\nu}. \tag{5}$$

Now define the stream function ψ' of the disturbing flow by $\psi=\int^y U_0 dy+\psi'$, where $U_0(y)$ stands for the laminar velocity. On substituting in (5) and dropping products in ψ' and its derivatives, we obtain

$$\frac{\partial \nabla^2\psi'}{\partial t}+U_0\frac{\partial \nabla^2\psi'}{\partial x}-\frac{d^2U_0}{dy^2}\frac{\partial \psi'}{\partial x}=\frac{1}{R}\nabla^2\nabla^2\psi'. \tag{6}$$

With $\psi'=e^{i\alpha(Ct-x)}f(y)$, (6) becomes

$$f^{\mathrm{iv}}-2\alpha^2\ddot{f}+\alpha^4 f+i\alpha R\,[2f+(1-y^2-C)(\ddot{f}-\alpha^2 f)]=0, \tag{7}$$

$$f^{\mathrm{iv}}-2\alpha^2\ddot{f}+\alpha^4 f+i\alpha R\,(y-C)(\ddot{f}-\alpha^2 f)=0, \tag{8}$$

for the parabolic and linear flows respectively. The boundary conditions for the vanishing of the velocities are

$$f(1)=f(-1)=\dot{f}(1)=\dot{f}(-1)=0. \tag{9}$$

For given values of α and R there is a set of eigenvalues of C for which the systems (7) and (9), or (8) and (9) are satisfied. The motion is then proved to be stable for small oscillations, if the imaginary parts of the C's are positive, and vice versa.

If

$$\phi\equiv\ddot{f}-\alpha^2 f, \tag{10}$$

then the solution for f which vanishes, together with its first derivatives, at $y=-1$, is

$$f=\frac{1}{\alpha}\int_{-1}^{y}\phi(t)\sinh\alpha(y-t)\,dt. \tag{11}$$

The vanishing of f and $\dot{f}$ at $y=1$ requires that

$$\int_{-1}^{1}\phi(t)\sinh\alpha(1-t)\,dt=\int_{-1}^{1}\phi(t)\cosh\alpha(1-t)\,dt=0,$$

which is equivalent to the conditions

$$\int_{-1}^{1}\phi(t)\sinh\alpha t\,dt=0, \tag{12}$$

$$\int_{-1}^{1}\phi(t)\cosh\alpha t\,dt=0. \tag{13}$$

It should be noticed that orthogonality to $\sinh\alpha y$ and $\cosh\alpha y$ are also necessary conditions for the solutions for $\nabla^2\psi$ of the non-linearized equation (4), if ψ is assumed to have the form $\psi=e^{i\alpha x}f(y,t)$. Conversely, if ϕ satisfies (12) and (13), then a solution f of (10), which vanishes together with its first derivative at one

* In the latter case the laminar velocity is assumed to be given by yU/h, where y ranges from -1 to 1.

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of the boundaries, will vanish also at the other boundary. This follows from the identities

$$\int_{-1}^{1} \phi(y) \sinh \alpha y \, dy = \sinh \alpha [\dot{f}(1) + \dot{f}(-1)] - \alpha \cosh \alpha [f(1) - f(-1)], \quad (14)$$

$$\int_{-1}^{1} \phi(y) \cosh \alpha y \, dy = \cosh \alpha [\dot{f}(1) - \dot{f}(-1)] - \alpha \sinh \alpha [f(1) + f(-1)], \quad (15)$$

where the determinant of the values of f and $\dot{f}$ at the other boundary is seen to be α^*. Let us try to satisfy the systems (7), (8) and (9) by the following expansions:

$$f = f_0 + (i\alpha R) f_1 + (i\alpha R)^2 f_2 + \ldots, \quad (16)$$

$$\phi = \phi_0 + (i\alpha R) \phi_1 + (i\alpha R)^2 \phi_2 + \ldots, \quad (17)$$

$$C = \frac{C_{-1}}{i\alpha R} + C_0 + (i\alpha R) C_1 + (i\alpha R)^2 C_2 \ldots. \quad (18)$$

It will be shown that the f_i, ϕ_i and C_i can be determined uniquely. We thus obtain solutions for f and C which are valid in the domain of convergence of the series on the right-hand sides of (16) and (18) respectively.

