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On subshift presentations

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Abstract. We consider partitioned graphs, by which we mean finite directed graphs with a partitioned edge set $\mathcal{E} = \mathcal{E}^- \cup \mathcal{E}^+$. Additionally given a relation $\mathcal{R}$ between the edges in $\mathcal{E}^-$ and the edges in $\mathcal{E}^+$, and under the appropriate assumptions on $\mathcal{E}^-$, $\mathcal{E}^+$ and $\mathcal{R}$, denoting the vertex set of the graph by $\mathfrak{P}$, we speak of an $\mathcal{R}$-graph $\mathcal{G}_\mathcal{R}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$. From $\mathcal{R}$-graphs $\mathcal{G}_\mathcal{R}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ we construct semigroups (with zero) $\mathcal{S}_\mathcal{R}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ that we call $\mathcal{R}$-graph semigroups. We write a list of conditions on a topologically transitive subshift with property (A) that together are sufficient for the subshift to have an $\mathcal{R}$-graph semigroup as its associated semigroup.

Generalizing previous constructions, we describe a method of presenting subshifts by means of suitably structured finite labeled directed graphs $(\mathcal{V}, \Sigma, \lambda)$ with vertex set $\mathcal{V}$, edge set Σ, and a label map that assigns to the edges in Σ labels in an $\mathcal{R}$-graph semigroup $\mathcal{S}_\mathcal{R}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^-)$. We denote the presented subshift by $X(\mathcal{V}, \Sigma, \lambda)$ and call $X(\mathcal{V}, \Sigma, \lambda)$ an $\mathcal{S}_\mathcal{R}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^-)$-presentation.

We introduce a property (B) of subshifts that describes a relationship between contexts of admissible words of a subshift, and we introduce a property (c) of subshifts that in addition describes a relationship between the past and future contexts and the context of admissible words of a subshift. Property (B) and the simultaneous occurrence of properties (B) and (c) are invariants of topological conjugacy.

We consider subshifts in which every admissible word has a future context that is compatible with its entire past context. Such subshifts we call right instantaneous. We introduce a property RI of subshifts, and we prove that this property is a necessary and sufficient condition for a subshift to have a right instantaneous presentation. We consider also subshifts in which every admissible word has a future context that is compatible with its entire past context, and also a past context that is compatible with its entire future context. Such subshifts we call bi-instantaneous. We introduce a property BI of subshifts, and we prove that this property is a necessary and sufficient condition for a subshift to have a bi-instantaneous presentation.

We define a subshift as strongly bi-instantaneous if it has for every sufficiently long admissible word a an admissible word c, that is contained in both the future context of a

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and the past context of a, and that is such that the word ca is a word in the future context of a that is compatible with the entire past context of a, and the word ac is a word in the past context of a, that is compatible with the entire future context of a. We show that a topologically transitive subshift with property (A), and associated semigroup a graph inverse semigroup $\mathcal{S}$, has an $\mathcal{S}$-presentation, if and only if it has properties (c) and BI, and a strongly bi-instantaneous presentation, if and only if it has properties (c) and BI, and all of its bi-instantaneous presentations are strongly bi-instantaneous.

We construct a class of subshifts with property (A), to which certain graph inverse semigroups $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ are associated, that do not have $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentations.

We associate to the labeled directed graphs $(\mathcal{V}, \Sigma, \lambda)$ topological Markov chains and Markov codes, and we derive an expression for the zeta function of $X(\mathcal{V}, \Sigma, \lambda)$ in terms of the zeta functions of the topological Markov shifts and the generating functions of the Markov codes.

1. *Introduction*

We study symbolic dynamical systems, which we construct by means of finite directed graphs. We will use the symbol s (t) to denote the source (target) vertex of an edge, or of a finite path, or of a right (left) infinite path in a directed graph.

Let there be given a finite directed graph with vertex set $\mathfrak{P}$ and edge set $\mathcal{E}$, and also a partition

$$\mathcal{E} = \mathcal{E}^- \cup \mathcal{E}^+.$$

We set

$$\mathcal{E}^-(\mathfrak{q}, \mathfrak{r}) = \{e^- \in \mathcal{E}^- : s(e^-) = \mathfrak{q},\ t(e^-) = \mathfrak{r}\},$$
$$\mathcal{E}^+(\mathfrak{q}, \mathfrak{r}) = \{e^+ \in \mathcal{E}^+ : s(e^+) = \mathfrak{r},\ t(e^+) = \mathfrak{q}\}, \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P}.$$

We assume that $\mathcal{E}^-(\mathfrak{q}, \mathfrak{r}) \neq \emptyset$ if and only if $\mathcal{E}^+(\mathfrak{q}, \mathfrak{r}) \neq \emptyset$, $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$. We also assume that the directed graph $(\mathfrak{P}, \mathcal{E}^-)$ is strongly connected, or, equivalently, that the directed graph $(\mathfrak{P}, \mathcal{E}^+)$ is strongly connected. We call the structure $(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ a partitioned graph. Let there also be given relations

$$\mathcal{R}(\mathfrak{q}, \mathfrak{r}) \subset \mathcal{E}^-(\mathfrak{q}, \mathfrak{r}) \times \mathcal{E}^+(\mathfrak{q}, \mathfrak{r}), \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P},$$

and set

$$\mathcal{R} = \bigcup_{\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}} \mathcal{R}(\mathfrak{q}, \mathfrak{r}).$$

The structure that is given in this way, for which we use the notation $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, we call an $\mathcal{R}$-graph. From an $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ we construct a semigroup (with zero) $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ that contains idempotents $\mathbf{1}_{\mathfrak{p}}$, $\mathfrak{p} \in \mathfrak{P}$, and that has $\mathcal{E}$ as a generating set. Besides $\mathbf{1}_{\mathfrak{p}}^2 = \mathbf{1}_{\mathfrak{p}}$, $\mathfrak{p} \in \mathfrak{P}$, the defining relations of $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ are:

$$\mathbf{1}_{\mathfrak{q}} e^- = e^- \mathbf{1}_{\mathfrak{r}} = e^-, \quad e^- \in \mathcal{E}^-(\mathfrak{q}, \mathfrak{r}),$$
$$\mathbf{1}_{\mathfrak{r}} e^+ = e^+ \mathbf{1}_{\mathfrak{q}} = e^+, \quad e^+ \in \mathcal{E}^+(\mathfrak{q}, \mathfrak{r}),\ \mathfrak{q}, \mathfrak{r} \in \mathfrak{P},$$
$$f^- g^+ = \begin{cases} \mathbf{1}_{\mathfrak{q}} & \text{if } (f^-, g^+) \in \mathcal{R}(\mathfrak{q}, \mathfrak{r}), \\ 0 & \text{if } (f^-, g^+) \notin \mathcal{R}(\mathfrak{q}, \mathfrak{r}), \end{cases} \quad f^- \in \mathcal{E}^-(\mathfrak{q}, \mathfrak{r}),\ g^+ \in \mathcal{E}^+(\mathfrak{q}, \mathfrak{r}),\ \mathfrak{q}, \mathfrak{r} \in \mathfrak{P},$$

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and

$$\mathbf{1}_{\mathfrak{q}}\mathbf{1}_{\mathfrak{r}} = 0, \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P}, \mathfrak{q} \neq \mathfrak{r}.$$

We call $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ an $\mathcal{R}$-graph semigroup. The graph inverse semigroups (the generalized polycyclic semigroups; see [**AH**] and [**L**, §10.7] and compare [**CK**]) are a special case of $\mathcal{R}$-graph semigroups. The graph inverse semigroup of a finite directed graph with vertex set $\mathfrak{P}$ and edge set $\mathcal{E}$ is obtained by taking a copy of the graph $(\mathfrak{P}, \mathcal{E})$ with vertex set $\mathfrak{P}$ and edge set $\mathcal{E}^- = \{e^- : e \in \mathcal{E}^\circ\}$ and a copy $(\mathfrak{P}, \mathcal{E}^+)$ of the reversed graph of $(\mathfrak{P}, \mathcal{E})$ and by constructing the $\mathcal{R}$-graph semigroup of the partitioned graph $(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ with the relations

$$\mathcal{R}(\mathfrak{q}, \mathfrak{r}) = \{(e^-, e^+) : e \in \mathcal{E}, s(e) = \mathfrak{q}, t(e) = \mathfrak{r}\}, \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P}.$$

The relation being understood, we suppress the $\mathcal{R}$ and denote the graph inverse semigroup of the finite directed graph by $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$. The Dyck inverse monoids (the polycyclic monoids) $\mathcal{D}_N$, $N > 1$, [**NP**] are obtained in this way from the one-vertex graph with N loops. For the $\mathcal{R}$-graph semigroups that are obtained from a one-vertex graph see also [**HK1**, §4].

In §2 we show that the isomorphism of $\mathcal{R}$-graph semigroups $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ implies the isomorphism of the $\mathcal{R}$-graphs $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$.

For a semigroup (with zero) $\mathcal{S}$, and for $F \in \mathcal{S}$ set

$$\Gamma(F) = \{(G^-, G^+) \in \mathcal{S} \times \mathcal{S} : G^- F G^+ \neq 0\}$$

and

$$[F] = \{F' \in \mathcal{S} : \Gamma(F') = \Gamma(F)\}.$$

The set $[\mathcal{S}] = \{[F] : F \in \mathcal{S}\}$ with the product given by

$$[G][H] = [GH], \quad G, H \in \mathcal{S},$$

is a semigroup (e.g. see [**P**, §2.2]). In §2 we also write a list of conditions on a semigroup (with zero) $\mathcal{S}$ that together are necessary and sufficient for the semigroup to be an $\mathcal{R}$-graph semigroup, such that the projection of $\mathcal{S}$ onto $[\mathcal{S}]$ is an isomorphism.

In symbolic dynamics one studies subshifts (X, S_X), where X is a shift invariant closed subset of the shift space $\Sigma^{\mathbb{Z}}$ and S_X is the restriction of the shift on $\Sigma^{\mathbb{Z}}$ to X. For an introduction to the theory of subshifts see [**Ki**] and [**LM**]. The study of topological conjugacy and its invariants occupies a central place in symbolic dynamics. Any subshift $\widetilde{X}$ that is topologically conjugate to a given subshift X is called a presentation of X. A property (A) of subshifts, an invariant of topological conjugacy, was introduced in [**Kr2**], and for a subshift X with property (A) a semigroup (with zero) $\mathcal{S}(X)$ was constructed that is also an invariant of topological conjugacy. Next to topological conjugacy, flow equivalence and its invariants are studied in symbolic dynamics. It was shown by Costa and Steinberg [**CS**] that property (A) and the associated semigroup are also invariants of flow equivalence.

In §3 we translate all but the last of the conditions on the list of §2 into conditions on a subshift that are invariant under topological conjugacy. For the last condition we obtain a possibly stronger version for subshifts that is also invariant under topological conjugacy.

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For a subshift X with property (A) these conditions together imply that $\mathcal{S}(X)$ is an $\mathcal{R}$-graph semigroup.

We now describe a way to present subshifts by means of $\mathcal{R}$-graph semigroups. Here we closely follow [**HIK**, §3] where this method of presenting subshifts was introduced for the case of graph inverse semigroups of directed graphs in which every vertex has at least two incoming edges. These presentations were introduced for the purpose of extending the criterion for the existence of an embedding of an irreducible subshift of finite type into a Dyck shift [**HI**] to a wider class of target shifts. Given an $\mathcal{R}$-graph semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, denote by $\mathcal{S}^-_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-)$ $(\mathcal{S}^+_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^+))$ the subset of $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ that contains the non-zero elements of the subsemigroup of $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ that is generated by $\mathcal{E}^-(\mathcal{E}^+)$, and consider a finite strongly connected labeled directed graph with vertex set $\mathcal{V}$ and edge set Σ, and a labeling map λ that assign to every edge $\sigma \in \Sigma$ a label

$$\text{(G1)} \qquad \lambda(\sigma) \in \mathcal{S}^-_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-) \cup \{\mathbf{1}_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P}\} \cup \mathcal{S}^+_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^+).$$

The label map λ extends to finite paths $(\sigma_i)_{1\le i\le I}$, $I \in \mathbb{N}$, in the graph $(\mathcal{V}, \Sigma)$ by

$$\lambda((\sigma_i)_{1\le i\le I}) = \prod_{1\le i\le I} \lambda(\sigma_i).$$

We denote, for $\mathfrak{p} \in \mathfrak{P}$, by $\mathcal{V}(\mathfrak{p})$ the set of $V \in \mathcal{V}$ such that there is a cycle $(\sigma_i)_{1\le i\le I}$, $I \in \mathbb{N}$, in the graph $(\mathcal{V}, \Sigma)$ from V to V such that

$$\lambda((\sigma_i)_{1\le i\le I}) = \mathbf{1}_{\mathfrak{p}}.$$

We impose the following four conditions:

(G2) $\mathcal{V}(\mathfrak{p}) \neq \emptyset$, $\mathfrak{p} \in \mathfrak{P}$.

(G3) $\{\mathcal{V}(\mathfrak{p}) : \mathfrak{p} \in \mathfrak{P}\}$ is a partition of $\mathcal{V}$.

(G4) For $V \in \mathcal{V}(\mathfrak{p})$, $\mathfrak{p} \in \mathfrak{P}$, and for all edges e that leave V, $\mathbf{1}_{\mathfrak{p}}\lambda(e) \neq 0$, and for all edges e that enter V, $\lambda(e)\mathbf{1}_{\mathfrak{p}} \neq 0$.

(G5) For $f \in \mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$, such that $\mathbf{1}_{\mathfrak{q}} f \mathbf{1}_{\mathfrak{r}} \neq 0$, and for $U \in \mathcal{V}(\mathfrak{q})$, $W \in \mathcal{V}(\mathfrak{r})$, there exists a path b in the labeled directed graph $(\mathcal{V}, \Sigma, \lambda)$ from U to W such that $\lambda(b) = f$.

A finite labeled directed graph $(\mathcal{V}, \Sigma, \lambda)$, that satisfies conditions (G1)–(G5), gives rise to a subshift $X(\mathcal{V}, \Sigma, \lambda)$ that has as its language of admissible words the set of finite non-empty paths b in the graph $(\mathcal{V}, \lambda)$ such that $\lambda(b) \neq 0$. We call the subshift $X(\mathcal{V}, \Sigma, \lambda)$ an $S_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation, and we say that a subshift X has an $S_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation if it is topologically conjugate to an $S_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation. The $S_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentations with a labeling map λ such that

$$\lambda(\sigma) \in \mathcal{E}^- \cup \{\mathbf{1}_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P}\} \cup \mathcal{E}^+, \quad \sigma \in \Sigma,$$

belong to the class of sofic-Dyck shifts as described in [**BBD3**, **BBD2**]. In [**HIK**, §6] and [**IK2**, §4] $\mathcal{D}_2$-presentations are described that arise as intersections of $\mathcal{D}_2$ with subshifts of finite type.

The special cases of $S_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentations that are obtained for $\mathcal{V} = \mathfrak{P}$, $\Sigma = \mathcal{E}$, and with the identity map as the labeling map, are the $\mathcal{R}$-graph shifts (see [**HK1**, §4] and [**HK2**]). In the case of directed graphs the $\mathcal{R}$-graph shifts are the Markov–Dyck shifts

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[**M3**], and in the case of one-vertex directed graphs the $\mathcal{R}$-graph shifts are the Dyck shifts [**Kr1**]. The $\mathcal{R}$-graph shifts belong to the class of FT-Dyck shifts as described in [**BBD1**]. For $\mathcal{R}$-graph shifts see also [**Kr6**]. Adding a loop at each vertex $\mathfrak{p} \in \mathfrak{P}$ with label $\mathbf{1}_{\mathfrak{p}}$, one obtains what can be called the $\mathcal{R}$-graph Motzkin shifts. In the case of directed graphs the $\mathcal{R}$-graph Motzkin shifts are the Markov–Motzkin shifts [**KM3**, §4.1], and in the case of one-vertex directed graphs the $\mathcal{R}$-graph Motzkin shifts are the Motzkin shifts [**M2**, **I**].

In §4 we introduce a property (B) of subshifts that describes a relationship between contexts of admissible words of a subshift. In §5 we introduce a property (c) of subshifts that describes a relationship between the past and future contexts and the context of an admissible word of a subshift. One is led to the formulation of properties (B) and (c) by observing the behavior of the Dyck shifts and by abstracting essential dynamical properties. We prove that property (B) is an invariant of topological conjugacy, and that the simultaneous occurrence of properties (B) and (c) is also an invariant of topological conjugacy. An $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation that has property (A), has property (B) precisely if $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ is a graph inverse semigroup.

In §6 we consider subshifts in which every admissible word has a future context that is compatible with its entire past context. Such subshifts we call right instantaneous. We introduce a property RI of subshifts, and we prove that this property is a necessary and sufficient condition for the subshift to have a right instantaneous presentation. We consider also subshifts in which every admissible word has a future context that is compatible with its entire past context, and also a past context that is compatible with its entire future context. Such subshifts we call bi-instantaneous. We introduce a property BI of subshifts, and we prove that this property is a necessary and sufficient condition for the subshift to have a bi-instantaneous presentation. Sofic systems have bi-instantaneous presentations.

