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10.1017/mag.2019.24

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103.10 Around the nine-point circle

For a non-right scalene triangle ABC we will calculate relative angular position on its nine-point circle of centres of the Jerabek and Kiepert rectangular hyperbolae ([1, 2]). A sum of squares will show that this angular separation cannot exceed $\frac{1}{3}\pi$ radians.

The coordinates

With the area of the triangle of reference denoted by ΔABC, a point $P[x, y, z]$ has actual areal coordinates $x = \Delta PBC/\Delta ABC$, $y = \Delta PCA/\Delta ABC$, $z = \Delta PAB/\Delta ABC$ $(x + y + z = 1)$. We use the cotangent notation $\alpha = \cot A$, $\beta = \cot B$, $\gamma = \cot C$ $(\beta\gamma + \gamma\alpha + \alpha\beta = 1)$, $\Sigma = \alpha + \beta + \gamma$, $\Pi = \alpha\beta\gamma$ and employ $[\,]$ for actual and $(\,)$ for relative coordinates (e.g. vertex $A[1, 0, 0]$ and centroid $G(1, 1, 1)$).

Note also the formulae $\alpha(\beta + \gamma) = 1 - \beta\gamma$, $\alpha^2 + \beta^2 + \gamma^2 = \Sigma^2 - 2$, $\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2 = 1 - 2\Sigma\Pi$, $\alpha^3 + \beta^3 + \gamma^3 = \Sigma^3 - 3\Sigma + 3\Pi$ and $\beta^3\gamma^3 + \gamma^3\alpha^3 + \alpha^3\beta^3 = 1 - 3\Sigma\Pi + 3\Pi^2$.

We first develop a symmetrical function of the cotangents α, β, γ.

The function Φ

Now since

$$\Phi = \alpha^2(\beta - \gamma) + \beta^2(\gamma - \alpha) + \gamma^2(\alpha - \beta) \tag{1}$$

would appear to be intractable, we will express its square as a polynomial in Σ and Π. Then summing the squared terms we have

$$\begin{aligned}
&\alpha^4(\beta - \gamma)^2 + \beta^4(\gamma - \alpha)^2 + \gamma^4(\alpha - \beta)^2 \\
&= \alpha^4(\beta + \gamma)^2 + \beta^4(\gamma + \alpha)^2 + \gamma^4(\alpha + \beta)^2 - 4\Pi\left(\alpha^3 + \beta^3 + \gamma^3\right) \\
&= (\alpha(1 - \beta\gamma))^2 + (\beta(1 - \gamma\alpha))^2 + (\gamma(1 - \alpha\beta))^2 - 4\Pi\left(\Sigma^3 - 3\Sigma + 3\Pi\right) \\
&= \left(\Sigma^2 - 2\right) - 2\Sigma\Pi + 3\Pi^2 - 4\Pi\left(\Sigma^3 - 3\Sigma + 3\Pi\right) \\
&= -4\Sigma^3\Pi + 10\Sigma\Pi + \Sigma^2 - 9\Pi^2 - 2.
\end{aligned}$$

https://doi.org/10.1017/mag.2019.25 Published online by Cambridge University Press

For the cross terms we have

$$\begin{aligned}
&2\beta^2\gamma^2(\gamma-\alpha)(\alpha-\beta)+2\gamma^2\alpha^2(\alpha-\beta)(\beta-\gamma)+2\alpha^2\beta^2(\beta-\gamma)(\gamma-\alpha)\\
&=2\beta^2\gamma^2\left(-\alpha^2-2\beta\gamma+1\right)+2\gamma^2\alpha^2\left(-\beta^2-2\gamma\alpha+1\right)+2\alpha^2\beta^2\left(-\gamma^2-2\alpha\beta+1\right)\\
&=-6\alpha^2\beta^2\gamma^2-4\left(\beta^3\gamma^3+\gamma^3\alpha^3+\alpha^3\beta^3\right)+2\left(\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2\right)\\
&=-6\Pi^2-4\left(1-3\Sigma\Pi+3\Pi^2\right)+2\left(1-2\Sigma\Pi\right)\\
&=8\Sigma\Pi-18\Pi^2-2.
\end{aligned}$$

Hence

$$\begin{aligned}
\Phi^2 &= \left(-4\Sigma^3\Pi+10\Sigma\Pi+\Sigma^2-9\Pi^2-2\right)+\left(8\Sigma\Pi-18\Pi^2-2\right)\\
&= -4\Sigma^3\Pi+\Sigma^2+18\Sigma\Pi-27\Pi^2-4.
\end{aligned}$$

Now Φ^2 is positive semi-definite (vanishing for an isosceles triangle) but a sum of squares in the variables Σ and Π has not been found. In this context note the constraints $\Sigma > \sqrt{3}$ and $\Pi < 1/(3\sqrt{3})$ (the former arising from Weitzenbeck's inequality).