Now substitute in (7) and (8) and equate the coefficient of each power of $i\alpha R$ to zero. The term independent of $i\alpha R$ yields in both equations the relation

$$f_0^{iv} - 2\alpha^2 \ddot{f}_0 + \alpha^4 f_0 - C_{-1}(\ddot{f}_0 - \alpha^2 f_0) = f_0^{iv} + (p^2 - \alpha^2) \ddot{f}_0 - \alpha^2 p^2 f_0 = 0, \quad (19)$$

or

$$\ddot{\phi}_0 + p^2 \phi_0 = 0, \quad (20)$$

where

$$p^2 = -\alpha^2 - C_{-1}. \quad (21)$$

The solutions of the system (20), (12) and (13), being clearly either even or odd, are

$$\phi_0 = \cos pt, \quad (22)$$

with

$$p \tan p = -\alpha \tanh \alpha \quad (23)$$

and

$$\phi_0 = \sin pt, \quad (24)$$

with

$$\frac{\tan p}{p} = \frac{\tanh \alpha}{\alpha}. \quad (25)$$

The corresponding solutions for f_0 are

$$f_0 = \frac{1}{\gamma}\left(\frac{\cos p}{\cosh \alpha} \cosh \alpha y - \cos py\right) \quad (26)$$

and

$$f_0 = \frac{1}{\gamma}\left(\frac{\sin p}{\sinh \alpha} \sinh \alpha y - \sin py\right), \quad (27)$$

where

$$\gamma = \alpha^2 + p^2 = -C_{-1}. \quad (28)$$

Equations (23) and (25) each have an infinite number of positive real roots p_n, the two sets of roots separating each other. The above is the well-known solution of Rayleigh† for the small oscillations of a fluid at rest.

* It can easily be shown that a disturbance which is independent of x ($\alpha = 0$) is damped.

† Rayleigh, *Scientific Papers*, Vol. 3, p. 582.

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Proceeding now with the parabolic flow we obtain, on annulling the coefficient of $(i\alpha R)$, the equation

$$f_1^{iv} + (p^2 - \alpha^2)\ddot{f}_1 - \alpha^2 p^2 f_1 = \ddot{\phi}_1 + p^2\phi_1 = -[2f_0 + (1 - y^2 - C_0)\phi_0] = 0, \quad (29)$$

which serves to determine f_1 and C_0. The latter can be determined without solving (29), as follows. Multiply (29) by f_0 and (19) by f_1, subtract and integrate from -1 to 1. It will be found that the left-hand side of the resulting equation vanishes on account of the identities

$$f_1^{iv} f_0 - f_0^{iv} f_1 = \frac{d}{dy}(\dddot{f}_1 f_0 - \ddot{f}_1 \dot{f}_0 - f_1 \dddot{f}_0 + \dot{f}_1 \ddot{f}_0); \quad \ddot{f}_1 f_0 - \ddot{f}_0 f_1 = \frac{d}{dy}(\dot{f}_1 f_0 - f_1 \dot{f}_0)$$

and the boundary conditions. Hence

$$\int_{-1}^{1} [2f_0 + (1 - y^2 - C_0)\phi_0] f_0 \, dy = 0, \quad (30)$$

i.e.

$$C_0\left(\int_0^1 f_0\phi_0 \, dy\right) = \left[2\int_0^1 f_0^2 \, dy + \int_0^1 (1 - y^2)\phi_0 f_0 \, dy\right], \quad (31)$$

since f and ϕ are either even or odd. Similarly we obtain

$$\ddot{\phi}_2 + p^2\phi_2 = -[2f_1 + (1 - y^2 - C_0)\phi_1 - C_1\phi_0], \quad (32)$$

$$C_1\left(\int_0^1 f_0\phi_0 \, dy\right) = \left[2\int_0^1 f_1 f_0 \, dy + \int_0^1 (1 - y^2 - C_0)\phi_1 f_0 \, dy\right]. \quad (33)$$

Clearly, the equation determining ϕ_n is of the form

$$\ddot{\phi}_n + p^2\phi_n = F_n(y), \quad (34)$$

where $F_n(y)$ depends on the preceding f's and ϕ's and the unknown C_{n-1}. C_{n-1} can then be determined by the condition $\int_0^1 F_n f_0 \, dy = 0$. Furthermore, $F_n(y)$ consists of terms of the type $P(y)(\sin py, \cos py, \sinh \alpha y, \cosh \alpha y)$, $P(y)$ being a polynomial in y, so that the integration of equation (34) can be conveniently performed by assuming a form for the solution and then determining its coefficients. Having obtained a solution for ϕ_n, we obtain f_n similarly by integrating the equation $\ddot{f}_n - \alpha^2 f_n = \phi_n$ and adding a term $A \sinh \alpha y$, when f_n is odd, and $A \cosh \alpha y$, when f_n is even. If A is then determined by the condition $f_n(1) = 0$, it will be found that $\dot{f}_n(1) = 0$, this having been secured by the orthogonality of F_n to f_0. It should be noticed that in integrating equation (34) we do not use the solutions ϕ_0 of the homogeneous equation, since such a solution will simply generate another multiple of f, as defined in (16), and will not affect the C_i. To illustrate this point assume a solution of (29) of the form $\phi_1 = \phi_1^* + A\phi_0$, where ϕ_1^* is independent of ϕ_0. Then $f_1 = f_1^* + Af_0$ and the equation determining ϕ_2 is