In §7 we introduce a notion of strong bi-instantaneity. We show that a topologically transitive subshift with property (A), and associated semigroup a graph inverse semigroup $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, has an $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation, if and only if it has properties (c) and BI and a strongly bi-instantaneous presentation, if and only if it has properties (c) and BI, and a strongly bi-instantaneous presentation, and all of its bi-instantaneous presentations are strongly bi-instantaneous.

In §8 we construct for a graph $\mathcal{G}(\mathfrak{P}, \mathcal{E})$, such that for $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$, $\mathrm{card}(\mathcal{E}(\mathfrak{q}, \mathfrak{r})) \neq 1$, a subshift with property (A) and with the graph inverse semigroup of $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ as associated semigroup, that does not have an $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation.

In §9, applying methods of Keller [**Ke**], we associate to the labeled directed graphs $(\mathcal{V}, \Sigma, \lambda)$ topological Markov chains and circular Markov codes, and we derive an expression for the zeta function of $X(\mathcal{V}, \Sigma, \lambda)$ in terms of the zeta functions of the topological Markov shifts and the generating functions of the Markov codes.

2. *$\mathcal{R}$-graph semigroups*

Let $\mathcal{S}$ be a semigroup (with zero). We set

$$\mathcal{C}_-(F) = \{G \in \mathcal{S} : GF \neq 0\},$$
$$\mathcal{C}_+(F) = \{G \in \mathcal{S} : FG \neq 0\}, \quad F \in \mathcal{S}.$$

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We denote by $\mathcal{U}_{\mathcal{S}}$ the set of idempotents U of $\mathcal{S}$, such that one has for $F_- \in \mathcal{C}_-(U)$ that $F_- U = F_-$ and for $F_+ \in \mathcal{C}_+(U)$ that $U F_+ = F_+$. We set

$$\mathcal{S}^-(U) = \bigcap_{F_- \in \mathcal{C}_-(U)} \mathcal{C}_+(F_- U), \quad U \in \mathcal{U}_{\mathcal{S}},$$
$$\mathcal{S}^-(V, W) = \mathcal{S}^-(V)W \setminus \{0\}, \quad V, W \in \mathcal{U}_{\mathcal{S}},$$

and

$$\mathcal{S}^+(U) = \bigcap_{F_+ \in \mathcal{C}_+(U)} \mathcal{C}_-(U F_+), \quad U \in \mathcal{U}_{\mathcal{S}},$$
$$\mathcal{S}^+(V, W) = W\mathcal{S}^+(V) \setminus \{0\}, \quad W, V \in \mathcal{U}_{\mathcal{S}}.$$

We define subsemigroups (with zero) $\mathcal{S}^-$ and $\mathcal{S}^+$ of $\mathcal{S}$ by setting

$$\mathcal{S}^- = \bigcup_{U \in \mathcal{U}_{\mathcal{S}}} \mathcal{S}^-(U), \quad \mathcal{S}^+ = \bigcup_{U \in \mathcal{U}_{\mathcal{S}}} \mathcal{S}^+(U).$$

We say that an element $F \in \mathcal{S}^- \setminus \mathcal{U}_{\mathcal{S}}$ is indecomposable in $\mathcal{S}^-$ if $F = GH$, $G, H \in \mathcal{S}^-$, implies that G or H is in $\mathcal{U}_{\mathcal{S}}$. The indecomposable elements in $\mathcal{S}^+$ are defined symmetrically.

THEOREM 2.1. *The isomorphism of the $\mathcal{R}$-graph semigroups $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ implies the isomorphism of the $\mathcal{R}$-graphs $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$.*

Proof. Let

$$\mathcal{S} = \mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+).$$

With vertices

$$\mathfrak{p}_i \in \mathfrak{P}, \quad 0 \le i \le I, \ I \in \mathbb{Z}_+,$$

and edges

$$e_i^+ \in \mathcal{E}^+(\mathfrak{p}_{i-1}, \mathfrak{p}_i), \quad I \ge i \ge 1,$$
$$e_i^- \in \mathcal{E}^-(\mathfrak{p}_{i-1}, \mathfrak{p}_i), \quad 1 \le i \le I,$$

such that

$$(e_i^-, e_i^+) \in \mathcal{R}(\mathfrak{p}_{i-1}, \mathfrak{p}_i), \quad 1 \le i \le I,$$

the normal forms of the idempotents of $\mathcal{S}$ are given by

$$\Big(\prod_{I \ge i \ge 1} e_i^+\Big)\mathbf{1}_{\mathfrak{p}_0}\Big(\prod_{1 \le i \le I} e_i^-\Big).$$

From this expression it can be seen that

$$\mathcal{U}_{\mathcal{S}} = \{\mathbf{1}_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P}\},$$

which then means that one can identify $\mathfrak{P}$ with $\mathcal{U}_{\mathcal{S}}$, and then also $\mathcal{E}^-(\mathfrak{q}, \mathfrak{r})$ with the set of indecomposable elements f of $\mathcal{S}^-$, such that $\mathbf{1}_{\mathfrak{q}} f \ne 0$, $f\mathbf{1}_{\mathfrak{r}} \ne 0$, and $\mathcal{E}^+(\mathfrak{q}, \mathfrak{r})$ with the set of indecomposable elements of $\mathcal{S}^+$, such that $\mathbf{1}_{\mathfrak{r}} f \ne 0$, $f\mathbf{1}_{\mathfrak{q}} \ne 0$. Also, for $e^- \in \mathcal{E}^-(\mathfrak{q}, \mathfrak{r})$, $e^+ \in \mathcal{E}^+(\mathfrak{q}, \mathfrak{r})$, $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$, one has $(e^-, e^+) \in \mathcal{R}(\mathfrak{q}, \mathfrak{r})$ if and only if $e^- e^+ \ne 0$. □

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We have a condition (AP1) on a semigroup $\mathcal{S}$:

(AP1) $\mathcal{U}_{\mathcal{S}}$ is a finite set.

We also have a condition (AP2) on a semigroup $\mathcal{S}$, that comes in two parts that are symmetric to one another, of which we write only one part:

(AP2−) $\mathcal{S}^-(V, W) \neq \emptyset$, $V, W \in \mathcal{U}_{\mathcal{S}}$.

We also have conditions (AQ1)–(AQ2) on a semigroup $\mathcal{S}$:

(AQ1) $\mathcal{S}^-(V, U)\mathcal{S}^+(W, U) \subset \mathcal{S}^-(V) \cup \mathcal{S}^+(W)$, $U, V, W \in \mathcal{U}_{\mathcal{S}}$.

(AQ2) $\mathcal{S} = \bigcup_{U \in \mathcal{U}_{\mathcal{S}}} \mathcal{S}^+(U)\mathcal{S}^-(U)$.

We also have conditions (AQ3)–(AQ6) on a semigroup $\mathcal{S}$, that similarly come in two parts that are symmetric to one another. We only write one part of these conditions:

(AQ3−) For $U, V \in \mathcal{U}_{\mathcal{S}}$ one has that for $F^- \in \mathcal{S}^-(U, V)$ there exists an $F^+ \in \mathcal{S}^+(U, V)$ such that

$$F^-F^+ = U.$$

(AQ4−) For $U, V, W \in \mathcal{U}_{\mathcal{S}}$ one has that for

$$F^- \in \mathcal{S}^-(U, V) \setminus \{U\}, \quad G^+ \in \mathcal{S}^+(W, V) \setminus \{V\},$$

such that

$$F^-G^+ \in \mathcal{S}^-(U, W),$$

there exists an $H^- \in \mathcal{S}^-(W, V) \setminus \{W\}$ such that

$$F^-G^+H^- = F^-.$$

(AQ5−) $\mathcal{S}^-$ has finitely many indecomposable elements.

(AQ6−) $\mathcal{S}^-$ is generated by its indecomposable elements.

Note that conditions (AP2−) and (AQ3−) together imply condition (AP2+), and, symmetrically, conditions (AP2+) and (AQ3+) together imply condition (AP2−).

For an $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^-)$ we set

$$\Omega^+(e^-) = \{e^+ \in \mathcal{E}^+(\mathfrak{q}, \mathfrak{r}) : (e^-, e^+) \in \mathcal{R}(\mathfrak{q}, \mathfrak{r})\}, \quad e^- \in \mathcal{E}^-(\mathfrak{q}, \mathfrak{r}),$$
$$\Omega^-(e^+) = \{e^- \in \mathcal{E}^-(\mathfrak{q}, \mathfrak{r}) : (e^-, e^+) \in \mathcal{R}(\mathfrak{q}, \mathfrak{r})\}, \quad e^+ \in \mathcal{E}^+(\mathfrak{q}, \mathfrak{r}), \ \mathfrak{q}, \mathfrak{r} \in \mathfrak{P}.$$

We introduce condition (a) on an $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^-)$, which consists of two parts that are symmetric to one another:

$$(a-) \quad \Omega^+(e^-) \neq \Omega^+(\widetilde{e}^-), \quad e^-, \widetilde{e}^- \in \mathcal{E}^-(\mathfrak{q}, \mathfrak{r}), e^- \neq \widetilde{e}^-, \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P},$$
$$(a+) \quad \Omega^-(e^+) \neq \Omega^-(\widetilde{e}^+), \quad e^+, \widetilde{e}^+ \in \mathcal{E}^+(\mathfrak{q}, \mathfrak{r}), e^+ \neq \widetilde{e}^+, \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P}.$$

The following theorem characterizes the $\mathcal{R}$-graph semigroups among the semigroups $\mathcal{S}$ such that the projection of $\mathcal{S}$ onto $[\mathcal{S}]$ is an isomorphism. For the proof of the theorem compare [**Kr4**, §3].

THEOREM 2.2. *Let the semigroup $\mathcal{S}$ be such that the projection of $\mathcal{S}$ onto $[\mathcal{S}]$ is an isomorphism. Then $\mathcal{S}$ is an $\mathcal{R}$-graph semigroup if and only if $\mathcal{S}$ satisfies conditions (AP1)–(AP2) and conditions (AQ1)–(AQ6).*

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Proof. We prove sufficiency. By (AP1), (AQ1), (AQ5), and (AQ6) we can use a partitioned graph with vertex set $\mathcal{U}_{\mathcal{S}}$ and edge sets

$$\mathcal{E}^- = \bigcup_{V,W\in\mathcal{U}_{\mathcal{S}}} \mathcal{E}^-(V,W), \quad \mathcal{E}^+ = \bigcup_{V,W\in\mathcal{U}_{\mathcal{S}}} \mathcal{E}^+(V,W),$$

with the set of indecomposable elements in $\mathcal{S}^-(V,W)$ as the set $\mathcal{E}^-(V,W)$, and the set of indecomposable elements in $\mathcal{S}^+(V,W)$ as the set $\mathcal{E}^+(V,W)$, $V, W \in \mathcal{U}_{\mathcal{S}}$.

Let $V, W \in \mathcal{U}_{\mathcal{S}}$, and let $E^- \in \mathcal{S}^-(V,W)$ be indecomposable in $\mathcal{S}^-$. By (AQ3−) there exists an $E^+ \in \mathcal{S}^+(V,W)$ such that

$$E^-E^+ = V. \tag{2.1}$$

We prove that E^+ is indecomposable. Assume the contrary, and let $U \in \mathcal{U}_{\mathcal{S}}$ and

$$F^+ \in \mathcal{S}^+(U,W), \quad G^+ \in \mathcal{S}^+(V,U)$$

be such that

$$E^+ = F^+G^+, \quad G^+ \neq V. \tag{2.2}$$

By (AQ1) then either

$$E^-F^+ \in \mathcal{S}^-(V) \setminus \{V\},$$

in which case there would exist by (AQ4−) an $F^- \in \mathcal{S}^-(U,W) \setminus \{U\}$ such that

$$E^- = E^-F^+F^-,$$

contradicting the indecomposability of E^-, or

$$E^-F^+ \in \mathcal{S}^+(W),$$

in which case

$$E^-F^+G^+ \in \mathcal{S}^+(V) \setminus \{V\},$$

contradicting (2.1) and (2.2). The symmetric argument shows also for $U, W \in \mathcal{U}_{\mathcal{S}}$, and for an $E^+ \in \mathcal{S}^+(U,W)$ that is indecomposable in $\mathcal{S}^+$, that there exists an $E^- \in \mathcal{S}^+(U,W)$ that is indecomposable in $\mathcal{S}^-$, such that

$$E^-E^+ = U.$$

It follows from (AP2) that the directed graph $(\mathcal{U}_{\mathcal{S}}, \mathcal{E}^-)$ is strongly connected. We define relations $\mathcal{R}(V,W) \subset \mathcal{E}^-(V,W) \times \mathcal{E}^+(V,W)$ by

$$\mathcal{R}(V,W) = \{(E^-,E^+) \in \mathcal{E}^-(V,W) \times \mathcal{E}^+(V,W) : E^-E^+ \neq 0\}, \quad V, W \in \mathcal{U}_{\mathcal{S}},$$

and set

$$\mathcal{R} = \bigcup_{V,W\in\mathcal{U}_{\mathcal{S}}} \mathcal{R}(V,W).$$

By (AQ4) one has

$$E^-E^+ = \begin{cases} U & \text{if } U = W,\ (E^-,E^+) \in \mathcal{R}(U,V), \\ 0 & \text{if } U = W,\ (E^-,E^+) \notin \mathcal{R}(U,V), \\ 0 & \text{if } U \neq W, \end{cases}$$

$$E^- \in \mathcal{E}^-(U,V), \quad E^+ \in \mathcal{E}^+(W,V), \quad U, V, W \in \mathcal{U}_{\mathcal{S}}.$$

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It follows from (AQ2), (AQ5), and (AQ6) that for $F \in \mathcal{S}$ there exist $I(-), I(+) \in \mathbb{Z}_+$ and

$$U_{i_+}(+) \in \mathcal{U}_{\mathcal{S}}, \quad I_+ \geq i_+ \geq 1, \ U \in \mathcal{U}, \quad U_{i_-}(-) \in \mathcal{U}_{\mathcal{S}}, \quad 1 \leq i_- \leq I_-,$$

and, setting

$$U_0(+) = U_0(-) = U,$$

also

$$E^+_{i_+} \in \mathcal{E}^+(U_{i_+}, U_{i_+-1}), \quad I_+ \geq i_+ \geq 1, \quad E^-_{i_-} \in \mathcal{E}^+(U_{i_--1}, U_{i_-}), \quad 1 \leq i_- \leq I_-,$$

such that

$$F = \Big(\prod_{I_+ \geq i_+ \geq 1} E^+_{i_+} \Big) U \Big(\prod_{1 \leq i_- \leq I_-} E^-_{i_-} \Big). \tag{2.3}$$

It follows from the assumption that the projection of $\mathcal{S}$ onto $[\mathcal{S}]$ is an isomorphism, that the $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathcal{U}_{\mathcal{S}}, \mathcal{E}^-, \mathcal{E}^+)$ satisfies (a), since for $E, \widetilde{E} \in \mathcal{E}^-(U, V)$ one has that $\Omega^+(E) = \Omega^+(\widetilde{E})$ would imply that $[E] = [\widetilde{E}]$, $U, V \in \mathcal{U}_{\mathcal{S}}$. Applying $(a-)$ and $(a+)$ repeatedly, and keeping in mind that one has for $H \in \mathcal{S}$, $U \in \mathcal{U}_{\mathcal{S}}$ that $[H] = [U]$ implies that $H = U$, one finds that the presentation (2.3) of F is in fact unique. □

3. *Subshifts with property (A) to which $\mathcal{R}$-graph semigroups are associated*

We introduce terminology and notation for subshifts. Given a subshift $X \subset \Sigma^{\mathbb{Z}}$, we set

$$x_{[i,k]} = (x_j)_{i \leq j \leq k}, \quad x \in X, \ i, k \in \mathbb{Z}, \ i \leq k,$$

and

$$X_{[i,k]} = \{x_{[i,k]} : x \in X\}, \quad i, k \in \mathbb{Z}, i \leq k.$$

We use similar notation also for blocks,

$$b_{[i',k']} = (b_j)_{i' \leq j \leq k'}, \quad b \in X_{[i,k]}, \ i \leq i' \leq k' \leq k,$$

and also if indices range in semi-infinite intervals. The symbol that denotes a block is also used to denote the word that is carried by the block. We identify the elements of $X_{(-\infty,0)}$ with the left infinite words that they carry, and we identify the elements of $X_{(0,\infty)}$ with the right infinite words that they carry. The set of admissible words of X is denoted by $\mathcal{L}(X)$. For the N-block system of a subshift $X \subset \Sigma^{\mathbb{Z}}$ we use the notation

$$x^{\langle (m,m+N] \rangle} = (x_{(i+m,i+m+N]})_{i \in \mathbb{Z}}, \quad x \in X,$$
$$X^{\langle (m,m+N] \rangle} = \{x^{\langle (m,m+N] \rangle} : x \in X\}, \quad m \in \mathbb{Z}, \ N \in \mathbb{N}.$$

We set

$$\Gamma(a) = \{(x^-, x^+) \in X_{(-\infty,0)} \times X_{(0,\infty)} : (x^-, a, x^+) \in X\},$$
$$\Gamma^+_n(a) = \{b \in X_{(k,k+n]} : (a, b) \in X_{[i,k+n]}\}, \quad n \in \mathbb{N},$$
$$\Gamma^+_\infty(a) = \{y^+ \in X_{(k,\infty)} : (a, y^+) \in X_{[i,\infty)}\},$$
$$\Gamma^+(a) = \bigcup_{n \in \mathbb{N}} \Gamma^+_n(a), \quad a \in X_{[i,k]}, \ i, k \in \mathbb{Z}, \ i \leq k.$$

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Γ^- has the time symmetric meaning. We set

$$\omega_n^+(a) = \bigcap_{x^- \in \Gamma_\infty^-(a)} \{b \in X_{(k,k+n]} : (x^-, a, b) \in X_{(-\infty,k+n]}\},$$
$$\omega_\infty^+(a) = \bigcap_{x^- \in \Gamma_\infty^-(a)} \{y^+ \in X_{(k,\infty)} : (x^-, a, y^+) \in X\},$$
$$\omega^+(a) = \bigcup_{n\in\mathbb{N}} \omega_n^+(a), \quad a \in X_{[i,k]},\ i, k \in \mathbb{Z},\ i \le k.$$

ω^- has the time symmetric meaning.

