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# Tail pressure and the tail entropy function

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*Abstract*. Burguet [A direct proof of the tail variational principle and its extension to maps. *Ergod. Th. & Dynam. Sys.* **29** (2009), 357–369] presented a direct proof of the variational principle of tail entropy and extended Downarowicz's results to a non-invertible case. This paper defines and discusses tail pressure, which is an extension of tail entropy for continuous transformations. This study reveals analogs of many known results of topological pressure. Specifically, a variational principle is provided and some applications of tail pressure, such as the investigation of invariant measures and equilibrium states, are also obtained.

## 1. *Introduction*

In academic studies, a dynamical system is a concept in which a fixed rule describes the time dependence of a point in a special space. Entropy is an elementary, but important, factor since this invariant quantifies the complexity of the orbit structure of a dynamical system. This study discusses this invariant from topological and measure-theoretical viewpoints. These two notions are related by a variational principle, which states that the supremum of the metric entropy over all of the invariant probability measures is equal to the topological entropy; this principle has gained a lot of attention. A good discussion of references for these entropy invariants appears in [**11, 20**], which contains many of the early references.

In recent years, researchers have characterized the local structure of maps by defining entropy pairs, entropy tuples, entropy sets, or entropy points in both topological and measure-theoretical situations. Several studies examine the connection between measure-theoretical entropy notions and topological entropy ones. These studies investigate the local behavior of these entropies, and, in many cases, establish new variational principles. See [**6–8, 19**] for related propositions and results.

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Furthermore, entropy can appear locally at arbitrarily small scales. Previous researchers [**1, 2, 5**] have investigated tail entropy $h^*$, which can capture the entropy near any single orbit. From the measure-theoretical point of view, the local complexity of any orbit can be estimated by the defect of uniformity in the convergence of the measure-theoretical entropy function computed at increasingly fine scales. Downarowicz [**5**] showed the relation, the so-called tail variational principle, for homeomorphisms. Burguet [**2**] provided an elementary proof of this variational principle and extended the related results to a non-invertible case.

As a natural generalization of topological entropy, topological pressure is a quantity which belongs to one of the concepts in thermodynamic formalism. Thermodynamic formalism itself is a generalization of the concepts of statistical physics to the area of mathematical dynamical systems theory. Theories of topological pressure, variational principles, equilibrium states and related topics play a fundamental role in statistical mechanics, ergodic theory, and dynamical systems. For references, see [**9, 10, 16–18**].

In the present paper, we pursue the connection between tail pressure and the tail entropy function. For a dynamical system admitting a refining sequence of essential partitions, the proof proposed here is inspired by Misiurewicz's work on the variational principle for topological pressure (see [**13**]). The key point of the proof is that the tail pressure can be written in terms of these partitions. A refining sequence of essential partitions does not always exist, but Lindenstrauss [**12**] proved that the product of a homeomorphism of finite entropy with an irrational rotation exhibits such sequences.

This paper first describes the notion of entropy structures, which was introduced by Downarowicz [**5**], and also states the tail variational principle by Burguet [**2**]. Section 3 shows that the tail pressure is invariant under a natural extension and product with an irrational rotation of the circle. This section also examines the power rule and product rule for tail pressure. In particular, §4 gives a proof of the tail pressure variational principle by natural extension and shows that the tail pressure determines invariant measures. The last section gives some applications of tail pressure to equilibrium states.

## 2. *Preliminary*

Misiurewicz [**13**] was the first to introduce the quantity of tail entropy using open covers. For convenience, this paper uses the notations adopted by Burguet [**2**]. Let $T : X \to X$ be a continuous self-map of a compact metric space $(X, d)$ with finite topological entropy. Recall that the set $M(X)$ of all Borel probability measures is compact for the weak*-topology, and $M(X, T) \subseteq M(X)$ is the subset of all $T$-invariant measures. The notion of tail entropy according to Bowen's balls is as follows.

Let $\epsilon > 0$, $n \in \mathbb{N}\backslash\{0\}$; the notation $B(x, n, \epsilon)$ for the Bowen's ball of order $n$, of radius $\epsilon$ and of center $x$ is defined by

$$B(x, n, \epsilon) := \{y \in X \mid \text{for all } 0 \leq k < n,\ d(T^k(x), T^k(y)) < \epsilon\}.$$

A set $E \subset X$ is said to be a $(n, \delta)$-spanning subset of $K$ if $K \subset \bigcup_{x \in E} B(x, n, \delta)$ and a $(n, \delta)$-separated subset of $K$ if, for all $x, y \in E$, there exists an integer $0 \leq k < n$ such that $d(T^k(x), T^k(y)) \geq \delta$. We use the notation $r(n, \delta, K)$ (respectively, $s(n, \delta, K)$) for the minimal cardinality of a $(n, \delta)$-spanning subset of a set $K \subset X$ (the maximal cardinality

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of a $(n, \delta)$-separated subset of $K$, respectively). The Bowen metric is usually denoted by $d_n(x, y) := \max_{0\le i\le n-1} d(T^i x, T^i x)$ for all $x, y \in X$. The tail entropy of $(X, T)$ is defined as follows:

$$h^*(T) := \lim_{\epsilon\to 0} \lim_{\delta\to 0} \limsup_n \frac{1}{n} \sup_{x\in X} \log r(n, \delta, B(x, n, \epsilon)).$$

The tail entropy does not depend on the metric $d$, but only on the topology of $X$ and, moreover, the same as in topological entropy, $h^*$ is a topological conjugacy invariant.