$$\ddot{\phi}_2 + p^2\phi_2 = -[2f_1^* + (1 - y^2 - C_0)\phi_1^* - C_1\phi_0] - A[2f_0 + (1 - y^2 - C_0)\phi_0]. \quad (35)$$

The coefficient of A, being orthogonal to f_0, will clearly not affect C_1. On integrating (35), we obtain, in view of (29), $\phi_2 = \phi_2^* + A\phi_1$, and so on. Thus, instead of the original solution f, we obtain $(1 + A)f$. An inspection of the structure of

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equation (34) shows that the same results if a multiple of ϕ_0 is added to any of the later ϕ's.

In the following we give expressions for C_0, ϕ_1, f_1 and C_1 for the even and odd solutions of the parabolic flow.

I. *Even solutions*

$$C_0 = 1 - \frac{2}{\gamma} + \frac{8p^2}{\gamma^2} - \frac{2p}{(p+\sin p\cos p)}\left\{\frac{\cos^2 p}{\alpha\gamma\cosh^2\alpha}(\sinh\alpha\cosh\alpha+\alpha) + \frac{2\cos^2 p}{\gamma} + \frac{1}{6} + \frac{1}{8p^3}\right.$$
$$\left. \times [2p\cos 2p + (2p^2-1)\sin 2p]\right\}, \quad (36)$$

$$\phi_1 = \sin py\left[\left(\frac{1}{6p}\right)y^3 + \left(-\frac{1}{2p} + \frac{C_0}{2p} + \frac{1}{\gamma p} - \frac{1}{4p^3}\right)y\right]$$
$$+ \left(\frac{1}{4p^2}\right)y^2\cos py - \left(\frac{2\cos p}{\gamma^2\cosh\alpha}\right)\cosh\alpha y, \quad (37)$$

$$f_1 = \sin py\left[-\left(\frac{1}{6\gamma p}\right)y^3 + \left(-\frac{1}{p\gamma^2} + \frac{1}{4\gamma p^3} + \frac{1}{2\gamma p} - \frac{C_0}{2\gamma p} + \frac{4p}{\gamma^3}\right)y\right]$$
$$+ \cos py\left[-\left(\frac{1}{\gamma^2} + \frac{1}{4\gamma p^2}\right)y^2 + \left(-\frac{4}{\gamma^3} + \frac{1}{\gamma^2} - \frac{C_0}{\gamma^2} + \frac{8p^2}{\gamma^4}\right)\right]$$
$$- \left(\frac{\cos p}{\alpha\gamma^2\cosh\alpha}\right)y\sinh\alpha y - A\cosh\alpha y, \quad (38)$$

$$A\cosh\alpha \equiv -\left(\frac{1}{\alpha\gamma^2}\right)\cos p\tanh\alpha + \left[-\frac{1}{p\gamma^2} + \frac{1}{4\gamma p^3} - \frac{1}{6\gamma p} + \frac{4p}{\gamma^3} + \frac{1}{2\gamma p}(1-C_0)\right]\sin p$$
$$+ \left[-\frac{4}{\gamma^3} - \frac{1}{\gamma^2} + \frac{8p^2}{\gamma^4} - \frac{1}{4\gamma p^2} + \frac{(1-C_0)}{\gamma^2}\right]\cos p, \quad (39)$$