We recall that, given subshifts $X \subset \Sigma^{\mathbb{Z}}$, $\widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$, and a topological conjugacy $\varphi : X \to \widetilde{X}$, there is for some $L \in \mathbb{Z}_+$ a block mapping

$$\Phi : X_{[-L,L]} \to \widetilde{\Sigma}$$

such that

$$\varphi(x) = (\Phi(x_{[i-L,i+L]}))_{i\in\mathbb{Z}}.$$

We say then that φ is given by Φ, and we write

$$\Phi(a) = (\Phi(a_{[j-L,j+L]}))_{i+L\le j\le k-L}, \quad a \in X_{[i,k]},\ i, k \in \mathbb{Z},\ k - i \ge 2L,$$

and use similar notation if indices range in semi-infinite intervals.

For a subshift $X \subset \Sigma^{\mathbb{Z}}$, set

$$A_n^-(X) = \bigcap_{i\in\mathbb{Z}} \{x \in X : x_i \in \omega^+(x_{[i-n,i)})\}, \quad n \in \mathbb{N},$$

and

$$A^-(X) = \bigcup_{n\in\mathbb{N}} A_n^-(X).$$

Define $A_n^+(X)$, $n \in \mathbb{N}$, and $A^+(X)$ symmetrically and set

$$A_n(X) = A_n^-(X) \cap A_n^+(X), \quad n \in \mathbb{N},$$

and

$$A(X) = \bigcup_{n\in\mathbb{N}} A_n(X).$$

Denote the set of periodic points in $A(X)$ by $P(A(X))$. We write $q \succeq r$, for $q, r \in P(A(X))$, if there exists a point in $A(X)$ that is left asymptotic to q and right asymptotic to a point in the orbit of r. The equivalence relation that results from the preorder relation $\succeq$ we write as $\approx$ and we denote the order relation that results from $\succeq$ also by $\succeq$. We denote the set of $\approx$-equivalence classes by $\mathfrak{P}(X)$. The order structure $(\mathfrak{P}(X), \succeq)$ is invariantly associated to X [**Kr2**]. We denote by $Y(X)$ the set of points $x \in X$ such that there are $q, r \in P(A(X))$ such that x is left asymptotic to q and right asymptotic to r.

We write a condition (DP0) on a subshift $X \subset \Sigma^{\mathbb{Z}}$ that is invariant under topological conjugacy [**Kr2**]:

(DP0) $P(A(X))$ is dense in X.

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The following conditions (DP1) and (DP2) on a subshift $X \subset \Sigma^{\mathbb{Z}}$ are invariant under topological conjugacy. Condition (DP1) is the translation of condition (AP1) and Condition (DP2−) is the translation of condition (AP2−):
(DP1) $\mathfrak{P}(X)$ is a finite set.
(DP2−) For $q, r \in P(A(X))$ there exists an $x \in A^-(X)$ that is left asymptotic to q and right asymptotic to r.

We introduce a condition (DQ1) and a condition (DQ2) on a subshift $X \subset \Sigma^{\mathbb{Z}}$. Condition (DQ1) is the translation of condition (AQ1) and condition (DQ2) is the translation of condition (AQ2):
(DQ1) For $p \in P(A(X))$, there exists an $H \in \mathbb{N}$ such that if

$$y \in A^-(X) \cap Y(X), \quad y_{(-H,\infty)} = p_{(-H,\infty)},$$
$$z \in A^+(X) \cap Y(X), \quad z_{(-\infty,H]} = p_{(-\infty,H]},$$

and

$$(y_{(-\infty,0]}, z_{(0,\infty)}) \in X,$$

then

$$(y_{(-\infty,0]}, z_{(0,\infty)}) \in A^-(X) \cup A^+(X).$$

(DQ2) Given $q, r \in P(A(X))$, there exists an $H \in \mathbb{N}$ such that the following holds: for $x \in X$ and $K > H$ such that

$$x_{(-\infty,-K]} = q_{(-\infty,0]}, \quad x_{(K,\infty)} = r_{(0,\infty)},$$

and for $M \in \mathbb{N}$, there exist $p \in P(A(X))$, $I, J > M$, and

$$y \in A^+(X) \cap Y(X), \quad z \in A^-(X) \cap Y(X),$$

such that

$$y_{(-\infty,-I]} = q_{(-\infty,0]}, \quad y_{(M,\infty)} = p_{(M,\infty)},$$
$$z_{(-\infty,M]} = p_{(-\infty,M]}, \quad z_{(J,\infty)} = r_{(0,\infty)},$$

and such that

$$\Gamma(x_{(-K-M,K+M]}) = \Gamma((y_{(-I-H,M]}, z_{(M,J+H]})).$$

Next we introduce a condition (DQ3) and a condition (DQ4) on a subshift $X \subset \Sigma^{\mathbb{Z}}$. Again these conditions come in two symmetric parts, of which we write only one. Condition (DQ3) is the translation of condition (AQ3) and condition (DQ4) is the translation of condition (AQ4):
(DQ3−) For $p, q \in P(A(X))$ and $x \in A^-(X) \cap Y(X)$ that is left asymptotic to p, and such that $x_{(0,\infty)} = p_{(0,\infty)}$, and for $M \in \mathbb{N}$, there exists a $y \in A^+(X) \cap Y(X)$ that is right asymptotic to a point in the orbit of q such that

$$y_{(-\infty,M]} = p_{(-\infty,M]},$$

and such that

$$(x_{(-\infty,0]}, y_{(0.\infty)}) \in A(X).$$

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(DQ4−) For $q, r \in P(A(X))$ there exist an $H \in \mathbb{N}$ such that for K_+, $M > H$, and for $p \in P(A(X))$, $x \in A^-(X)$, $y \in A^+(X)$ and I, $K_- \in \mathbb{N}$, such that

$$x_{(-\infty,-K_-]} = p_{(-\infty,-K_-]}, \quad x_{(0,\infty)} = q_{(0,\infty)},$$
$$y_{(-\infty,K_+]} = q_{(-\infty,K_+]}, \quad y_{(K_++I,\infty)} = r_{(K_++I,\infty)},$$

and

$$(x_{(-\infty,K_+]}, y_{(K_+,\infty)}) \in A^-(X),$$

there exist a $z \in A^-(X)$, and J, $D \in \mathbb{N}$, such that

$$z_{(-\infty,K_++I+M]} = r_{(-\infty,0]}, \quad z_{(K_++I+M+J,\infty)} = q_{(0,\infty)},$$

and

$$\Gamma(x_{(-K_--D,K_+]}, y_{(K_+,K_++I]}, z_{(K_++I,K_++I+M+J+D]}) = \Gamma(x_{(-K_--D,K_++D]}).$$

We call a point $x \in A^-(X) \cap Y(X)$ indecomposable if there is an $H \in \mathbb{N}$, such that the following holds for $K > H$: for $p \in P(A(X))$ and $I \in \mathbb{Z}$ such that

$$x_{(I,\infty)} = p_{(I,\infty)},$$

and

$$y \in A^+(X) \setminus A(X)$$

such that

$$y_{(-\infty,I+K]} = p_{(-\infty,I+K]},$$

one has that

$$(x_{(-\infty,I]}, y_{(I,\infty)}) \in X,$$

implies that

$$(x_{(-\infty,I]}, y_{(I,\infty)}) \in A^+(X).$$

The indecomposable points in $A^+(X) \cap Y(X)$ are defined symmetrically.

With the notion of an indecomposable point we translate condition (AQ5) into a condition (DQ5):

(DQ5−) There exists an $M \in \mathbb{N}$ such that for $q, r \in P(A(X))$ and an indecomposable point $x \in A^-(X) \cap Y(X)$ and for I_-, $I_+ \in \mathbb{Z}$, $I_- < I_+$, such that

$$x_{(-\infty,I_-]} = q_{-\infty,0]}, \quad x_{(I_+,\infty)} = r_{(0,\infty)},$$

there exist an indecomposable point $y \in A^-(X) \cap Y(X)$ and J_-, $J_+ \in \mathbb{Z}$, $J_- < J_+$, and $D \in \mathbb{N}$, such that

$$J_+ - J_- \le M,$$
$$y_{(-\infty,J_-]} = q_{-\infty,0]}, \quad y_{(J_+,\infty)} = r_{(0,\infty)},$$

and

$$\Gamma(x_{(I_--D,I_++D]}) = \Gamma(y_{(J_--D,J_++D]}).$$

We have a condition (DQ6) that also consists of two parts (DQ6−) and (DQ6+) that are symmetric to one another:

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(DQ6−) If $x^- \in X_{(-\infty,0]}$ is such that for all $M \in \mathbb{N}$ there is an $I > M$ such that

$$x^-_{-I} \notin \omega_1^-(x^-_{(-I,0]}),$$

then for $y^-_{(-\infty,M]} \in X_{(-\infty,J}$, $J \in \mathbb{N}$, such that

$$y^-_{(-\infty,0]} = x^-_{(-\infty,0]},$$

and for all $M \in \mathbb{N}$ there is an $I > M$ such that

$$y^-_{-I} \notin \omega_1^-(y^-_{(-I,J]}).$$

Condition (DQ6−) appeared in connection with the Cantor horizon of the Dyck shift [**KM1**]. Inspection shows that conditions (DQ1−) and (DQ2−), and therefore also conditions (DQ1+) and (DQ2+), are invariant under topological conjugacy. Also, a topological conjugacy maps indecomposable points to indecomposable points. As a consequence, condition (DQ5−), and therefore also condition (DQ5+), is invariant under topological conjugacy. We prove that condition (DQ6−) is invariant under topological conjugacy.

PROPOSITION 3.1. *Let $X \subset \Sigma^{\mathbb{Z}}$, $\widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ be subshifts and let $\varphi : X \to \widetilde{X}$ be a topological conjugacy. Let X satisfy condition* (DQ6−)*. Then $\widetilde{X}$ also satisfies condition* (DQ6−)*.*

Proof. Let $L \in \mathbb{Z}_+$ be such that $[-L, L]$ is a coding window for φ and φ^{-1}, and let φ^{-1} be given by the block map $\Phi : \widetilde{X}_{[-L.L]} \to \Sigma$.

Let $\widetilde{x}^- \in \widetilde{X}_{(-\infty,0]}$, and assume that, for all $\widetilde{M} \in \mathbb{N}$, there is an $\widetilde{I} > \widetilde{M}$ such that

$$\widetilde{x}^-_{\widetilde{I}} \notin \omega_1^-(\widetilde{x}^-_{(-\widetilde{I},0]}). \tag{3.1}$$

Set

$$x^- = \Phi(\tilde{x}^-).$$

For $M \in \mathbb{N}$, setting $\widetilde{M} = M + L$, one has as a consequence of (3.1) that there is an $\widetilde{I} > M + L$ such that

$$x_{[\widetilde{I}-L,\widetilde{I}+L]} \notin \omega^-_{2L+1}(x_{(\widetilde{I}+L,-L]}),$$

and it follows that there is an

$$I \in [\widetilde{I} - L, \widetilde{I} + L]$$

such that

$$x_{-I} \notin \omega_1^-(x_{(-I,-L]}).$$

To confirm (DQ6−) for $\widetilde{X}$, let

$$\widetilde{y}^- \in \widetilde{X}_{(-\infty,J]}, \quad J \in \mathbb{N},$$

such that

$$\widetilde{y}^-_{(\infty,0]} = \widetilde{x}^-,$$

and let $\widetilde{M} \in \mathbb{N}$. Set

$$y^- = \Phi(\widetilde{y}^-).$$

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As we have seen at the beginning of the proof, we can apply (DQ6−) of X to x^-, and we find that there is an

$$I > \widetilde{M} + L \tag{3.2}$$

such that

$$y^-_{-I} \notin \omega_1^-(y^-_{(-I,J-L]}).$$

This implies that

$$\widetilde{y}^-_{(-I-L,-I+L]} \notin \omega^-_{2L+1}(\widetilde{y}^-_{(-I+L,J]}).$$

It follows that there exists an

$$\widetilde{I} \in [-I - L, -I + L] \tag{3.3}$$

such that

$$\widetilde{y}^-_{\widetilde{I}} \notin \omega_1^-(\widetilde{y}^-_{(-\widetilde{I},J]}). \tag{3.4}$$

By (3.2), (3.3), and (3.4), (DQ6−) is satisfied by $\widetilde{X}$. □

We recall from [**Kr2**] the definition of property (A). For $n \in \mathbb{N}$, a subshift $X \subset \Sigma^{\mathbb{Z}}$ such that $A_1(X) \neq \emptyset$, has property (a, n, H), $H \in \mathbb{N}$, if for $h, \widetilde{h} \geq 3H$ and for

$$a \in A_n(X)_{[1,h]}, \quad \widetilde{a} \in A_n(X)_{[1,\widetilde{h}]},$$

such that

$$a_{[1,H]} = \widetilde{a}_{[1,H]}, \quad a_{(h-H,h]} = \widetilde{a}_{(\widetilde{h}-H,\widetilde{h}]},$$

one has that a and $\tilde{a}$ have the same context. A subshift $X \subset \Sigma^{\mathbb{Z}}$ has property (A) if there are H_n, $n \in \mathbb{N}$, such that X has the properties (a, n, H_n), $n \in \mathbb{N}$.

We also recall the construction of the associated semigroup. Let $y, \widetilde{y} \in Y(X)$, let y be left asymptotic to $q \in P(A(X))$ and right asymptotic to $r \in P(A(X))$, and let $\widetilde{y}$ be left asymptotic to $\widetilde{q} \in P(A(X))$ and right asymptotic to $\widetilde{r} \in P(A(X))$. Given that X has the properties (a, n, H_n), $n \in \mathbb{N}$, we say that y and $\widetilde{y}$ are equivalent, $y \approx \widetilde{y}$, if $q \approx \widetilde{q}$ and $r \approx \widetilde{r}$, and if for $n \in \mathbb{N}$ such that $q, r, \widetilde{q}, \widetilde{r} \in A_n(X)$ and for $I, J, \widetilde{I}, \widetilde{J} \in \mathbb{Z}$, $I < J$, $\widetilde{I} < \widetilde{J}$, such that

$$y_{(-\infty,I]} = q_{(-\infty,0]}, \quad y_{(J,\infty)} = r_{(0,\infty)},$$
$$\widetilde{y}_{(-\infty,\widetilde{I}]} = \widetilde{q}_{(-\infty,0]}, \quad \widetilde{y}_{(\widetilde{J},\infty)} = \widetilde{r}_{(0,\infty)},$$

one has for $h \geq 3H_n$ and for

$$a \in X_{(I-h,J+h]}, \quad \widetilde{a} \in X_{(\widetilde{I}-h,\widetilde{J}+h]},$$

such that

$$a_{(I-H_n,J+H_n]} = y_{(I-H_n,J+H_n]}, \quad \widetilde{a}_{(\widetilde{I}-H_n,\widetilde{J}+H_n]} = \tilde{y}_{(\tilde{I}-H_n,\tilde{J}+H_n]},$$
$$a_{(I-h,I-h+H_n)} = \widetilde{a}_{(\widetilde{I}-h,\widetilde{I}-h+H_n)},$$
$$a_{(J+h-H_n,J+h]} = \widetilde{a}_{(\widetilde{J}+h-H_n,\widetilde{J}+h]},$$

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and such that

$$a_{(I-h,I]} \in A_n(X)_{(I-h,I]}, \quad \widetilde{a}_{(\widetilde{I}-h,\widetilde{I}]} \in A_n(X)_{(\widetilde{I}-h,\widetilde{I}]},$$
$$a_{(J,J+h]} \in A_n(X)_{(J,J+h]}, \quad \widetilde{a}_{(\widetilde{J},\widetilde{J}+h]} \in A_n(X)_{(\widetilde{J},\widetilde{J}+h]},$$

that a and $\widetilde{a}$ have the same context. To give $[Y(X)]_{\approx}$ the structure of a semigroup $\mathcal{S}(X)$ that is invariantly associated to X, let $u, v \in Y(X)$, let u be right asymptotic to $q \in P(A(X))$ and let v be left asymptotic to $r \in P(A(X))$. If here $q \succeq r$, then set $[u]_{\approx}[v]_{\approx}$ equal to $[y]_{\approx}$, where y is any point in $Y(X)$ such that there are $n \in \mathbb{N}$, I, J, $\widehat{I}$, $\widehat{J} \in \mathbb{Z}$, $I < J$, $\widehat{I} < \widehat{J}$, such that $q, r \in A_n(X)$, and such that

$$u_{(I,\infty)} = q_{(I,\infty)}, \quad v_{(-\infty,J]} = r_{(-\infty,J]},$$
$$y_{(-\infty,\widehat{I}+H_n]} = u_{(-\infty,I+H_n]}, \quad y_{(\widehat{J}-H_n,\infty)} = v_{(J-H_n,\infty)},$$

and

$$y_{(\widehat{I},\widehat{J}]} \in A_n(X)_{(\widehat{I},\widehat{J}]},$$

provided that such a point y exists. If such a point y does not exist, set $[u]_{\approx}[v]_{\approx}$ equal to zero. Also, in the case that $q \not\succeq r$, set $[u]_{\approx}[v]_{\approx}$ equal to zero.