However, we can find a sum of squares to show that

$$4\Phi^2 < \left(\Sigma-\Pi\right)\left(\Sigma-9\Pi\right)\left(\Sigma^2-3\right)$$

where $\Sigma-\Pi>0$, $\Sigma-9\Pi>0$, $\Sigma^2-3>0$.

Thus

$$\begin{aligned}
&\left(\Sigma-\Pi\right)\left(\Sigma-9\Pi\right)\left(\Sigma^2-3\right)-4\Phi^2\\
&=\left(\Sigma^4-10\Sigma^3\Pi+9\Sigma^2\Pi^2-3\Sigma^2+30\Sigma\Pi-27\Pi^2\right)\\
&\qquad\qquad-\left(-16\Sigma^3\Pi+4\Sigma^2+72\Sigma\Pi-108\Pi^2-16\right)\\
&=\Sigma^4+6\Sigma^3\Pi+9\Sigma^2\Pi^2-7\Sigma^2-42\Sigma\Pi+81\Pi^2+16\\
&=\left(\Sigma^2+3\Sigma\Pi-4\right)^2+\left(\Sigma-9\Pi\right)^2>0, \qquad (2)
\end{aligned}$$

with equality in the equilateral limit.

The Jerabek-Kiepert angular separation

Now the distance between points $P[x, y, z]$ and $P'[x', y', z']$ is given by the cotangent metric

$$\frac{(PP')^2}{2\Delta} = \alpha(x-x')^2+\beta(y-y')^2+\gamma(z-z')^2$$

where

$$\Delta = \Delta ABC = 2R^2\sin A\sin B\sin C = \frac{2R^2}{\Sigma-\Pi}$$

(circumradius R) ([3]).

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After further algebra, we find the separation of the centres $J\big(\alpha(\beta-\gamma)^2, \beta(\gamma-\alpha)^2, \gamma(\alpha-\beta)^2\big)$ and $K\big((\beta-\gamma)^2, (\gamma-\alpha)^2, (\alpha-\beta)^2\big)$ to be given by

$$JK^2 = \frac{R^2\Phi^2}{(\Sigma - \Pi)(\Sigma - 9\Pi)(\Sigma^2 - 3)}.$$

Hence the angle ϕ subtended by the chord JK at the centre N of the nine-point circle (radius $\frac{1}{2}R$) is given by

$$\cos\phi = 1 - 2\sin^2\tfrac{1}{2}\phi = 1 - \frac{2JK^2}{R^2}$$

$$= 1 - \frac{2\Phi^2}{(\Sigma - \Pi)(\Sigma - 9\Pi)(\Sigma^2 - 3)},$$

leading to

$$\tfrac{1}{2} < \cos\phi < 1 \qquad \text{or} \qquad 0 < \phi < \tfrac{1}{3}\pi,$$

by the inequality (2).

Concluding remarks

Note that the Jerabek hyperbola

$$\alpha\big(\beta^2 - \gamma^2\big)yz + \beta\big(\gamma^2 - \alpha^2\big)zx + \gamma\big(\alpha^2 - \beta^2\big)xy = 0$$

is degenerate for a right triangle.

For the Archimedean triangle (angles $\frac{1}{7}\pi$, $\frac{2}{7}\pi$, $\frac{4}{7}\pi$) we have $\Sigma = \sqrt{7}$ and $\Pi = -1/\sqrt{7}$ leading to the mid-range value $\cos\phi = \frac{3}{4}$.

To ground this scenario, note finally that both centres J and K lie on the Kiepert line $\dfrac{x}{\beta-\gamma} + \dfrac{y}{\gamma-\alpha} + \dfrac{z}{\alpha-\beta} = 0$ by inspection.

The Kiepert line is normal to the Fermat line
$(\beta-\gamma)\big(3\alpha^2-1\big)x + (\gamma-\alpha)\big(3\beta^2-1\big)y + (\alpha-\beta)\big(3\gamma^2-1\big)z = 0$ ([4]).

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10.1017/mag.2019.25

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https://doi.org/10.1017/mag.2019.25 Published online by Cambridge University Press