$$-\left(\frac{1}{2p}\right)(p+\sin p\cos p)\,C_1$$
$$= \frac{2\cos^2 p}{\gamma^2\cosh^2\alpha}\left\{\frac{1}{6} - \frac{1}{2}(1-C_0) + \sinh 2\alpha\left[\frac{1}{4\alpha} + \frac{1}{4\alpha^3} - \frac{1}{4\alpha}(1-C_0)\right] - \left(\frac{1}{2\alpha^2}\right)\cosh 2\alpha\right\}$$
$$+ \left(\frac{2\cos^2 p}{\alpha\gamma^3\cosh^2\alpha} - \frac{A\cos p}{\alpha\cosh\alpha}\right)(\sinh\alpha\cosh\alpha+\alpha) + (p+\sin p\cos p)$$
$$\times\left[\frac{12}{p\gamma^3} - \frac{256p}{\gamma^4} + \frac{320p^3}{\gamma^5} + \frac{3}{p\gamma^2} - \frac{40p}{3\gamma^3} + \frac{1}{6\gamma p} - \frac{2}{p^3\gamma^2} + \frac{1}{6\gamma p^3} + \frac{19}{32p^7} - \frac{19}{48p^5} + \frac{1}{15p^3} - \frac{3}{4\gamma p^5}\right.$$
$$\left. + (1-C_0)\left(-\frac{6}{p\gamma^2} + \frac{16p}{\gamma^3} - \frac{2}{3\gamma p} - \frac{1}{4\gamma p^3} + \frac{3}{8p^5} - \frac{5}{24p^3}\right) + \left(\frac{1}{8p^3} + \frac{1}{2\gamma p}\right)(1-C_0)^2\right]$$
$$+ \cos^2 p\left[\frac{84}{\gamma^3} - \frac{160p^2}{\gamma^4} + \frac{20}{3\gamma^2} + \frac{4}{p^2\gamma^2} + \frac{1}{3\gamma p^2} - \frac{1}{12p^2} + \frac{19}{24p^4} - \frac{19}{16p^6} + \frac{3}{2\gamma p^4}\right.$$
$$\left. + (1-C_0)\left(-\frac{8}{\gamma^2} + \frac{1}{2\gamma p^2} + \frac{1}{3p^2} - \frac{3}{4p^4}\right) - \frac{1}{4p^2}(1-C_0)^2\right]$$
$$+ \sin p\cos p\left[\frac{6}{p\gamma^2} - \frac{80p}{3\gamma^3} + \frac{2}{3\gamma p} + \frac{1}{3\gamma p^3} + \frac{4}{15p^3} - \frac{19}{24p^5} - (1-C_0)\left(\frac{4}{3\gamma p} + \frac{5}{12p^3}\right)\right]. \quad (40)$$

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II. *Odd solutions*

$$C_0=\frac{2}{3}-\frac{2}{\gamma}+\frac{1}{2p^2}+\frac{8p^2}{\gamma^2}+\frac{1}{(p-\sin p\cos p)}\left[\frac{2}{3}\sin p\cos p-\left(\frac{1}{p}+\frac{4p}{\gamma}\right)\sin 2p-\left(\frac{2p\sin^2 p}{\alpha\gamma\sinh^2\alpha}\right)(\sinh\alpha\cosh\alpha-\alpha)\right], \quad (41)$$

$$\phi_1=\cos py\left[-\left(\frac{1}{6p}\right)y^3+\left(\frac{1}{4p^3}-\frac{1}{\gamma p}+\frac{1}{2p}-\frac{C_0}{2p}\right)y\right]+\left(\frac{1}{4p^2}\right)y^2\sin py-\left(\frac{2\sin p}{\gamma^2\sinh\alpha}\right)\sinh\alpha y, \quad (42)$$

$$f_1=\cos py\left[\left(\frac{1}{6\gamma p}\right)y^3+\left(-\frac{4p}{\gamma^3}+\frac{1}{p\gamma^2}-\frac{1}{4p^3\gamma}-\frac{1}{2\gamma p}+\frac{C_0}{2\gamma p}\right)y\right]+\sin py\left[-\left(\frac{1}{\gamma^2}+\frac{1}{4\gamma p^2}\right)y^2+\left(\frac{8p^2}{\gamma^4}-\frac{4}{\gamma^3}+\frac{1}{\gamma^2}-\frac{C_0}{\gamma^2}\right)\right]-\left(\frac{\sin p}{\alpha\gamma^2\sinh\alpha}\right)y\cosh\alpha y-A\sinh\alpha y, \quad (43)$$

$$A\sinh\alpha\equiv-\left(\frac{1}{\alpha\gamma^2}\right)\sin p\coth\alpha+\left[\frac{1}{p\gamma^2}-\frac{1}{4p^3\gamma}+\frac{1}{6\gamma p}-\frac{4p}{\gamma^3}-\frac{1}{2\gamma p}(1-C_0)\right]\cos p+\left[-\frac{4}{\gamma^3}-\frac{1}{\gamma^2}+\frac{8p^2}{\gamma^4}-\frac{1}{4\gamma p^2}+\frac{1}{\gamma^2}(1-C_0)\right]\sin p, \quad (44)$$