For a subshift with property (A), conditions (AP1), (AP2), (AQ1), (AQ2), and (AQ4) are equivalent to their translations. We prove that, for a subshift with property (A), conditions (DQ4) and (DQ6) together imply condition (AQ6).

PROPOSITION 3.2. *Let the subshift X have property (A) and let conditions* (DQ3)*,* (DQ4)*, and* (DQ6) *be satisfied by X. Then $\mathcal{S}(X)$ satisfies condition* (AQ6).

Proof. If $\mathcal{S}(X)$ does not satisfy (AQ6−) then there exist

$$H_m^-, G_m^- \in \mathcal{S}^-(X) \setminus \mathcal{U}_{\mathcal{S}(X)}, \quad m \in \mathbb{Z}_+,$$

such that

$$H_m^- = H_{m+1}^- G_m^-, \quad m \in \mathbb{Z}_+, \tag{3.5}$$

or such that

$$H_m^- = G_m^- H_{m+1}^-, \quad m \in \mathbb{Z}_+. \tag{3.6}$$

Assume (3.5). Let $U \in \mathcal{U}_{\mathcal{S}}$ be given by

$$U H_0^- = H_0^-,$$

and let $U_m \in \mathcal{U}_{\mathcal{S}}$, $m \in \mathbb{Z}_+$, be given by

$$U_{m+1} G_m^- = G_m^-, \quad m \in \mathbb{Z}_+.$$

By (DQ3−) there exist an $H^+ \in \mathcal{S}^+(X)(U, U_0) \setminus \mathcal{U}_{\mathcal{S}(X)}$ such that

$$H_0^- H^+ = U,$$

and also $G_m^+ \in \mathcal{S}^+(X)(U_{m+1}, U_m) \setminus \mathcal{U}_{\mathcal{S}(X)}$ such that

$$G_m^- G_m^+ = U_m, \quad m \in \mathbb{Z}_+. \tag{3.7}$$

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One has also $G_m \in \mathcal{S}(X) \setminus \mathcal{U}_{\mathcal{S}(X)}$, $m \in \mathbb{Z}_+$, such that

$$UG_m = G_m, \quad G_m^- G_m = 0, \quad m \in \mathbb{Z}_+, \tag{3.8}$$

for otherwise one would have, for some $m_\circ \in \mathbb{N}$, that $G_{m_\circ} \in \mathcal{U}_{\mathcal{S}(X)}$. Let $V_m \in \mathcal{U}_{\mathcal{S}(X)}$, $m \in \mathbb{Z}_+$, be given by

$$G_m V_m = G_m, \quad m \in \mathbb{Z}_+.$$

Let Δ be a system of representatives of $\approx$-equivalence in $P(A(X))$, let $n \in \mathbb{N}$ be such that every point in Δ has a period that is less or equal to n, let $H_n \in \mathbb{N}$ be such that X has property (a, n, H_n), and set $h = 3H_n$. In view of

$$\mathcal{U}_{\mathcal{S}(X)} = \mathfrak{P}(X),$$

we can choose a $p \in \Delta$ such that

$$U = [p]_\approx,$$

and also $p^{(m)}, q^{(m)} \in \Delta$, $m \in \mathbb{Z}_+$, such that

$$U_m = [p^{(m)}]_\approx, \quad V_m = [q^{(m)}]_\approx, \quad m \in \mathbb{Z}_+,$$

together with a $J > n$ and J_m^-, J_m^+, $K_m \in \mathbb{N}$, $m \in \mathbb{N}$, such that one has, setting

$$J_0^- = J_0^+ = 0,$$

that

$$J_m^- > J_{m-1}^- + h, \quad J_m^+ > J_{m-1}^+ + h, \quad m \in \mathbb{N},$$

and also

$$K_m > J_{m-1}^+ + h, \quad m \in \mathbb{N},$$

together with $a \in X_{(0,J+h]}$ such that

$$a_{(0,h]} = p^{(0)}_{(0,h]}, \quad a_{(J,J+h]} = p_{(0,h]}, \tag{3.9}$$
$$[(p^{(0)}_{(-\infty,h]}, a_{(h,J]}, p_{(0,\infty)})]_\approx = H_0^+,$$

and $x^- \in X_{(-\infty,h]}$ such that

$$x^-_{(J_m^- - h, J_m^-]} = p^{(m)}_{(-h,0]}, \quad m \in \mathbb{Z}_+, \tag{3.10}$$
$$[(p^{(m)}_{(-\infty,h]}, x^-_{(-J_m^- + h, -J_{m-1}]}, p^{(m-1)}_{(0,\infty)})]_\approx = G_m^-, \quad m \in \mathbb{N},$$

and $a^{(m)} \in X_{(0,K_m+h]}$, $m \in \mathbb{N}$, such that

$$a^{(m)}_{(J_\ell^+, J_\ell^+ + h]} = p^{(\ell)}_{(0,h]}, \tag{3.11}$$
$$[(p^{(\ell)}_{(-\infty,h]}, a^{(m)}_{(J_\ell^+ + h, J_{\ell+1}^+]}, p^{(\ell+1)}_{(0,\infty)})]_\approx = G_\ell^+, \quad 0 \le \ell < m,$$
$$a^{(m)}_{(K_m, K_m + h]} = q^{(m)}_{(0,h]}, \tag{3.12}$$
$$[(p^{(m)}_{(-\infty,h]}, a^{(m)}_{(J_m^+ + h, K_m]}, q^{(m)}_{(0,\infty)})]_\approx = G_m, \quad m \in \mathbb{N}.$$

By (3.7), (3.10), and (3.11), and by the choice of h,

$$(x^-_{(-J_{m-1}^-, 0]}, a^{(m)}) \in X_{[-J_{m-1}^-, K_m + h]},$$

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and by (3.8), (3.10) and (3.12), and by the choice of h,

$$(x^-_{(-J^-_m,0]}, a^{(m)}) \notin X_{(-J^-_m, K_m+h]}.$$

It follows that there is an $i \in (J^-_{m-1}, -J^-_m]$ such that $x^-_{-i} \notin \omega^-_1(x^-_{(-i,0]})$. On the other hand, it follows from

$$H^-_m\Big(\prod_{m\le\ell\le 0} G^-_\ell\Big)H^+ = U$$

that

$$\Big(\prod_{m\le\ell\le 0} G^-_\ell\Big)H^+ \in \mathcal{S}^+(X),$$

and by (3.9) and (3.10) then

$$x_{-i} \in \omega^-_1(x_{(-i,0]}, a), \quad i \in \mathbb{N},$$

and we have a contradiction to (DQ6−). The case (3.6) reduces by (DQ4+) to the case (3.5). The proof that (DQ3+), (DQ4−), and (DQ6+) together imply (DQ6−) is symmetric. □

COROLLARY 3.3. *A subshift with property* (A)*, that satisfies conditions (DP1)–(DP2) and conditions (DQ1)–(DP6) has as its associated semigroup an* $\mathcal{R}$*-graph semigroup.*

Proof. By [**HK2**, Theorem 2.3] for a semigroup $\mathcal{S}$ that is associated to a subshift with property (A), the projection of $\mathcal{S}$ onto $[\mathcal{S}]$ is an isomorphism. Taking into account that (DP1) and (DP2) are the translations of (AP1) and (AP2) and that (DQ3), (DQ4), and (D5) are the translations of (AQ3), (AQ4), and (AQ5), apply Proposition 3.2 and Theorem 2.1. □

4. *Property* (B)

We say that a subshift $X \subset \Sigma^{\mathbb{Z}}$ has property (B) with respect to the parameter $M \in \mathbb{N}$ if the following holds: for $I_-, I_+, J_-, J_+ \in \mathbb{Z}$ such that

$$I_+ - I_-,\ J_+ - J_- \ge M,$$

and

$$a \in X_{(I_-,I_+]}, \quad b \in X_{(J_-,J_+]}$$

such that

$$a_{(I_-,I_-+M]} = b_{(J_-,J_-+M]}, \quad a_{(I_+-M,I_+]} = b_{(J_+-M,J_+]},$$

and $R \in \mathbb{N}$ and

$$x^- \in \omega^-_\infty(a) \cap \omega^-_\infty(b), \quad x^+ \in \omega^+_\infty(a) \cap \omega^+_\infty(b)$$

such that

$$\Gamma(x^-_{(I_--R,I_-]}, a, x^+_{(I_+,I_++R]}) = \Gamma(x^-_{(J_--R,J_-]}, b, x^+_{(J_+,J_++R]}),$$

one has that

$$\Gamma(a) = \Gamma(b).$$

We say that a subshift $X \subset \Sigma^{\mathbb{Z}}$ has property (B) if it has property (B) with respect to some parameter.

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For a graph inverse semigroup $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^-)$ an $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^-)$-presentation has property (B), as can be seen by inspection. Conversely, if an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^-)$ has properties (A) and (B), then the semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^-)$ is necessarily a graph inverse semigroup.

THEOREM 4.1. *Property (B) is an invariant of topological conjugacy.*

Proof. If a subshift $X \subset \Sigma^{\mathbb{Z}}$ has property (B) with respect to some parameter, then its higher block systems also have property (B). To prove the proposition it is therefore enough to consider the situation that one is given subshifts $X \subset \Sigma^{\mathbb{Z}}$, $\widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ and a topological conjugacy $\varphi : X \to \widetilde{X}$ that is given by a one-block map $\Phi : \Sigma \to \widetilde{\Sigma}$ with φ^{-1} given for some $L \in \mathbb{Z}_+$ by a block map $\widetilde{\Phi} : \widetilde{X}_{[-L,L]} \to \Sigma$, where X has property (B) with respect to the parameter M, and to prove that the subshift $\widetilde{X}$ has also property (B). We set

$$\widetilde{M} = M + 2L,$$

and we prove that $\widetilde{X}$ has property (B) with respect to the parameter $\widetilde{M}$. For this let $I_-, I_+, J_-, J_+ \in \mathbb{Z}$,

$$I_+ - I_-,\ J_+ - J_- \geq \widetilde{M},$$

and let

$$\widetilde{a} \in \widetilde{X}_{(I_-,I_+]}, \quad \widetilde{b} \in \widetilde{X}_{(J_-,J_+]}$$

be such that

$$\widetilde{a}_{(I_-,I_-+\widetilde{M}]} = \widetilde{b}_{J_-,J_-+\widetilde{M}]}, \quad \widetilde{a}_{(I_+-\widetilde{M},I_+]} = \widetilde{b}_{(J_+-\widetilde{M},J_+]}, \tag{4.1}$$

and let $R \in \mathbb{N}$ and

$$\widetilde{x}^- \in \omega^-_\infty(\widetilde{a}) \cap \omega^-_\infty(\widetilde{b}), \quad \widetilde{x}^+ \in \omega^+_\infty(\widetilde{a}) \cap \omega^+_\infty(\widetilde{b}) \tag{4.2}$$

be such that

$$\Gamma(\widetilde{x}^-_{(I_--R,I_-]}, \widetilde{a}, \widetilde{x}^+_{(I_+,I_++R]}) = \Gamma(\widetilde{x}^-_{(J_--R,J_-]}, \widetilde{b}, \widetilde{x}^+_{(J_+-,J_++R]}). \tag{4.3}$$

We have to prove that

$$\Gamma(\widetilde{a}) = \Gamma(\widetilde{b}). \tag{4.4}$$

We let

$$a \in X_{(I_-+L,I_+-L]}, \quad b \in X_{(J_-+L,J_+-L]}$$

be given by

$$a = \widetilde{\Phi}(\widetilde{a}), \quad b = \widetilde{\Phi}(\widetilde{b}).$$

By (4.2),

$$a_{(I_-+L,I_-+L+M]} = b_{(J_-+L,J_-+L+M]}, \quad a_{(I_+-L-M,I_+-L]} = b_{(J_--L-M,J_+-L]}. \tag{4.5}$$

We set also

$$x^- = \widetilde{\Phi}(\widetilde{x}^-, \widetilde{a}_{(I_-,I_-+2L]}), \quad x^+ = \widetilde{\Phi}(\widetilde{a}_{(I_+-2L,I_+]}, \widetilde{x}^+).$$

It follows from (4.1) and (4.2) that

$$x^- \in \omega^-_\infty(a) \cap \omega^-_\infty(b), \quad x^+ \in \omega^+_\infty(a) \cap \omega^+_\infty(b). \tag{4.6}$$

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One has

$$\Gamma(x^-_{(I_--R-L,I_-+L]}, a, x^+_{(I_+-L,I_++R+L]}) = \{(\widetilde{\Phi}(\widetilde{u}^-), \widetilde{\Phi}(\widetilde{u}^+) : (\widetilde{u}^-, \widetilde{u}^+) \in \Gamma(\widetilde{x}^-_{(I_--R,I_-]}, \widetilde{a}, \widetilde{x}^+_{(I_+,I_++R]})\},$$

with a similar expression for $\Gamma(x^-_{(J_--R-L,J_-+L]}, b, x^+_{(J_+-L,J_++R+L]})$, and it is seen that (4.3) implies that

$$\Gamma(x^-_{(I_--R-L,I_-+L]}, a, x^+_{(I_+-L,I_++R+L]}) = \Gamma(x^-_{(J_--R-L,J_-+L]}, b, x^+_{(J_+-L,J_++R+L]}). \tag{4.7}$$

By (4.5), (4.6), and (4.7) we can apply property (B) of X to obtain

$$\Gamma(a) = \Gamma(b). \tag{4.8}$$

One has

$$\Gamma(\widetilde{a}) = \{(\Phi(u^-_{(-\infty,I_-]}), \Phi(u^+_{(I_+,\infty)})) : (u^-, u^+) \in \Gamma(a), \Phi(u^-_{(I_-,I_-+L]}) = \widetilde{a}_{(I_-,I_-+L]}, \Phi(u^+_{(I_+,I_++L]}) = \widetilde{a}_{(I_+,I_++L]}\},$$

with a similar expression for $\Gamma(\widetilde{b})$, from which it is seen that (4.8) implies (4.4). □

LEMMA 4.2. *Let $X \subset \Sigma^{\mathbb{Z}}$, $\widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ be subshifts, and let $\varphi : X \to \widetilde{X}$ be a topological conjugacy that is given by a one-block map $\Phi : \Sigma \to \widetilde{\Sigma}$ with φ^{-1} given for some $L \in \mathbb{Z}_+$ by a block map $\widetilde{\Phi} : \widetilde{X}_{[-L,L]} \to \Sigma$. Let $\widetilde{X}$ have property (B) with respect to the parameter L. Let I_-, I_+, J_-, $J_+ \in \mathbb{Z}$ be such that*

$$I_+ - I_-,\ J_+ - J_- \geq L,$$

and let

$$\widetilde{a} \in \widetilde{X}_{(I_-,I_+]}, \quad \widetilde{b} \in \widetilde{X}_{(J_-,J_+]}$$

be such that

$$\widetilde{a}_{(I_-,I_-+L]} = \widetilde{b}_{(J_-,J_-+L]}, \quad \widetilde{a}_{(I_+-L,I_+]} = \widetilde{b}_{(J_+-L,J_+]},$$

and let

$$\widetilde{x}^- \in \omega^-_\infty(\widetilde{a}) \cap \omega^-_\infty(\widetilde{b}), \quad \widetilde{x}^+ \in \omega^+_\infty(\widetilde{a}) \cap \omega^+_\infty(\widetilde{b}).$$

Set

$$a = \widetilde{\Phi}(\widetilde{x}^-_{(I_--L,I_-]}, \widetilde{a}, \widetilde{x}^+_{(I_+,I_++L]}), \quad b = \widetilde{\Phi}(\widetilde{x}^-_{(J_--L,J_-]}, \widetilde{b}, \widetilde{x}^+_{(J_+,J_++L]}).$$

Then

$$\Gamma(a) = \Gamma(b) \tag{4.9}$$

implies

$$\Gamma(\widetilde{a}) = \Gamma(\widetilde{b}). \tag{4.10}$$

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Proof. We have

$$\begin{aligned}&\Gamma(\widetilde{x}^-_{(I_--L,I_-]},\widetilde{a},\widetilde{x}^+_{(I_+,I_++L]}))\\&\quad=\{(\Phi(u^-_{(-\infty,I_--L]}),\Phi(u^+_{(I_++L,\infty)})):(u^-,u^+)\in\Gamma(a),\\&\qquad\Phi(u^-_{(I_--L,I_-]})=\widetilde{x}^-_{(I_--L,I_-]},\Phi(u^+_{(I_+,I_++L]})=\widetilde{x}^+_{(I_+,I_++L]}\},\end{aligned}$$

and replacing a by b in the right-hand side yields the corresponding expression for $\Gamma(\widetilde{x}^-_{(J_--L,J_-]},\widetilde{b},\widetilde{x}^+_{(J_+,J_++L]})$. From this it is seen that (4.9) implies

$$\Gamma(\widetilde{x}^-_{(I_--L,I_-]},\widetilde{a},\widetilde{x}^+_{(I_+,I_++L]}))=\Gamma(\widetilde{x}^-_{(I_--L,I_-]},\widetilde{b},\widetilde{x}^+_{(I_+,I_++L]})),$$

which then by property (B) of $\widetilde{X}$ implies (4.10). □

5. *Property* (c)

We say that a subshift $X\subset\Sigma^{\mathbb{Z}}$ has property (c) with respect to the parameter $Q\in\mathbb{N}$ if the following holds: for I_-, I_+, J_-, $J_+\in\mathbb{Z}_+$,

$$I_+-I_-,\ J_+-J_-\geq Q,$$

and for

$$a\in X_{(I_-,I_+]},\quad b\in X_{(J_-,J_+]},$$

such that

$$\omega^-_\infty(a)\cap\omega^-_\infty(b)\neq\emptyset,\quad \omega^+_\infty(a)\cap\omega^+_\infty(b)\neq\emptyset,$$

and such that

$$a_{(I_-,I_-+Q]}=b_{(J_-,J_-+Q]},\quad a_{(I_+-Q,I_+]}=b_{(J_+-Q,J_+]},$$

and

$$\Gamma^-(a)=\Gamma^-(b),\quad \Gamma^+(a)=\Gamma^+(b),$$

one has that

$$\Gamma(a)=\Gamma(b).$$

We say that a subshift $X\subset\Sigma^{\mathbb{Z}}$ has property (c) if it has property (c) with respect to some parameter. Inspection shows that $S_{\mathcal{R}}(\mathfrak{P},\mathcal{E}^-,\mathcal{E}^+)$-presentations have property (c).

PROPOSITION 5.1. *The simultaneous occurrence of properties* (B) *and* (c) *is an invariant of topological conjugacy.*

Proof. By Theorem 4.1, property (B) is an invariant of topological conjugacy. Also, if a subshift $X\subset\Sigma^{\mathbb{Z}}$ has property (c), then its higher block systems also have property (c). To prove the proposition it is therefore enough to consider the situation that one is given subshifts $X\subset\Sigma^{\mathbb{Z}}$, $\widetilde{X}\subset\widetilde{\Sigma}^{\mathbb{Z}}$ and a topological conjugacy $\varphi:X\to\widetilde{X}$ that is given by a one-block map $\Phi:\Sigma\to\widetilde{\Sigma}$ with φ^{-1} given for some $L\in\mathbb{Z}_+$ by a block map $\widetilde{\Phi}:\widetilde{X}_{[-L,L]}\to\Sigma$, where X has property (B) with respect to the parameter L and has property (c) with respect to the parameter Q, and to prove that the subshift $\widetilde{X}\subset\widetilde{\Sigma}^{\mathbb{Z}}$ has property (c). We set

$$\widetilde{Q}=Q+L.$$

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We prove that $\widetilde{X}$ has property (c) with respect to the parameter $\widetilde{Q}$. For this, let $I_-, I_+, J_-, J_+ \in \mathbb{Z}_+$ be such that

$$I_+ - I_-,\ J_+ - J_- \geq \widetilde{Q},$$

and let

$$\widetilde{a} \in \widetilde{X}_{(I_-,I_+]}, \quad \widetilde{b} \in \widetilde{X}_{(J_-,J_+]}$$

be such that

$$\omega_\infty^-(\widetilde{a}) \cap \omega_\infty^-(\widetilde{b}) \neq \emptyset, \quad \omega_\infty^+(\widetilde{a}) \cap \omega_\infty^+(\widetilde{b}) \neq \emptyset, \tag{5.1}$$

and such that

$$\widetilde{a}_{(I_-,I_-+\widetilde{Q}]} = \widetilde{b}_{(J_-,J_-+\widetilde{Q}]}, \quad \widetilde{a}_{(I_+-\widetilde{Q},I_+]} = \widetilde{b}_{(J_+-\widetilde{Q},J_+]}, \tag{5.2}$$

$$\Gamma^-(\widetilde{a}) = \Gamma^-(\widetilde{b}), \tag{5.3}$$

and

$$\Gamma^+(\widetilde{a}) = \Gamma^+(\widetilde{b}).$$

We have to prove that

$$\Gamma(\widetilde{a}) = \Gamma(\widetilde{b}). \tag{5.4}$$

By (5.1) we can choose

$$\widetilde{x}^- \in \omega_\infty^-(\widetilde{a}) \cap \omega_\infty^-(\widetilde{b}) \tag{5.5}$$

and

$$\widetilde{x}^+ \in \omega_\infty^+(\widetilde{a}) \cap \omega_\infty^+(\widetilde{b}). \tag{5.6}$$

We set

$$a = \widetilde{\Phi}(\widetilde{x}^-_{(I_--L,I_-]}, \widetilde{a}, \widetilde{x}^+_{(I_+,I_++L]}), \quad b = \widetilde{\Phi}(\widetilde{x}^-_{(J_--L,J_-]}, \widetilde{b}, \widetilde{x}^+_{(J_+,J_++L]}).$$

By Lemma 4.2 and by property (B) of $\widetilde{X}$ we will have proved (5.4) once we have shown that

$$\Gamma(a) = \Gamma(b). \tag{5.7}$$

We set

$$x^- = \widetilde{\Phi}(\widetilde{x}^-, \widetilde{a}_{(I_-,I_-+L]}), \quad x^+ = \widetilde{\Phi}(\widetilde{a}_{(I_+-L,I_+]}, \widetilde{x}^-).$$

By (5.2) then also

$$x^- = \widetilde{\Phi}(\widetilde{x}^-, \widetilde{b}_{(J_-,J_-+L]}), \quad x^+ = \widetilde{\Phi}(\widetilde{b}_{(J_+-L,J_+]}, \widetilde{x}^-),$$

and by (5.5) and (5.6) then

$$x^- \in \omega_\infty^-(a) \cap \omega_\infty^-(b), \quad x^+ \in \omega_\infty^+(a) \cap \omega_\infty^+(b). \tag{5.8}$$

By (5.2),

$$\begin{aligned} a_{(I_-,I_-+Q]} &= \widetilde{\Phi}(\widetilde{x}^-_{(I_--L,I_-]}, \widetilde{a}_{(I_-,I_-+\widetilde{Q}]}) \\ &= \widetilde{\Phi}(\widetilde{x}^-_{(I_--L,I_-]}, \widetilde{b}_{(J_+-\widetilde{Q},J_+]}) = b_{(J_-,J_-+Q]}, \end{aligned} \tag{5.9}$$

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and symmetrically

$$a_{(I_-,I_-+Q]} = b_{(J_-,J_-+Q]}. \tag{5.10}$$

We prove that

$$\Gamma^-(a) = \Gamma^-(b). \tag{5.11}$$

For this let $u^- \in \Gamma^-(a)$ and set $\widetilde{u} = \Phi(u^-)$. One has $\widetilde{u} \in \Gamma^-(\widetilde{a})$, which by (5.3) implies that $\widetilde{u} \in \Gamma^-(\widetilde{b})$. By (5.6) then $(\widetilde{u}, \widetilde{b}, \widetilde{x}^+) \in \widetilde{X}$, and therefore $(u, b, x^+) \in X$ and (5.10) is confirmed. Symmetrically one has that

$$\Gamma^+(a) = \Gamma^+(b). \tag{5.12}$$

We use (5.8)–(5.12) and property (c) of X to show (5.7). □

6. *Instantaneous presentations*

We say that a subshift $X \subset \Sigma^{\mathbb{Z}}$ is right instantaneous if $\omega_1^+(\sigma) \neq \emptyset$, $\sigma \in \Sigma$, equivalently, if $\omega_\infty^+(\sigma) \neq \emptyset$, $\sigma \in \Sigma$. Left instantaneity is defined time symmetrically. The notion of left instantaneity was considered by Matsumoto in [**M1**, §4]. We say that a subshift is bi-instantaneous if it is left and right instantaneous. We give an example of a topologically transitive sofic system that is left instantaneous but not right instantaneous. This example is a variation of an example of Carlsen and Matsumoto in [**CM**]. Let $\Sigma = \{0, 1, \alpha, \beta\}$ and exclude from $\Sigma^{\mathbb{Z}}$ the points that contain one of the following words: 10, 11, $\alpha 0^n 1\beta$, $\beta 0^n 1\alpha$, $n \in \mathbb{N}$. In this way one obtains a left instantaneous sofic system in which the words of the form $0^n 1$, $n \in \mathbb{N}$, do not have a future that is compatible with their entire past. By a product construction one can obtain examples of this type of topologically transitive sofic system that are neither left nor right instantaneous.

THEOREM 6.1. *A sofic system has a bi-instantaneous presentation.*

Proof. For the construction of a bi-instantaneous presentation of a sofic system $X \subset \Sigma^{\mathbb{Z}}$ set

$$\mathcal{V} = \{\Gamma_\infty^+(x^-) : x^- \in X_{(-\infty,0]}\},$$

and denote, for an admissible word a of X, by $\mathcal{V}(a)$ the set of $V \in \mathcal{V}$ that contain a sequence that starts with a. A word $a \in \mathcal{L}(X)$ determines a partial mapping τ_a of $\mathcal{V}$ into itself that has $\mathcal{V}(a)$ as its domain of definition, and that is given by

$$\tau_a(V) = \{y^+ \in X_{[1,\infty)} : (a, y^+) \in V(a)\}, \quad V \in \mathcal{V}(a).$$

For $x \in X$ there are $J, K \in \mathbb{N}$ such that

$$\tau_{x_{[1,J]}} = \tau_{x_{[1,J+K]}},$$

in which case

$$\tau_{x_{(J,J+K]}} \restriction \tau_{x_{[1,J]}}(\mathcal{V}(x_{[1,J]})) = \mathrm{id}.$$

For a point $x \in X$ we can therefore define an $I^+(x) \in \mathbb{N}$ as the smallest $I \in \mathbb{N}$ such that there is an $i \in (1, I)$ such that

$$\tau_{x_{(i,I]}} \restriction \tau_{x_{[1,i]}}(\mathcal{V}_{x_{[1,i]}}) = \mathrm{id}, \tag{6.1}$$

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and an $i^+(x) \in (1,\ I^+(x))$ as the uniquely determined $i \in (1,\ I^+(x))$ such that (6.1) holds. One has

$$I^+(S_X(x)) + 1 \geq I^+(x), \quad x \in X. \tag{6.2}$$

To see this, note that

$$\tau_{x_1}(\mathcal{V}(x_{[1,i^+(S_X(x))]})) \subset \mathcal{V}(x_{(1,i^+(S_X(x))]})$$

and

$$\tau_{x_{(i^+(S_X(x)),I^+(S_X(x))]}} \restriction \tau_{x_{(1,i^+(S_X(x))]}}(\mathcal{V}(x_{(1,i^+(S_X(x))]}) = \mathrm{id}$$

imply that

$$\tau_{x_{(i^+(S_X(x))+1,I^+(S_X(x))+1]}} \restriction \tau_{x_{[1,i^+(S_X(x))+1]}}(\mathcal{V}_{x_{[1,i^+(S_X(x))+1]}}) = \mathrm{id},$$

which means that the contrary of (6.1) would contradict the definition of I^+. Denote by z^+ the point in $X_{(I^+(x),\infty)}$ that carries the right infinite concatenation of the word $x_{x_{(i^+,I^+]}}$. It follows from $\mathcal{V}_{x_{[1,i^+(x)]}} = \mathcal{V}_{x_{[1,I^+(x)]}}$ that

$$z^+ \in \omega^+(x_{[1,I^+(x)]}). \tag{6.3}$$

With I^- and z^- defined time symmetrically one has that

$$I^-(S_X(x)) - 1 \leq I^-(x), \quad x \in X, \tag{6.4}$$

and

$$z^- \in \omega^-(x_{[I^-(x),0)}). \tag{6.5}$$

We set

$$\Xi(x) = (x_{[I^-(x),0)},\ 0,\ x_{[1,I^+(x)]}), \quad x \in X,$$

and with M denoting a bound for $\{|I^-(x)|,\ I^+(x) : x \in X\}$ we can by (6.2) and (6.4) define an embedding ξ of X into $(\cup_{1\leq m\leq M}(\mathcal{L}_m(X) \times \Sigma \times \mathcal{L}_m(X)))^{\mathbb{Z}}$ by

$$\xi(x) = (\Xi(S^{-i}x))_{i\in\mathbb{Z}}, \quad x \in X.$$

Set

$$\Delta = \Xi(X), \quad Y = \xi(X).$$

We prove that Y is a bi-instantaneous presentation of X. For this let

$$(a(-),\ \sigma,\ a(+)) \in \Delta,$$

and let $x \in X$ be such that

$$\Xi(x) = (a(-),\ \sigma,\ a(+)).$$

By (6.3),

$$\Xi(S_X^{-1}((x_{(-\infty,I^+(x)]},\ z^+(x))) \in \omega_1^+(a(-),\ \sigma,\ a(+)),$$

which confirms the right instantaneity of Y. The proof that Y is left instantaneous is time symmetric and uses (6.5). □

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We give an example of a semisynchronizing (see [**Kr2**, **Kr5**]) right instantaneous non-sofic system that is not left instantaneous, but has a left instantaneous presentation. For this, take as alphabet the set

$$\Sigma = \{\mathbf{1}, \alpha_\lambda, \alpha_\rho, \beta_\lambda, \beta_\rho, \gamma_\lambda, \gamma_\rho\},$$

and view Σ as a generating set of $\mathcal{D}_3$ with relations

$$\alpha_\lambda\alpha_\rho = \beta_\lambda\beta_\rho = \gamma_\lambda\gamma\rho = \mathbf{1}, \quad \alpha_\lambda\beta_\rho = \beta_\lambda\alpha_\rho = \alpha_\lambda\gamma_\rho = \gamma_\lambda\alpha_\rho = \beta_\lambda\gamma_\rho = \gamma_\lambda\beta_\rho = 0.$$

Intersect the Motzkin shift

$$M_3 = \left\{x \in \Sigma^{\mathbb{Z}} : \prod_{I_- \le i < I_+} x_i \ne 0,\ I_-, I_+ \in \mathbb{Z},\ I_- < I_+\right\}$$

with the subshift of finite type that is obtained by excluding from $\Sigma^{\mathbb{Z}}$ all points that contain one of the words $\gamma_\lambda\gamma_\lambda$, $\alpha_\lambda\alpha_\lambda\gamma_\lambda$, $\beta_\lambda\beta_\lambda\gamma_\lambda$ to obtain a subshift X. There is a topological conjugacy of X onto a subshift $\bar{X}$ that is given by a three-block mapping Φ,

$$\Phi(\alpha_\lambda\beta_\lambda\gamma_\lambda) = \Phi(\beta_\lambda\alpha_\lambda\gamma_\lambda) = \gamma_\lambda,$$
$$\Phi(\sigma\sigma'\sigma'') = \sigma', \quad \sigma\sigma'' \notin \{\alpha_\lambda\gamma_\lambda, \beta_\lambda\gamma_\lambda\}.$$

Whereas the subshift X is bi-instantaneous, the subshifts $\bar{X}^{\langle[0,n)\rangle}$, $n \in \mathbb{N}$, are right instantaneous but not left instantaneous: the words $\gamma_\lambda\gamma_\lambda\mathbf{1}^n$, $n \in \mathbb{N}$, do not have a past that is compatible with their entire future context.

For a subshift $X \subset \Sigma^{\mathbb{Z}}$, for $L \in \mathbb{Z}_+$, and for mappings

$$\Psi^{(+)} : X_{[-L,L]} \to X_{[1,L+1]},$$

we formulate a condition (RI) that comes in two parts:

$$(RIa)\text{: } \Psi^{(+)}(a) \in \Gamma^+(x^-, a_{[-L,0]}), \qquad a \in X_{[-L,L]},\ x^- \in \Gamma^-(a).$$
$$(RIb)\text{: } \Psi^{(+)}(b_{(-2L+\ell,-L)}, a_{[-L,\ell]}) = \Psi^{(+)}(b_{(-2L+\ell,-L)}, a_{[-L,0]}, \Psi^{(+)}(a)_{[1.\ell]}),$$
$$b \in X_{(-2L,L)},\ a \in X_{[-L,L]},\ 1 \le \ell \le L.$$

We say that a mapping $\Psi^{(+)} : X_{[-L,L]} \to X_{[1,L+1]}$, $L \in \mathbb{Z}_+$ that satisfies condition (RI) is an RI-mapping of X.