$$-\left(\frac{1}{2p}\right)(p-\sin p\cos p)\,C_1=\frac{2\sin^2 p}{\gamma^2\sinh^2\alpha}\left\{-\frac{1}{6}+\frac{1}{2}(1-C_0)+\left[\frac{1}{4\alpha}+\frac{1}{4\alpha^3}-\frac{1}{4\alpha}(1-C_0)\right]\sinh 2\alpha-\left(\frac{1}{2\alpha^2}\right)\cosh 2\alpha\right\}+\left(\frac{2\sin^2 p}{\alpha\gamma^3\sinh^2\alpha}-\frac{A\sin p}{\alpha\sinh\alpha}\right)(\sinh\alpha\cosh\alpha-\alpha)+(p-\sin p\cos p)\left[\frac{12}{\gamma^3 p}-\frac{256p}{\gamma^4}+\frac{320p^3}{\gamma^5}+\frac{3}{\gamma^2 p}-\frac{40p}{3\gamma^3}+\frac{1}{6\gamma p}-\frac{2}{\gamma^2p^3}+\frac{1}{6\gamma p^3}-\frac{3}{4\gamma p^5}+\frac{1}{15p^3}-\frac{19}{48p^5}+\frac{19}{32p^7}+(1-C_0)\left(-\frac{6}{\gamma^2 p}+\frac{16p}{\gamma^3}-\frac{1}{4\gamma p^3}-\frac{2}{3\gamma p}-\frac{5}{24p^3}+\frac{3}{8p^5}\right)+(1-C_0)^2\left(\frac{1}{8p^3}+\frac{1}{2\gamma p}\right)\right]+\sin^2 p\left[\frac{84}{\gamma^3}-\frac{160p^2}{\gamma^4}+\frac{20}{3\gamma^2}+\frac{4}{p^2\gamma^2}+\frac{3}{2\gamma p^4}+\frac{1}{3\gamma p^2}-\frac{1}{12p^2}+\frac{19}{24p^4}-\frac{19}{16p^6}+(1-C_0)\left(-\frac{8}{\gamma^2}+\frac{1}{2\gamma p^2}+\frac{1}{3p^2}-\frac{3}{4p^4}\right)-\frac{1}{4p^2}(1-C_0)^2\right]+\sin p\cos p\left[-\frac{6}{\gamma^2 p}+\frac{80}{3}\frac{p}{\gamma^3}-\frac{2}{3\gamma p}-\frac{1}{3\gamma p^3}-\frac{4}{15p^3}+\frac{19}{24p^5}+(1-C_0)\left(\frac{4}{3\gamma p}+\frac{5}{12p^3}\right)\right]. \quad (45)$$

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The limiting values of the C's as α approaches zero or infinity are for the even solutions:

$$\underset{\alpha\to 0}{C_0}=\frac{2}{3}-\frac{5}{2p^2},\quad \underset{\alpha\to\infty}{C_0}=\frac{2}{3}+\frac{1}{2p^2},$$

$$\underset{\alpha\to 0}{C_1}=-\frac{1}{45p^2}-\frac{7}{12p^4}+\frac{77}{8p^6},\quad \underset{\alpha\to\infty}{C_1}=-\frac{1}{45p^2}+\frac{5}{12p^4}-\frac{7}{8p^6},$$

and for the odd solutions:

$$\underset{\alpha\to 0}{C_0}=\frac{2}{3}+\frac{5}{6p^2},\quad \underset{\alpha\to\infty}{C_0}=\frac{2}{3}+\frac{1}{2p^2},$$

$$\underset{\alpha\to 0}{C_1}=-\frac{1}{45p^2}+\frac{11}{4p^4}-\frac{355}{8p^6},\quad \underset{\alpha\to\infty}{C_1}=-\frac{1}{45p^2}+\frac{5}{12p^4}-\frac{7}{8p^6}.$$

We can obtain a partial check on the above formulae by developing f and C in the expansions

$$f=f_0+\alpha f_1+\alpha^2 f_2+\ldots,$$

$$C=C_0+\alpha C_1+\ldots,$$

and determining the f_i and C_i by the same method of the perturbation theory. The new C_i can also be obtained by expanding each of the C's in (18) in a power series in α and collecting the coefficients of each power of α.

Values of C_{-1}, C_0 and C_1 for several values of α are given in Table I.

Table I

α	$-C_{-1}$		C_0		C_1	
	Even	Odd	Even	Odd	Even	Odd
0	9·870	20·191	0·4134	0·7079	0·001772	0·0002539
1	9·314	20·571	0·5198	0·7122	0·000842	0·0002669
1·2566	9·274	20·829	0·5637	0·7145	0·000616	0·0002742
2·3956	11·291	—	0·7186	—	—	—
∞	∞	∞	0·8693	0·7173	0·001184	0·0011158

We notice that C_0, the only term which we have obtained for the phase velocity, is everywhere less than 1 (the velocity at the centre of the canal) and increases continuously with α. This variation of C_0 with α implies that the group velocity V which is given by

$$V=\frac{d(\alpha C)}{d\alpha}\simeq\frac{d(\alpha C_0)}{d\alpha},\qquad(46)$$

is, for sufficiently small values of R, *greater* than the phase velocity. The even C_1, like C_{-1}, goes through a minimum value for some value of α greater than 1. Written in ordinary units the dependence of ψ' on x and t is given by the factor $e^{i\alpha(Clt/h-x/h)}$. The imaginary part of C, which determines the damping, is by (18)

$$I(C)=-\frac{C_{-1}}{\alpha R}[1+(\alpha R)^2 C_1-(\alpha R)^4 C_3+\ldots].\qquad(47)$$

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The degree of convergence of this expansion can be illustrated in the case of even solutions and $\alpha = 1{\cdot}2566$ $(= 0{\cdot}4\pi)$, where $I(C) = \dfrac{9{\cdot}274}{\alpha R}[1 + 0{\cdot}0000664\,(\alpha R)^2 + \ldots]$. This function is tabulated in Table II for several values of s, defined by $s = \dfrac{16}{\pi^4}\alpha R$, and should be compared with the values of $-I(\lambda/k)$ in the example treated by Dr Goldstein in the accompanying paper*. Clearly, the neglected terms of the series become perceptible only at $s = 10$. The corresponding formula for the odd solutions is

$$I(C) = \frac{20{\cdot}829}{\alpha R}(1 + 0{\cdot}00001316\,(\alpha R)^2 + \ldots]. \tag{48}$$

Table II

s	0·2	0·4	0·6	1·0
αR	1·483	5·930	13·34	37·06
$6{\cdot}64 \times 10^{-5}\,(\alpha R)^2$	0·0000984	0·000394	0·000886	0·00246
$I(C)$	7·617	3·810	2·541	1·527
s	2·0	4·0	6·0	10·0
αR	148·3	593·0	1334	3706
$6{\cdot}64 \times 10^{-5}\,(\alpha R)^2$	0·00984	0·0394	0·0886	0·246
$I(C)$	0·7692	0·3958	0·2764	0·1898

If, guided by the results obtained above for the even solutions, we assume the validity of the approximate formula (48) up to the point where the second term is 0·25, we can infer from it stability for values of αR up to 140. When the wave-length of the disturbance is about three times the width of the canal (α of the order of unity), we can thus predict stability from our approximate formulae for values of R up to 50 in the case of even solutions and up to 100 in the case of odd solutions. For longer wave-lengths the corresponding ranges of R increase as the square of the wave-length. More definite information as to the domain of convergence of the expression (47) for $I(C)$ must await the development of additional terms in the series. This will be done in the near future.

Thus far we have dealt only with the lowest harmonic of the disturbance, corresponding to the first roots of p in (23) and (25). An inspection of the formulae for C_1 will show that it is of the order of $1/p^2$ for large values of p. Preliminary calculations of C_2 and C_3 for small α also indicate that they are respectively of the order of $1/p^4$ and $1/p^6$ for large values of p. Thus it is probable, although yet unproved, that whenever the lowest harmonic is stable the higher harmonics are also stable.

* S. Goldstein, *Proc. Camb. Phil. Soc.* 32 (1936), 40–54. I am indebted to Dr Goldstein for having shown me his unpublished results and for the opportunity to discuss the subject matter with him.

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2. *The linear flow.* The equation determining ϕ_1 and C_0 is now

$$\ddot{\phi}_1 + p^2\phi_1 = -(y - C_0)\phi_0. \tag{2·1}$$

C_0 is determined, as before, by making the right-hand side of (2·1) orthogonal to f_0. Now, it is clear that, whether we deal with the even or odd solutions, $y\phi_0 f_0$ is odd, and therefore its integral from -1 to 1 vanishes. It follows that C_0 vanishes. Similarly we conclude from the succeeding equations that all the C's with even subscripts must vanish. This implies that the phase velocity is zero, in agreement with the assumption made by Southwell and Chitty. Furthermore, the closure of the curves representing $I(C)$ for certain values of R, which was discovered by these authors, suggests that the series (18) for C converges only up to these values of R. The application of our method to the linear flow thus yields no new information about stability, but it offers a more direct means of computing $I(C)$ for small values of αR. In comparison with the parabolic flow the analysis is here simplified by the non-appearance of f in the differential equations. The expressions for C_1 for the initially even and odd solutions respectively are