LEMMA 6.2. *Let $\widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ be a right instantaneous subshift, and let φ be a topological conjugacy of $\widetilde{X}$ onto a subshift $X \subset \Sigma^{\mathbb{Z}}$ that is given by a one-block map $\widetilde{\Phi} : \widetilde{\Sigma} \to \Sigma$ with φ^{-1} given for some $L \in \mathbb{Z}_+$ by a block map*

$$\Phi : X_{[-L,L]} \to \widetilde{\Sigma}.$$

Then X has an RI-mapping $\Psi^{(+)} : X_{[-L,L]} \to X_{[1,L+1]}$.

Proof. Choose a mapping $\widetilde{\Psi}^{(+)}$ that selects for every $\widetilde{\sigma} \in \widetilde{\Sigma}$ an element of $\widetilde{X}_{[1,L+1]} \cap \omega^+(\widetilde{\sigma})$, and set

$$\Psi^{(+)} = \widetilde{\Phi}\widetilde{\Psi}^{(+)}\Phi.$$

We show that $\Psi^{(+)}$ is an RI-mapping of X. To see that $\Psi^{(+)}$ satisfies (RIa), let $a \in X_{[-L,L]}$ and let $x^- \in \Gamma^-(a)$. Then one has for

$$\widetilde{\sigma} = \Phi(a), \quad \widetilde{x}^- = \Phi(x^-, a),$$

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that

$$\widetilde{x}^- \in \Gamma^-(\widetilde{\sigma}),$$

and therefore

$$(\widetilde{x}^-, \widetilde{\sigma}, \widetilde{\Psi}^{(+)}(\widetilde{\sigma})) \in \widetilde{X},$$

and from

$$\widetilde{\Phi}(\widetilde{x}^-, \widetilde{\sigma}, \widetilde{\Psi}^{(+)}(\widetilde{\sigma})) = (x^-, a_{[-L,0]}, \Psi^{(+)}(a)),$$

one then has that

$$\Psi^{(+)}(a) \in \Gamma^+(x^-, a_{[-L,0]}),$$

and condition (RIa) is shown.

To show (RIb), let

$$a \in X_{[-L,L]}, \quad b \in X_{(-3L.L)} \cap \Gamma^-(a).$$

By (RIa),

$$\Phi(b, a) \in \Gamma^-(\Psi^{(+)}(a)),$$

and one has that

$$\widetilde{\Phi}(\Phi(b, a), \widetilde{\Psi}^{(+)}(\Phi(a))) = (b_{(-2L,-L)}, a_{[-L,0]}, \Psi^{(+)}(a)),$$

and therefore

$$\Phi(b_{(-2L+\ell,-L)}, a_{[-L,\ell]}) = \Phi(b_{(-2L+\ell,-L)}, a_{[-L,0]}, \Psi^{(+)}(a)_{[1,\ell]}), \quad 1 \le \ell \le L,$$

which implies (RIb). □

Let $X \subset \Sigma^{\mathbb{Z}}$ be a subshift with an RI-mapping $\Psi^{(+)} : X_{[-L,L]} \to X_{[1,L+1]}$. Define a block map

$$\Theta^{(+)}_{\Psi^{(+)}} : X_{[-L,L]} \to X_{[-L,0]} \times X_{[1,L+1]}$$

by

$$\Theta^{(+)}_{\Psi^{(+)}}(d) = (d_{[-L,0]}, \Psi^{(+)}(d)), \quad d \in X_{[-L,L]}.$$

The block map $\Theta^{(+)}_{\Psi^{(+)}}$ determines an embedding

$$\vartheta^{(+)}_{\Psi^{(+)}} : X \to (X_{[-L,0]} \times X_{[1,L+1]})^{\mathbb{Z}}$$

by

$$\vartheta^{(+)}_{\Psi^{(+)}}(x) = (x_{[i-L,i]}, \Psi^{(+)}(x_{[i-L,i+L]}))_{i\in\mathbb{Z}}, \quad x \in X.$$

LEMMA 6.3. *Let $X \subset \Sigma^{\mathbb{Z}}$ be a subshift, let $L \in \mathbb{Z}_+$, and let*

$$\Psi^{(+)} : X_{[-L,L]} \to X_{[-L.0]} \times X_{[1,L+1]},$$

be an RI-mapping of X. Then $\vartheta^{(+)}_{\Psi^{(+)}}(X)$ is right instantaneous.

Proof. Let $(b, c) \in \Theta^{(+)}_{\Psi^{(+)}}(X_{[-L,L]})$. One has by (RIb) that

$$\begin{aligned}\Gamma^-(b, c) &= \{(\Theta^{(+)}_{\Psi^{(+)}}(x^-_{[i-L,i+L]})_{-\infty<i<0} : x^- \in X_{(-\infty,L]}, (b, c) = \Theta^{(+)}_{\Psi^{(+)}}(x_{[-L,L]})\} \\ &= \{(\Theta^{(+)}_{\Psi^{(+)}}(x^-_{[i-L,i+L]})_{-\infty<i<0} : x^- \in X_{(-\infty,L]}, x_{[-L,L]} = (b, c_{[1,L]})\},\end{aligned}$$

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and by (RIb) one has, for $x^- \in X_{(-\infty,L]}$ such that $x_{[-L,L]} = (b, c_{[1,L]})$, that

$$(x^-_{(-\infty,1]}, \Psi^{(+)}(b, c_{[1,L]})) \in X_{(-\infty,L+1]}.$$

It follows, for

$$b' = (b_{(-L,0]}, c_1), \quad c' = \Psi^{(+)}(b', c_{[1,L+1]}),$$

that

$$(b', c') = \Theta^{(+)}_{\Psi^{(+)}}(b', c) \in \omega_1^+(b, c_{[1,L+1]}). \qquad \square$$

We say that a subshift $X \subset \Sigma^{\mathbb{Z}}$, for $L \in \mathbb{Z}_+$ has property RI if it has an RI-mapping. By the following theorem property RI is an invariant of topological conjugacy.

THEOREM 6.4. *A subshift has a right instantaneous presentation if and only if it has property RI.*

Proof. Apply Lemmas 6.2 and 6.3. $\square$

For a subshift $X \subset \Sigma^{\mathbb{Z}}$, for $L \in \mathbb{Z}_+$, and for mappings

$$\Psi^{(+)} : X_{[-L,L]} \to X_{[1,L+1]},$$

we have a condition (LI) that is symmetric to condition (RI):

$$(LIa)\colon \Psi^{(-)}(a) \in \Gamma^-(a_{[0,L]}, x^+), \qquad a \in X_{[-L,L]}, x^+ \in \Gamma^+(a).$$
$$(LIb)\colon \Psi^{(-)}(a_{[-\ell,L]}, b_{(L,2L-\ell)}) = \Psi^{(-)}(\Psi^{(-)}(a)_{[-\ell,-1]}, a_{[0,L]}, b_{(L,2L-\ell)}),$$
$$b \in X_{(L,2L)}, a \in X_{[-L,L]}, 1 \le \ell \le L.$$

We say that a mapping $\Psi^{(-)} : X_{[-L,L]} \to X_{[1,L+1]}$, $L \in \mathbb{Z}_+$, that satisfies condition (LI) is an LI-mapping of X. The symmetric analogues of Lemma 6.2, Lemma 6.3 and Theorem 6.4 have the symmetric proofs.

We say that a pair (Ψ_-, Ψ_+) that consists of an LI-mapping Ψ_- and a LR-mapping Ψ_+ is a pair of BI-mappings if also the following condition (BI), which comes in two parts, is satisfied:

$$(LBI)\colon \Psi^{(-)}(b_{(-2L+\ell,-L)}, a_{[-L,\ell]}) = \Psi^{(-)}(b_{(-2L+\ell,-L)}, a_{[-L,0]}, \Psi^{(+)}(a)_{[1.\ell]}),$$
$$b \in X_{(-2L,L)}, a \in X_{[-L,L]},$$
$$(RBI)\colon \Psi^{(+)}(a_{[-\ell,L]}, b_{(L,2L-\ell)}) = \Psi^{(+)}(\Psi^{(-)}(a)_{[-\ell,-1]}, a_{[0,L]}, b_{(L,2L-\ell)}),$$
$$b \in X_{(L,2L)}, a \in X_{[-L,L]}, 1 \le \ell \le L.$$

LEMMA 6.5. *Let $\widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ be a bi-instantaneous subshift, and let φ be a topological conjugacy of $\widetilde{X}$ onto a subshift $X \subset \Sigma^{\mathbb{Z}}$ that is given by a one-block map $\widetilde{\Phi} : \widetilde{\Sigma} \to \Sigma$ with φ^{-1} given for some $L \in \mathbb{Z}_+$ by a block map*

$$\Phi : X_{[-L,L]} \to \widetilde{\Sigma}.$$

Then X has a pair $(\Psi^{(-)}, \Psi^{(+)})$ of BI-mappings,

$$\Psi^{(-)} : X_{[-L,L]} \to X_{[-L-1,-1]}, \quad \Psi^{(+)} : X_{[-L,L]} \to X_{[1,L+1]}.$$

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Proof. Choose simultaneously a mapping $\widetilde{\Psi}^{(-)}$ that selects for every $\widetilde{\sigma} \in \widetilde{\Sigma}$ an element of $\widetilde{X}_{[-L-1,0)} \cap \omega^-(\widetilde{\sigma})$ and a mapping $\widetilde{\Psi}^{(+)}$ that selects for every $\widetilde{\sigma} \in \widetilde{\Sigma}$ an element of $\widetilde{X}_{[1,L+1]} \cap \omega^+(\bar{\sigma})$, and set

$$\Psi^{(-)} = \widetilde{\Phi}\widetilde{\Psi}^{(-)}\Phi, \quad \Psi^{(+)} = \widetilde{\Phi}\bar{\Psi}^{(+)}\Phi.$$

One confirms that $(\Psi^{(-)}, \Psi^{(+)})$ is a pair of BI-mappings for X. □

Let $X \subset \Sigma^{\mathbb{Z}}$ be a subshift with a pair $(\Psi^{(-)}, \Psi^{(+)})$,

$$\Psi^{(-)} : X_{[-L,L]} \to X_{[-L-1,-1]}, \quad \Psi^{(+)} : X_{[-L,L]} \to X_{[1,L+1]},$$

of BI-mappings for X. Define a block map

$$\Theta_{(\Psi^{(+)},\Psi^{(+)})} : X_{[-L,L]} \to X_{[-L-1,-1]} \times \Sigma \times X_{[1,L+1]}$$

by

$$\Theta_{(\Psi^{(+)},\Psi^{(+)})}(a) = (\Psi^{(-)}(a), a_0, \Psi^{(+)}(a)), \quad a \in X_{[-L,L]}.$$

The block map $\Theta_{(\Psi^{(+)},\Psi^{(+)})}$ determines an embedding

$$\vartheta_{(\Psi^{(+)},\Psi^{(+)})} : X \to (X_{[-L,0]} \times \Sigma \times X_{[1,L+1]})^{\mathbb{Z}}$$

by

$$\vartheta_{(\Psi^{(+)},\Psi^{(+)})}(x) = (\Psi^{(-)}(x_{[i-L,i+L]}), x_i, \Psi^{(+)}(x_{[i-L,i+L]}))_{i\in\mathbb{Z}}, \quad x \in X.$$

The proof of the following lemma is a twofold repeat of the preceding one, that uses the components of the pair of BI-mappings simultaneously.

LEMMA 6.6. *Let $X \subset \Sigma^{\mathbb{Z}}$, let $L \in \mathbb{Z}_+$, and let $(\Psi^{(-)}, \Psi^{(+)})$,*

$$\Psi^{(-)} : X_{[-L,L]} \to X_{[-L-1,-1]}, \quad \Psi^{(+)} : X_{[-L,L]} \to X_{[1,L+1]},$$

be a pair of BI-mappings of X. Then $\vartheta_{(\Psi^{(+)},\Psi^{(+)})}(X)^{\langle[1,L]\rangle}$ is bi-instantaneous.

Proof. Let $a \in X_{[-L,L]}$ and set

$$(b^-(i), d_i, b^+(i))_{-L\le i\le 0} = (\Theta(a_{[i-L.i+L]}))_{-L\le i\le 0}.$$

Set

$$\bar{a} = (a_{[-2L.0]}, \Psi^{(+)}(a_{[i-L.i+L]}).$$

By (LRb) and (LBI),

$$(b^-(i), d_i, b^+(i)) = (\Theta(\bar{a}_{[i-L.i+L]}))_{-L\le i\le 0}.$$

It follows by (LIa) that

$$(\Theta(\bar{a}_{[i-L.i+L]}))_{-L\le i\le 0} \in \omega_1^+((b^-(i), d_i, b^+(i))_{-L\le i\le 0}).$$

The proof of left instantaneity is symmetric. □

We say that a subshift $X \subset \Sigma^{\mathbb{Z}}$ has property BI if it has a pair of BI-mappings. The following theorem implies that property BI is an invariant of topological conjugacy.

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THEOREM 6.7. *A subshift has a bi-instantaneous presentation if and only if it has property BI.*

Proof. Apply Lemmas 6.5 and 6.6. □

The coded system (see [**BH**]) with code

$$\mathcal{C} = \{0\alpha^n\beta^n : n \in \mathbb{N}\}$$

is an example of a synchronizing system that has neither a left instantaneous presentation nor a right instantaneous presentation. We give an example of a semisynchronizing (see [**Kr3**]) non-synchronizing subshift that has neither a left instantaneous presentation nor a right instantaneous presentation. For this, take as the alphabet the set

$$\Sigma = \{\mathbf{1}, \alpha_\lambda, \alpha_\rho, \beta_\lambda, \beta_\rho\},$$

and view Σ as a generating set of $\mathcal{D}_2$ with relations

$$\alpha_\lambda\alpha_\rho = \beta_\lambda\beta_\rho = \mathbf{1}, \quad \alpha_\lambda\beta_\rho = \beta_\lambda\alpha_\rho = 0.$$

We let X be the subshift in $\Sigma^{\mathbb{Z}}$ that contains all $x \in M_2$ that are also label sequences of bi-infinite paths on the directed graph that has vertices v, $v(+)$, $v(-)$, four loops at v, one with labels α_λ, α_ρ, β_λ, β_ρ, a loop at $v(-)$ with label β_λ, a loop at $v(+)$ with label β_ρ, an edge from v to $v(-)$ with label α_λ, an edge from $v(-)$ to $v(+)$ with label $\mathbf{1}$, and an edge from $v(+)$ to v with label α_ρ. Adding a loop at vertex $v(+)$ that carries the label $\mathbf{1}$, one obtains a semi-synchronizing non-synchronizing right instantaneous subshift that does not have a left instantaneous presentation.

7. *Presenting a class of subshifts*

We say that a subshift $X \subset \Sigma^{\mathbb{Z}}$ is strongly bi-instantaneous if there is an $R \in \mathbb{N}$ such that, for any word $a \in \mathcal{L}(X)$ of length at least R, there is a word $c \in \Gamma^+(a) \cap \Gamma^-(a)$ such that $ca \in \omega^+(a)$ and $ac \in \omega^-(a)$.

PROPOSITION 7.1. *Let $X \subset \Gamma^{\mathbb{Z}}$ be a strongly bi-instantaneous topologically transitive subshift. Let X have property (A) with associated semigroup the $\mathcal{R}$-graph semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, and let X have property (c). Then X has an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation.*

Proof. Let $R \in \mathbb{N}$ be such that, for such that for any word $a \in \mathcal{L}(X)$ of length at least R, there is a word $c \in \Gamma^+(a) \cap \Gamma^-(a)$ such that $ca \in \omega^+(a)$ and $ac \in \omega^-(a)$. Assume that X has property (c) with respect to the parameter Q. Let $k_\circ \geq \max\{R, Q\}$ be such that every element of $\mathfrak{P}(X)$ has an $\approx$-representative in $A_{k_\circ}(X)$. By the strong bi-instantaneity of X there exist, for $a \in \mathcal{L}_k(X)$, $k \geq k_\circ$, words $c \in \Gamma^+(a) \cap \Gamma^-(a)$ such that

$$ca \in \omega^+(a), \quad ac \in \omega^-(a). \tag{7.1}$$

Then

$$\Gamma^-(a) = \Gamma^-(a(ca)^q), \quad \Gamma^+(a) = \Gamma^+((ac)^q a), \quad q > 1,$$

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and, in view of the length of a, it follows by property (c) of X that

$$\Gamma(a) = \Gamma(a(ca)^q), \quad q > 1,\ a \in \mathcal{L}_k(X),\ k \geq k_\circ. \tag{7.2}$$

This allows one to assign to a word $a \in \mathcal{L}_k(X)$, $k \geq k_\circ$, a $\approx$-class $\mathfrak{p}(a) \in \mathfrak{P}(X)$ that contains the points that carry the bi-infinite concatenation of the word ac, which by (7.1) belong to $A(X)$, since, as a consequence of (7.2), the $\approx$-class of these points does not depend on the choice of the word c.