$$C_1 = -\frac{1}{\gamma^2} + \frac{12p^2}{\gamma^3} - \frac{1}{2\gamma} - \frac{p}{(p + \sin p \cos p)}\left\{\frac{1}{12p^2} + \frac{1}{32p^5}[(10p - 4p^3)\cos 2p + (10p^2 - 5)\sin 2p] + \cos^2 p\left(\frac{6}{\gamma^2} + \frac{1}{\gamma p^2}\right) + \frac{1}{\gamma p}\cos p \sin p\right\}, \tag{2·2}$$

$$C_1 = -\frac{1}{\gamma^2} + \frac{12p^2}{\gamma^3} - \frac{1}{2\gamma} - \frac{p}{(p - \sin p \cos p)}\left\{\frac{1}{12p^2} - \frac{1}{32p^5}[(10p - 4p^3)\cos 2p + (10p^2 - 5)\sin 2p] + \sin^2 p\left(\frac{6}{\gamma^2} + \frac{1}{\gamma p^2}\right) - \frac{1}{\gamma p}\sin p \cos p\right\}. \tag{2·3}$$

We do not put down here the expressions for C_3, which are rather long. We find that, when $\alpha = 1$,

$$(\alpha R)\, I(C) = 9{\cdot}314\,(1 - 0{\cdot}0010667\,R^2 - 1{\cdot}723 \times 10^{-6} R^4 + \ldots), \tag{2·4}$$

$$(\alpha R)\, I(C) = 20{\cdot}571\,(1 - 0{\cdot}00050228\,R^2 - 7{\cdot}35 \times 10^{-8} R^4 + \ldots), \tag{2·5}$$

for the initially even and odd solutions. The corresponding values given by Southwell and Chitty are represented respectively by the upper and lower halves of the curve $\gamma = 1$ of their Fig. 1*. As far as they go, formulae (2·4) and (2·5) indicate an initial dipping of the curve downwards, which is not shown in Fig. 1.

3. In this section we outline an alternative method of computing the damping constant C, which is closely related to the perturbation method of the preceding sections. It is based essentially on the fact that the characteristic

* For a given state of laminar flow Southwell and Chitty assign a Reynolds number which is four times the value of R in (2·4) and (2·5).

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functions $\cos p_n y$ and $\sin p_n y$ for the disturbance of a fluid at rest form an orthogonal set. The orthogonal set $\sin p_n t$, where p_n are the roots of the equation

$$\frac{\tan p}{p} = \frac{1}{1-ah}, \tag{3·1}$$

occurs in the problem of the cooling of a homogeneous sphere* and in other problems involving surface cooling by radiation. Equation (3·1), in which a and h are positive, differs, however, from (25) in that $0 \leqslant \tanh \alpha/\alpha \leqslant 1$, whereas $1/(1-ah)$ ranges in the remaining part of the real axis. As a consequence, equation (25) has an additional imaginary root $i\alpha$ and associated with it the characteristic function $\sinh \alpha y$. Now it has been proved† that a function which can be represented by a Fourier series in the interval (0, 1) can also be represented by the Fourier series formed from the characteristic functions of the cooling sphere. This result can immediately be carried over to our case where y ranges from -1 to 1, and where the orthogonal set is formed from $\sinh \alpha y$, $\cosh \alpha y$, $\sin p_n y$ of (24) and $\cos p_n y$ of (22).

Let us assume that equation (4) has a solution, periodic in x, of the form $\psi = e^{i\alpha x} f(y, t)$. Then

$$\nabla^2 \psi = e^{i\alpha x}\left(\frac{\partial^2 f}{\partial y^2} - \alpha^2 f\right) \equiv e^{i\alpha x} \phi(y, t), \tag{3·2}$$

where ϕ has to be orthogonal to $\sinh \alpha y$ and $\cosh \alpha y$. Such a function can be represented by

$$\phi = \sum_1^\infty A_n(t) \sin p_n y + \sum_1^\infty B_n(t) \cos p_n y, \tag{3·3}$$

with
$$A_n(t) = \int_{-1}^{1} \phi(y, t) \sin p_n y \, dy, \quad B_n(t) = \int_{-1}^{1} \phi(y, t) \cos p_n y \, dy. \tag{3·4}$$

Furthermore, equation (3·3) can be integrated term by term to form f:

$$f(y, t) = \sum_1^\infty A_n(t) \frac{1}{\gamma_n}\left(\frac{\sin p_n}{\sinh \alpha} \sinh \alpha y - \sin p_n y\right) + \sum_1^\infty B_n(t) \frac{1}{\gamma_n}\left(\frac{\cos p_n}{\cosh \alpha} \cosh \alpha y - \cos p_n y\right)‡. \tag{3·5}$$

In the linearized equations (7) and (8), where ψ' was assumed to be periodic in t, f and ϕ stand for the y factors of the functions appearing in (3·5) and (3·3). Hence in their Fourier developments the A's and B's are constants. For simplicity let

* J. Fourier, *Analytical theory of heat* (translated by A. Freeman, 1878), pp. 268, 441.

† R. Fujisawa, *Journ. Coll. Sci. Imp. Univ. Japan*, 2 (1889), 1; W. B. Ford, *Studies in divergent series and summability* (1916), p. 142.

‡ This solves the problem discussed by J. L. Synge, *Proc. Lond. Math. Soc.* (2), 40 (1935), 23.

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us confine our attention to the even solutions of (7). Write equations (7) and (20) in the form

$$\ddot{\phi}-\alpha^2\phi+i\alpha R[2f+(1-y^2-C)\phi]=0, \tag{3·6}$$

$$\ddot{\phi}_n-\alpha^2\phi_n+\gamma_n\phi_n=0. \tag{3·7}$$

Multiply (3·6) by f_n and (3·7) by f, subtract and integrate from -1 to 1, obtaining

$$\int_{-1}^{1}\left\{f_n[2f+(1-y^2-C)\phi]+\frac{i\gamma_n}{\alpha R}\phi_n f\right\}dy=0. \tag{3·8}$$

With $f=\Sigma B_n f_n$ and $\phi=\Sigma B_n\phi_n$, equation (3·8) represents an infinite system of equations for the determination of the B_n. The vanishing of the determinant of the coefficients serves to determine C. On account of

$$\int_{-1}^{1}f_n\phi_m dy=0 \quad (n\neq m), \tag{3·9}$$

C occurs only in the diagonal elements which are of the form

$$\int_{-1}^{1}f_n\left[2f_n+\left(1-y^2-C+\frac{i\gamma_n}{\alpha R}\right)\phi_n\right]dy. \tag{3·10}$$

The determinant is, however, not symmetrical, since*

$$\int_{-1}^{1}y^2\phi_n f_m dy\neq\int_{-1}^{1}y^2\phi_m f_n dy.$$

As $R\to 0$, the non-diagonal terms, which are independent of R, become insignificant in comparison with the diagonal terms. The determinant then becomes equal to the product of the diagonal elements, which, upon vanishing, yield the first two terms in the expansion (18) for C.

Note added January 8, 1936.

It can be shown that the two-dimensional laminar flow is stable for values of R which are less than the smaller of the quantities 2α and α^3. Multiply (7) by $\bar{f}$, the complex conjugate of f, and integrate from -1 to 1, obtaining

$$\int_{-1}^{1}\{\ddot{f}\bar{f}+2\alpha^2\dot{f}\dot{\bar{f}}+\alpha^4 f\bar{f}+i\alpha R[2f\bar{f}+\alpha^2y^2f\bar{f}+y^2\dot{f}\bar{f}+2y\dot{f}\bar{f}+(C-1)(\dot{f}\dot{\bar{f}}+\alpha^2f\bar{f})]\}dy=0. \tag{3·11}$$

Clearly $I(C)$, the imaginary part of C, will be positive if

$$\int_{-1}^{1}[\ddot{f}\bar{f}+2\alpha^2\dot{f}\dot{\bar{f}}+\alpha^4 f\bar{f}-2\alpha RI(y\dot{f}\bar{f})]dy>0. \tag{3·12}$$

Now

$$\left|I\int_{-1}^{1}y\dot{f}\bar{f}\right|<\left|\int_{-1}^{1}y\dot{f}\bar{f}\right|<\int_{-1}^{1}\left|\dot{f}\bar{f}\right|\leqslant\frac{1}{2}\int_{-1}^{1}(\dot{f}\dot{\bar{f}}+f\bar{f}).$$

Hence (3·12) holds provided that R is less than the smaller of the quantities 2α and α^3. This result holds also for the linear flow†.

* The asymmetry in the determinant can be traced to the fact that equation (7) is not self-adjoint.

† The stability of the linear flow for disturbances of vanishing wavelength ($\alpha\to\infty$) was proved by Rayleigh, *Scientific Papers*, Vol. 6, p. 347.

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