For a word $a \in \mathcal{L}_{k_\circ+1}(X) \cup \mathcal{L}_{k_\circ+2}(X)$ we denote by $a^{(-)}$ the prefix of a that is obtained by removing the last symbol, and by $a^{(+)}$ the suffix of a that is obtained by removing the first symbol. For $a \in \mathcal{L}_{k_\circ+1}(X)$, let

$$c^{(-)} \in \Gamma^+(a^{(-)}) \cap \Gamma^-(a^{(-)}), \quad c^{(+)} \in \Gamma^+(a^{(+)}) \cap \Gamma^-(a^{(+)}),$$

such that

$$c^{(-)}a^{(-)} \in \omega^+(a^{(-)}), \quad a^{(-)}c^{(-)} \in \omega^-(a^{(-)}),$$
$$c^{(+)}a^{(+)} \in \omega^+(a^{(+)}), \quad a^{(+)}c^{(+)} \in \omega^-(a^{(+)}),$$

and denote by $y[c^{(-)}, a, c^{(+)}]$ the point $y \in Y(X)$, where $y_{[1,k_\circ+1]} = a$ and where $y_{(-\infty,0]}$ carries the left infinite concatenation of the word $a^{(-)}c^{(-)}$ and $y_{(k_\circ+1,\infty)}$ carries the right infinite concatenation of the word $c^{(+)}a^{(+)}$.

Let ζ denote an isomorphism of $\mathcal{S}(X)$ onto $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$. In view of the length of a, again appealing to property (c) of X, one has that the image of the point $y[c^{(-)}, a, c^{(+)}]$ under ζ does not depend on the choice of $c^{(-)}$ and $c^{(+)}$. As a consequence one obtains well-defined mappings

$$f^{(-)} : \mathcal{L}_{k_\circ+1} \to \mathcal{S}^-_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+), \quad f^{(+)} : \mathcal{L}_{k_\circ+1} \to \mathcal{S}^+_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+),$$

by setting

$$\zeta(y[c^{(-)}, a, c^{(+)}]) = f^{(+)}(a) f^{(-)}(a), \quad a \in \mathcal{L}_{k_\circ+1}(X).$$

$X^{\langle [1,k_\circ+2]\rangle}$ has the $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation $(\mathcal{V}, \Sigma, \lambda)$, where

$$\mathcal{V} = \mathcal{L}_{k_\circ+1}(X), \quad \Sigma = \mathcal{L}_{k_\circ+2}(X),$$

and

$$s(a) = a^{(-)},\ t(a) = a^{(+)}, \quad \lambda(a) = f^{(-)}(a^{(-)}) f^{(+)}(a^{(+)}), \quad a \in \mathcal{L}_{k_\circ+2}(X). \tag{7.3}$$

One has

$$\mathcal{V}(\mathfrak{p}) = \{a \in \mathcal{L}_{k_\circ+1}(X) : \mathfrak{p}(a) = \mathfrak{p}\}, \quad \mathfrak{p} \in \mathfrak{P}. \tag{7.4}$$

By (7.3) and (7.4) we have (G1), (G2) and (G3) satisfied, and (G4) holds by construction, The surjectivity of ζ implies (G5), and the irreducibility of $(\mathcal{V}, \Sigma, \lambda)$. follows from the topological transitivity of X. □

We introduce a condition (*SBI*) on a subshift $X \subset \Sigma^{\mathbb{Z}}$ that, for topologically transitive subshifts, also describes a strengthening of bi-instantaneity:
(*SBI*) For $a \in X_{[1,n]}$, $n \in \mathbb{N}$, there exists an $x \in A(X) \cap Y(X)$ such that $x_{[1,n]} = a$ and

$$x_{(-\infty,0)} \in \omega^-(a_1), \quad x_{(n,\infty)} \in \omega^+(a_n).$$

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LEMMA 7.2. *Let $X \subset \Sigma^{\mathbb{Z}}$, $\widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ be bi-instantaneous subshifts and let $\varphi : \widetilde{X} \to X$ be a topological conjugacy that is given by a one-block map $\widetilde{\Phi} : \widetilde{\Sigma} \to \Sigma$, with φ^{-1} given with some $L \in \mathbb{Z}_+$ by a block map $\Phi : X_{[-L,L]} \to \widetilde{\Sigma}$. Assume that X satisfies condition (SBI). Then $\widetilde{X}$ also satisfies condition (SBI).*

Proof. Let $\widetilde{a} \in \widetilde{X}_{[1,n]}$, $n \in \mathbb{N}$. By the bi-instantaneity of $\widetilde{X}$ we can choose

$$\begin{aligned} \widetilde{b}^- &\in \omega_L^-(\widetilde{a}_1), \\ \widetilde{b}^+ &\in \omega_L^+(\widetilde{a}_n). \end{aligned} \tag{7.5}$$

Set

$$b^- = \widetilde{\Phi}(\widetilde{b}^-), \quad a = \widetilde{\Phi}(\widetilde{a}), \quad b^+ = \widetilde{\Phi}(\widetilde{b}^+).$$

By the assumption that X satisfies (*SBI*), we can choose

$$\begin{aligned} x^- &\in \omega_\infty^-(b_1^-), \\ x^+ &\in \omega_\infty^+(b_L^+) \end{aligned} \tag{7.6}$$

such that

$$x^- b^- a b^+ x^+ \in A(X) \cap Y(X). \tag{7.7}$$

Let

$$\widetilde{x}^- \in \Gamma_\infty^-(\widetilde{a}), \quad \widetilde{x}^+ \in \Gamma_\infty^+(\widetilde{a})$$

be given by setting

$$\widetilde{x}^- \widetilde{a} \widetilde{x}^+ = \Phi(x^- b^- a b^+ x^+).$$

By (7.7),

$$\widetilde{x}^- \widetilde{a} \widetilde{x}^+ \in A(\widetilde{X}) \cap Y(\widetilde{X}).$$

We prove that

$$\widetilde{x}^+ \in \omega_\infty^+(\widetilde{a}_n). \tag{7.8}$$

For this, let

$$\widetilde{d} \in \Gamma^-(\widetilde{a}_n), \quad \widetilde{d}' \in \Gamma_L^-(\widetilde{d}\,\widetilde{a}_n).$$

By (7.5),

$$\widetilde{d}' \widetilde{d} \widetilde{b}^+ \in \mathcal{L}(\widetilde{X}).$$

Therefore one has, for

$$d' = \widetilde{\Phi}(\widetilde{d}'), \quad d = \widetilde{\Phi}(\widetilde{d}),$$

that

$$d' d b^+ \in \mathcal{L}(X).$$

By (7.6),

$$d' d b^+ x^+ \in \mathcal{L}(X),$$

which implies that

$$\Phi(d' d b^+ x^+) = \widetilde{d} \widetilde{x}^+ \in \mathcal{L}(\widetilde{X}),$$

and (7.8) is proved. The proof of $\widetilde{x}^- \in \omega_\infty^-(\widetilde{a}_1)$ is symmetric. □

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For a finite directed graph $\mathcal{G}(\mathfrak{P}, \mathcal{E})$ an $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation has property (A), if and only if every vertex $\mathfrak{p} \in \mathfrak{P}$ has at least two incoming edges (see [**HK2**, §2]).

LEMMA 7.3. *Let $\mathcal{G}(\mathfrak{P}, \mathcal{E})$ be a finite directed graph such that every vertex $\mathfrak{p} \in \mathfrak{P}$ has at least two incoming edges, and let $X(\mathcal{V}, \Sigma, \lambda)$ be an $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation. Let $\widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ be a bi-instantaneous subshift that is topologically conjugate to $X(\mathcal{V}, \Sigma, \lambda)$. Then $\widetilde{X}$ is strongly bi-instantaneous.*

Proof. To prove the lemma, it is enough to consider the situation that we have a topological conjugacy $\varphi : \widetilde{X} \to X(\mathcal{V}, \Sigma, \lambda)$ that is given by a one-block map $\widetilde{\Phi} : \widetilde{\Sigma} \to \Sigma$, with φ^{-1} given with some $L \in \mathbb{Z}_+$ by a block map $\Phi : X_{[-L,L]} \to \widetilde{\Sigma}$, and to show that for any word $\widetilde{a} \in \mathcal{L}(\widetilde{X})$ there is a word $\widetilde{c} \in \Gamma^+(\widetilde{a}) \cap \Gamma^-(\widetilde{a})$ such that $\widetilde{c}\widetilde{a} \in \omega^+(\widetilde{a})$ and $\widetilde{a}\widetilde{c} \in \omega^-(\widetilde{a})$.

Inspection shows that an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation $X(\mathcal{V}, \Sigma, \lambda)$ satisfies (SBI). It follows from Lemma 7.2 that the subshift $\widetilde{X}$ satisfies (SBI). Let $\widetilde{a} \in \mathcal{L}(\widetilde{X})$, and let

$$\begin{aligned} \widetilde{x}^- &\in \omega_\infty^-(\widetilde{a}), \\ \widetilde{x}^+ &\in \omega_\infty^+(\widetilde{a}), \end{aligned} \tag{7.9}$$

such that

$$\widetilde{x}^-\widetilde{a}\widetilde{x}^+ \in A(\widetilde{X}) \cap Y(\widetilde{X}). \tag{7.10}$$

Set

$$x^- = \widetilde{\Phi}(\widetilde{x}^-), \quad a = \widetilde{\Phi}(\widetilde{a}), \quad x = \widetilde{\Phi}(\widetilde{x}^-\widetilde{a}\widetilde{x}^+).$$

By (7.10),

$$x^-ax^+ \in A(X) \cap Y(X).$$

Denote the periodic point to which $x^-(x^+)$ is left (right) asymptotic by $p(-)$ $(p(+))$.

The hypothesis on the graph $\mathcal{G}(\mathfrak{P}, \mathcal{E})$ ensures that an $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation has property (A). As a consequence of (7.8) there is a $\mathfrak{p} \in \mathfrak{P}(X(\mathcal{V}, \Sigma, \lambda))$ such that

$$\{p(-), p(+)\} \subset \mathfrak{p}.$$

Let $\pi(-)$ $(\pi(+))$ denote a period of $p(-)$ $(p(+))$. Let $I(-)$, $I(+) > L$, be such that

$$x^-_{(-\infty,I_-]} = p(-)_{(-\infty,-I_-]}, \quad x^+_{(n+I_+,\infty)} = p(+)_{(n+I_+,\infty)},$$

and

$$\lambda(p_{(-I_- - \pi(-),-I_-]} = \lambda(p_{(n+I_+,n+I_++\pi(+)]}) = \mathbf{1}_{\mathfrak{p}}. \tag{7.11}$$

Set

$$b^- = x_{(-I_-,0]}, \quad b^+ = x_{(n,n+I_+]}.$$

By (7.10),

$$\lambda(b^-ab^+) = \mathbf{1}_{\mathfrak{p}}. \tag{7.12}$$

Both the source vertex of b^- and the target vertex of b^+ are, by (7.10) and (7.11), in $\mathcal{V}(\mathfrak{p})$, and we can choose a path b in $(\mathcal{V}, \Sigma, \lambda)$ such that

$$s(b) = t(b^+), \quad t(b) = s(b^\cdot), \quad \lambda(b) = \mathbf{1}_{\mathfrak{p}}. \tag{7.13}$$

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By (7.10) and (7.13),

$$x^- ab^+ bb^- bx^+ \in X.$$

Let a $\widetilde{c} \in \mathcal{L}(\widetilde{X})$ be given by

$$\widetilde{x}^- \widetilde{a}\widetilde{c}\widetilde{a}\widetilde{x}^+ = \Phi(x^- ab^+ bb^- bx^+).$$

We prove that

$$\widetilde{c}\widetilde{a} \in \omega^+(\widetilde{a}). \tag{7.14}$$

For this, let

$$\widetilde{d} \in \Gamma^-(\widetilde{a}), \quad \widetilde{d}' \in \Gamma_L^-(\widetilde{d}\,\widetilde{a}), \tag{7.15}$$

and set

$$d = \widetilde{\Phi}(\widetilde{d}), \quad d' = \widetilde{\Phi}(\widetilde{d}'),$$

By (7.9) and (7.15),

$$\lambda(d'dab^+) \neq 0,$$

and then by (7.12) and (7.13),

$$\lambda(d'dab^+bb^+ab^+) \neq 0,$$

which means that

$$d'dab^+bb^-ab^+ \in \mathcal{L}(X).$$

It follows that

$$\widetilde{d}\widetilde{a}\widetilde{c}\widetilde{a} = \Phi(d'dab^+bb^+ab^+) \in \mathcal{L}(\widetilde{X}),$$

and this proves (7.14). The proof of $\widetilde{a}\widetilde{c} \in \omega^-(\widetilde{a})$ is symmetric. □

THEOREM 7.4. *The following are equivalent for a subshift with property* (*A*) *to which there is associated a graph inverse semigroup* $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$*:*

(a) *X has an* $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$*-presentation;*

(b) *X has properties* (*c*) *and BI, and X has a strongly bi-instantaneous presentation;*

(c) *X has properties* (*c*) *and BI, X has a strongly bi-instantaneous presentation, and all bi-instantaneous presentations of X are strongly bi-instantaneous.*

Proof. One has that $\mathcal{S}(\mathfrak{P},\mathcal{E}^-,\mathcal{E}^+)$-presentations have property (*B*). Apply Proposition 7.1 and Lemma 7.3. □

COROLLARY 7.5. *The following are equivalent for a subshift with property* (*A*) *to which there is associated a graph inverse semigroup* $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$*:*

(a) *X has an* $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$*-presentation;*

(b) *X has property* (*c*)*, and X has a pair* $(\Psi^{(-)}, \Psi^{(+)})$*,*

$$\Psi^{(-)} : X_{[-L,L]} \to X_{[-L-1,-1]}, \quad \Psi^{(+)} : X_{[-L,L]} \to X_{[1,L+1]},$$

of BI-mappings such that $\vartheta_{(\Psi^{(-)},\Psi^{(+)})}(X)$ *is strongly bi-instantaneous;*

(c) *X has property* (*c*)*, X has a pair of BI-mappings, and for all pairs* $(\Psi^{(-)}, \Psi^{(+)})$*,*

$$\Psi^{(-)} : X_{[-L,L]} \to X_{[-L-1,-1]}, \quad \Psi^{(+)} : X_{[-L,L]} \to X_{[1,L+1]},$$

of BI-mappings of X one has that $\vartheta_{(\Psi^{(-)},\Psi^{(+)})}(X)$ *is strongly bi-instantaneous.*

Proof. Apply Lemma 6.6 and Theorem 7.4. □

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8. *A class of examples*

Let there be given a finite directed graph $\mathcal{G}(\mathfrak{P}, \mathcal{E})$ with its Markov–Dyck shift $MD(\mathcal{G}(\mathfrak{P}, \mathcal{E}))$. With the identity labeling on $\mathcal{G}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ as λ, we denote, for $\mathfrak{p} \in \mathfrak{P}$, by $\mathcal{C}_\mathfrak{p}$ the code that contains the paths $c \in \mathcal{L}(MD(\mathcal{G}(\mathfrak{P}, \mathcal{E})))$ such that $s(c) = \mathfrak{p} = t(c)$, $\lambda(c) = \mathbf{1}_\mathfrak{p}$, and such that, for every proper prefix a of c, $\lambda(a) \neq \mathbf{1}_\mathfrak{p}$.

We assume that for $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$, one has that $\mathrm{card}(\mathcal{E}(\mathfrak{q}, \mathfrak{r})) \neq 1$. For $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$ such that $\mathcal{E}(\mathfrak{q}, \mathfrak{r}) \neq \emptyset$ we have the code

$$\mathcal{C}_{\mathfrak{q},\mathfrak{r}} = \bigcup_{e \in \mathcal{E}(\mathfrak{q},\mathfrak{r})} e^- \mathcal{C}_\mathfrak{r}^\star e^+,$$

and we denote by $\mathcal{C}^\circ_{\mathfrak{q},\mathfrak{r}}$ the code that is obtained by removing from $\mathcal{C}_{\mathfrak{q},\mathfrak{r}}$ all words that have a subword in

$$\bigcup_{\{\mathfrak{q}',\mathfrak{r}' \in \mathfrak{P} : \mathcal{E}(\mathfrak{q}',\mathfrak{r}') \neq \emptyset\}} \{e^-(\mathcal{C}_{\mathfrak{r}'} \cup \mathcal{C}_{\mathfrak{r}'}^2)e^+ : e \in \mathcal{E}(\mathfrak{q}', \mathfrak{r}')\}.$$

We let $\kappa^-_{\mathfrak{q},\mathfrak{r},1}$ denote the identity map on $\mathcal{E}(\mathfrak{q}, \mathfrak{r})$ and we let $\kappa^-_{\mathfrak{q},\mathfrak{r},-1}$ denote a permutation of $\mathcal{E}(\mathfrak{q}, \mathfrak{r})$ without fixed points. Also we denote by $\mathcal{D}^\circ_{\mathfrak{q}}$, $\mathfrak{r}$ the code that contains the words

$$(c^\circ, d) \in \mathcal{L}(MD(\mathcal{G}(\mathfrak{P}, \mathcal{E})) \times \{1, -1\}$$

with $c^\circ \in \mathcal{C}^\circ_{\mathfrak{q},\mathfrak{r}}$. Given a word $(c^\circ, d) \in \mathcal{D}^\circ_{\mathfrak{q},\mathfrak{r}}$ of length $2I$, $I \in \mathbb{N}$, let $K \in \mathbb{N}$, and c°_-, $c^\circ_+ \in \mathcal{C}_\mathfrak{r}$, and $c^\circ(k) \in \mathcal{C}_\mathfrak{r}$, $1 \leq k \leq K$, such that

$$c^\circ = c^\circ_- \Bigg(\prod_{1 \leq k \leq K} c^\circ(k) \Bigg) c^\circ_+,$$

and denote by $M_-(c^\circ, d)(M_+(c^\circ, d))$ the length of $c^\circ_-(c^\circ_+)$, and write

$$d = (d_i)_{1 \leq i \leq 2I},$$

and set

$$\delta^-(c^\circ, d) = d_{M_-(c^\circ,d)}, \quad \delta^+(c^\circ, d) = d_{M_+(c^\circ,d)}.$$

Moreover, we set

$$\mathcal{F}_{\mathfrak{q},\mathfrak{r}} = \{(c^\circ, d) \in \mathcal{D}^\circ_{\mathfrak{q},\mathfrak{r}} : \delta^-(c^\circ, d)\delta^+(c^\circ, d) = -1\},$$

and we denote by $\mathcal{D}^\circ_{\mathfrak{q},\mathfrak{r}}$ the code that is obtained by removing from $\mathcal{D}^\circ_{\mathfrak{q},\mathfrak{r}}$ all words that have a subword in

$$\bigcup_{\{\mathfrak{q}',\mathfrak{r}' \in \mathfrak{P} : \mathcal{E}(\mathfrak{q}',\mathfrak{r}') \neq \emptyset\}} \mathcal{F}_{\mathfrak{q}',\mathfrak{r}'}.$$

Finally, we set

$$\mathcal{D}_\mathfrak{p} = \bigcup_{\{\mathfrak{r} \in \mathfrak{P}\} : \mathcal{E}(\mathfrak{p},\mathfrak{r}) \neq \emptyset\}} \mathcal{D}^\circ_{\mathfrak{q},\mathfrak{r}}, \quad \mathfrak{p} \in \mathfrak{P},$$

and we denote by Y the coded system of any of the codes $\mathcal{D}_\mathfrak{p}$, $\mathfrak{p} \in \mathfrak{P}$, which all yield the same coded system.

THEOREM 8.1. *The subshift Y is bi-instantaneous and has property (A) with associated semigroup $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$. Y does not have an $\mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation.*

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Proof. The subshift Y inherits bi-instantaneity from $MD(\mathcal{G}(\mathfrak{P}, \mathcal{E})) \times \{1, -1\}^{\mathbb{Z}}$.

For all $\mathfrak{p} \in \mathfrak{P}$, all words in $\mathcal{D}_{\mathfrak{p}}$ have the same context. Also, one observes that $P(A(Y))$ contains precisely the points that carry a bi-infinite concatenation of a code word in $\bigcup_{\mathfrak{p}\in\mathfrak{P}} \mathcal{D}_{\mathfrak{p}}$. Property (A) of Y follows.

For the proof of $\mathcal{S}(Y) = \mathcal{S}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ one constructs for $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$, $\mathcal{E}(\mathfrak{q}, \mathfrak{r}) \neq \emptyset$, a representative of e^-, $e \in \mathcal{E}(\mathfrak{q}, \mathfrak{r})$, from a word $(b, d) \in \mathcal{D}_{\mathfrak{q},\mathfrak{r}}$ such that the initial symbol of b is equal to e^- and $\delta(\mathfrak{q}, \mathfrak{r}) = 1$, or from a word $(b, d) \in \mathcal{D}_{\mathfrak{q},\mathfrak{r}}$ such that the initial symbol of b is equal to $(\kappa_{\mathfrak{q},\mathfrak{r}}^{-1}(e))^-$ and $\delta(\mathfrak{q}, \mathfrak{r}) = -1$, by appending the right infinite concatenation of a word in $\mathcal{D}_{\mathfrak{r}}$ on the right of the prefix of length $M_-(b, d)$ of (b, d), and a left infinite concatenation of a word in $\mathcal{D}_{\mathfrak{q}}$ on the left of the prefix, and one has the symmetric construction for a representative of e^+, $e \in \mathcal{E}(\mathfrak{q}, \mathfrak{r})$.

We show that Y does not have property (c). For this, let $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$, $\mathcal{E}(\mathfrak{q}, \mathfrak{r}) \neq \emptyset$, let $M \in \mathbb{N}$, and let $(b^\circ, d^\circ) \in \mathcal{D}_{\mathfrak{q},\mathfrak{r}}$ be a word such that $M_-(b^\circ, d^\circ), M_+(b^\circ, d^\circ) > M$. Let $2J$ be the length of (b°, d°). Let e^- be the initial symbol of b°, and let e^- be its final symbol. Denote by (b_-, d_-) $((b_+, d_+))$ the word that is obtained from the word (b°, d°) by removing the first (last) pair of symbols. Set

$$(b, d) = (b_- b^\circ b_+, b_- b^\circ b_+),$$

and write

$$d = (d_i)_{1\le i\le 6J-2}.$$

Denote, for $\delta_-, \delta_+ \in \{1, -1\}$, by $d(\delta_-, \delta_+)$ the word that is obtained from the word d by simultaneously replacing the symbol $d_{M_-(b^\circ)-1}$, if this symbol is not δ_-, by δ_-, and the symbol $d_{6J-2-M_+(b^\circ)+1}$, if this symbol is not δ_+, by δ_+. One finds that the words $(b, d(\delta_-, \delta_+))$, $\delta_-, \delta_+ \in \{1, -1\}$, are admissible for Y and that their left context, as well as their right context, is the same. However, the words $(b, d(\delta_-, \delta_+))$, $\delta_-, \delta_+ \in \{1, -1\}$, have distinct contexts. Set

$$e^-(\delta_-) = \begin{cases} e^- & \text{if } \delta_- = 1, \\ (\kappa_{\mathfrak{q},\mathfrak{r}}^{-1}(e))^- & \text{if } \delta_- = -1, \end{cases}$$

$$e^-(\delta_+) = \begin{cases} e^- & \text{if } \delta_+ = 1, \\ (\kappa_{\mathfrak{q},\mathfrak{r}}^{-1}(e))^+ & \text{if } \delta_+ = -1. \end{cases}$$

One has

$$(e^-(\delta'_-), e^+(\delta'_+)) \in \Gamma(b, d(\delta_-, \delta_+))$$

if and only if $\delta'_- = \delta_-$, $\delta'_+ = \delta_+$. Apply Theorem 4.1 and Proposition 5.1 to conclude the proof. □

One notes that Y actually has property (B).

9. *Markov codes and zeta functions*

Denoting by $\Pi_n(X)$ the number of points of period n of a shift invariant set $X \subset \Sigma^{\mathbb{Z}}$, the zeta function of X is given by

$$\zeta_X(z) = e^{\sum_{n\in\mathbb{N}} (\Pi_n(X) z^n/n)}.$$

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We also recall from [**Ke**] the notion of a circular Markov code to the extent that is needed here. We let a Markov code be given by a code $\mathcal{C}$ of words in the symbols of a finite alphabet Σ together with a finite set $\mathcal{V}$ and mappings $t : \mathcal{C} \to \mathcal{V}$, $s : \mathcal{C} \to \mathcal{V}$. To a Markov code $(\mathcal{C}, t, s)$ there is associated the shift invariant set $X_{(\mathcal{C},t,s)} \subset \Sigma^{\mathbb{Z}}$ of points $x \in \Sigma^{\mathbb{Z}}$ such that there are indices I_k, $k \in \mathbb{Z}$,

$$I_0 \le 0 < I_1, \quad I_k < I_{k+1}, \ k \in \mathbb{Z},$$

such that

$$x_{[I_k, I_{k+1})} \in \mathcal{C}, \quad k \in \mathbb{Z}, \tag{9.1}$$

and

$$r(x_{[I_{k-1}, I_k)}) = s(x_{[I_k, I_{k+1})}), \quad k \in \mathbb{Z}. \tag{9.2}$$

$(\mathcal{C}, t, s)$ is said to be a circular Markov code if, for every periodic point x in $X_{(\mathcal{C},r,s)}$, the indices I_k, $k \in \mathbb{Z}$, such that (9.1) and (9.2) hold, are uniquely determined by x. Given a circular Markov code $(\mathcal{C}, s, r)$, denote by $\mathcal{C}(u, w)$ the set of words $c \in \mathcal{C}$ such that $s(c) = u$, $t(c) = w$, $u, w \in \mathcal{V}$. We set

$$g_{\mathcal{C}(u,v)} = \sum_{0 \le n < \infty} \operatorname{card}\{c \in \mathcal{C} : s(c) = u, \ t(c) = v, \ \ell(c) = n\},$$

and we introduce the matrix

$$H^{(\mathcal{C})}(z) = (g_{\mathcal{C}(u,v)}(z))_{u,v \in \mathcal{V}}.$$

For a circular Markov code $(\mathcal{C}, t, s)$, one has [**Ke**]

$$\zeta_{X_{(\mathcal{C},t,s)}}(z) = \det(I - H^{(\mathcal{C})}(z))^{-1}. \tag{9.3}$$

Given an $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$. we associate to an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation $X(\mathcal{V}, \Sigma, \lambda)$ the state spaces

$$\Sigma^- = \{\sigma \in \Sigma : \lambda(\sigma) \in \mathcal{S}^- \cup \{1_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P}\}\}$$

and

$$\Sigma^+ = \{\sigma \in \Sigma : \lambda(\sigma) \in \{\mathbf{1}_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P}\} \cup \mathcal{S}^+\},$$

together with the 0–1 transition matrices $(A^-(\rho, \tau))_{\rho,\tau \in \Sigma^-}$ and $(A^+(\rho, \tau))_{\rho,\tau \in \Sigma^+}$, where for ρ, τ in $\Sigma^-(\Sigma^+)$ we set $A^-(\rho, \tau)(A^+(\rho, \tau))$ equal to 1 if and only if $r(\rho) = s(\tau)$. We denote the (possibly empty) topological Markov shift with state space $\Sigma^-(\Sigma^+)$ and transition matrix $A^-(A^+)$ by $X(\Sigma^-, A^-)(X(\Sigma^+, A^+))$. Also we associate to the finite directed labeled graph $(\mathcal{V}, \Sigma, \lambda)$ the circular Markov code $(\mathcal{C}^0(\mathcal{V}, \Sigma, \lambda), r, s)$, where $\mathcal{C}^0(\mathcal{V}, \Sigma, \lambda)$ is the set of words

$$(\sigma_i)_{1 \le i \le I} \in \mathcal{L}(X(\mathcal{V}, \Sigma, \lambda)), \quad I > 1,$$

such that

$$\lambda((\sigma_i)_{1 \le i \le I}) \in \{\mathbf{1}_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P}\},$$
$$\lambda((\sigma_i)_{1 \le j \le J}) \notin \{\mathbf{1}_{\mathfrak{p}} : \mathfrak{p} \in \mathfrak{P}\}, \quad 1 < J < I,$$

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we let

$$\mathcal{C}_\circ^-(\mathcal{V}, \Sigma, \lambda) \quad (\mathcal{C}_\circ^+(\mathcal{V}, \Sigma, \lambda))$$

be the set of words

$$(\sigma_i)_{1\le i\le I} \in \mathcal{L}(X(\mathcal{V}, \Sigma, \lambda)), \quad I > 1,$$

such that

$$\lambda((\sigma_i)_{1\le i\le I}) \in \mathcal{S}_{\mathcal{R}}^-(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+) \cup \{\mathbf{1}_\mathfrak{p} : \mathfrak{p} \in \mathfrak{P}\},$$
$$\lambda((\sigma_i)_{J\le i\le I}) \in \mathcal{S}_{\mathcal{R}}^+(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+), \quad 1 < J \le I,$$
$$\lambda((\sigma_i)_{1\le i\le I}) \in \{\mathbf{1}_\mathfrak{p} : \mathfrak{p} \in \mathfrak{P}\} \cup \mathcal{S}_{\mathcal{R}}^+(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+),$$
$$\lambda((\sigma_i)_{1\le i\le J}) \in \mathcal{S}_{\mathcal{R}}^+(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+), \quad 1 \le J < I,$$

and we associate to the finite directed labeled graph $(\mathcal{V}, \Sigma, \lambda)$ the circular Markov codes $(\mathcal{C}^-(\mathcal{V}, \Sigma, \lambda), t, s)$ and $(\mathcal{C}^+(\mathcal{V}, \Sigma, \lambda), t, s)$, where $\mathcal{C}^-(\mathcal{V}, \Sigma, \lambda)(\mathcal{C}^+(\mathcal{V}, \Sigma, \lambda))$ is the set of words that contains the words that are in $\mathcal{C}_\circ^-(\mathcal{V}, \Sigma, \lambda)(\mathcal{C}_\circ^+(\mathcal{V}, \Sigma, \lambda))$ or that are concatenations of a word in $\mathcal{C}_\circ^-(\mathcal{V}, \Sigma, \lambda)(\mathcal{C}_\circ^+(\mathcal{V}, \Sigma, \lambda))$ and a word in $\mathcal{L}(\Sigma^-, A^-)(\mathcal{L}(\Sigma^+, A^+))$.

THEOREM 9.1.

$$\zeta_{X(\mathcal{V},\Sigma,\lambda)}(z) = \frac{\det(\mathbf{1} - H^{\mathcal{C}^0(\mathcal{V},\Sigma,\lambda))}(z))}{\det(\mathbf{1} - A^- z)\det(\mathbf{1} - H^{\mathcal{C}^-(\mathcal{V},\Sigma,\lambda))}(z))\det(\mathbf{1} - H^{\mathcal{C}^+(\mathcal{V},\Sigma,\lambda))}(z)))\det(\mathbf{1} - A^+ z)}.$$

Proof. Apply the formula for the zeta function of a topological Markov shift [**LM**], and note that $\det(\mathbf{1} - A^- z) = 1(\det(\mathbf{1} - A^+ z) = 1)$ if $X(\Sigma^-, A^-)(X(\Sigma^+, A^+))$ is empty. Apply formula (9.3) and collect the contributions to the zeta function of $X(\mathcal{V}, \Sigma, \lambda)$. □

Following [**Ke**], special cases of Theorem 7.1 appeared in [**I**, **KM2**, **IK1**].

We denote, for an $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentation $X(\mathcal{V}, \Sigma, \lambda)$ and for $\mathfrak{p} \in \mathfrak{P}$, by $P_\mathfrak{p}(X(\mathcal{V}, \Sigma, \lambda))$ the set of periodic points of $X(\mathcal{V}, \Sigma, \lambda)$ that carry for some $V \in \mathcal{V}$ a bi-infinite concatenation of a path b from V to V such that $\lambda(b) = \mathbf{1}_\mathfrak{p}$.

PROPOSITION 9.2. *Let $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ be an $\mathcal{R}$-graph in which every vertex has at least two incoming edges. Let $X(\mathcal{V}, \Sigma, \lambda)$ and $X(\tilde{\mathcal{V}}, \tilde{\Sigma}, \tilde{\lambda})$ be topologically conjugate $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$-presentations. Then*

$$\prod_{\mathfrak{p}\in\mathfrak{P}} (\xi - \zeta_{P_\mathfrak{p}(X(\mathcal{V},\Sigma,\lambda))}(z)) = \prod_{\mathfrak{p}\in\mathfrak{P}} (\xi - \zeta_{P_\mathfrak{p}(X(\tilde{\mathcal{V}},\tilde{\Sigma},\tilde{\lambda}))}(z)).$$

Proof. The hypothesis on the $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ implies that

$$P(A(X(\mathcal{V}, \Sigma, \lambda))) = \bigcup_{\mathfrak{p}\in\mathfrak{P}} P_\mathfrak{p}(X(\mathcal{V}, \Sigma, \lambda))$$

and, moreover, that this is in fact the partition of $P(A(X(\mathcal{V}, \Sigma, \lambda)))$ into its $\approx$-equivalence classes, which is invariant under topological conjugacy. □

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In the case of graph inverse semigroups of finite directed graphs, in which every vertex has at least two incoming edges, the set $P(A(X(\mathcal{V}, \Sigma, \lambda)))$ appeared in [**HI**, **HIK**] as the set of neutral periodic points. For the case of a Markov–Dyck shift X the coefficient sequence of the Taylor expansion of the coefficient of $\xi^{\mathrm{card}(\mathfrak{P})-1}$ in $\prod_{\mathfrak{p}\in\mathfrak{P}}(\xi - \zeta_{P_{\mathfrak{p}}(X)}(z))$ was introduced in [**M4**] as a generalization of the Catalan numbers. (With the Catalan numbers $C_k = (1/(n+1))\binom{2n}{n}$ this sequence is in the case of the Dyck shift D_N, $N > 1$, equal to $N^k C_k$, $k \in \mathbb{Z}_+$.)

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