The following discussion defines the notion of entropy structures, which was introduced by Downarowicz in [**5**].

Let $S$ be a continuous self-map of a compact metric space $Y$ and let $\pi : Y \to X$ be a continuous map that is not necessarily surjective. However, we assume that the induced map $\pi^* : M(Y, S) \to M(X, T)$ is surjective. This study refers to $\pi$ and also to $(Y, S)$ as an extension of $(X, T)$ if $T \circ \pi = \pi \circ S$. This extension is zero-dimensional if the topological dimension of $Y$ is zero or, equivalently, if $Y$ has a base consisting of clopen sets. An extension $\pi : (Y, S) \to (X, T)$ is said to be principal if, for any $\nu \in M(Y, S)$, we have $h(\pi^*\nu) = h(\nu)$, where $h(\nu)$ denotes the usual measure-theoretical entropy.

Recall that the natural extension of $(X, T)$ is defined as follows. Denote

$$Y := \{(x_n)_{n\in\mathbb{Z}} \in X^{\mathbb{Z}} \mid Tx_n = x_{n+1}\}$$

and let $\pi : Y \to X$ be the projection defined by $\pi((x_n)_{n\in\mathbb{Z}}) = x_0$. Endow $Y$ with the metric

$$\overline{d}((x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}}) := \sum_{n\in\mathbb{Z}} (d(x_n, y_n)/2^{|n|}).$$

The natural extension $\overline{T} : Y \to Y$ of $(X, T)$ is the shift map $\overline{T}((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$. Thus, the natural extension is an extension as defined above, and this extension is also principal.

The term of a partition in this study means a finite partition of $X$ by Borelian sets. We say that a sequence of partitions $\mathcal{P} := (P_k)_{k\in\mathbb{N}}$ is refining if $P_{k+1}$ is finer than $P_k$ for all $k$ and the diameter of $P_k$ goes to zero when $k$ goes to infinity. A partition is said to be essential if the boundaries of its elements have zero $\mu$-measure for all invariant probability measures $\mu$. If $P$ is a partition and $\mu$ is an invariant probability measure, then $h(\mu, P)$ denotes the usual entropy of $\mu$ with respect to $P$.

If $P$ is a finite partition of $X$, denote $P^n := \bigvee_{i=0}^{n-1} T^{-i}P$, then

$$H(\mu, P) = -\sum_{A\in P} \mu(A) \log \mu(A).$$

If $P_1$ and $P_2$ are two partitions of $X$, and $P_1$ is finer than $P_2$, write

$$H(\mu, P_1|P_2) = \sum_{B\in P_2} \mu(B) H(\mu_B, P_1) = H(\mu, P_1) - H(\mu, P_2),$$

where $\mu_B$ denotes the probability measure induced by $\mu$ on $B$. Since $H(\mu, P_1^n|P_2^n)$ is a subadditive sequence, divided $n$ converges to its infimum, then $h(\mu, P_1|P_2) = h(\mu, P_1) - h(\mu, P_2)$, which is the conditional entropy.

A non-decreasing sequence of real functions defined on $M(Y, S)$, $\mathcal{H} := (h_k)_{k\in\mathbb{N}}$, converging to a bounded limit $h_\infty := \lim_k h_k$, is called a candidate of $(Y, S)$.

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Let $\pi : (Y', S') \to (Y, S)$ be an extension and $\mathcal{H} := (h_k)_{k\in\mathbb{N}}$ be a candidate of $(Y, S)$. The lift $\pi^*\mathcal{H}$ of $\mathcal{H}$ to $M(Y', S')$ is the candidate of $(Y', S')$ defined by $\pi^*\mathcal{H} := (h_k(\pi^*\cdot))_{k\in\mathbb{N}}$.

Two candidates of $(Y, S)$ are said to be uniformly equivalent if, for all $\gamma > 0$ and for all $k \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that $h_l > g_k - \gamma$ and $g_l > h_k - \gamma$.

Consider a compact metric space $X$ and a continuous map $T : X \to X$ with finite topological entropy.

Let $(X', T')$ be a zero-dimensional principal extension of $(X, T)$ and $(P_k)_{k\in\mathbb{N}}$ be a refining sequence of clopen partitions of $X'$.

An entropy structure $\mathcal{H} := (h_k)_{k\in\mathbb{N}}$ is a candidate of $(X, T)$ such that the lift $\pi^*\mathcal{H}$ of $\mathcal{H}$ to $M(X', T')$ is uniformly equivalent to the candidate $(h(\cdot, P_k))_{k\in\mathbb{N}}$.

Let $\mathcal{H} = (h_k)_{k\in\mathbb{N}}$ be a candidate. For a function $f : M(X, T) \to \mathbb{R}^+$, denote the smallest upper semi-continuous function greater than $f$ by $\widetilde{f}$, that is, $\widetilde{f}(\mu) := \limsup_{\nu\to\mu} f(\nu)$. Let us define $u^{\mathcal{H}} := \lim_{k\to+\infty} \widetilde{h_\infty - h_k}$.

If $\mathcal{P} := (P_k)_{k\in\mathbb{N}}$ is a refining sequence of partitions and $\mathcal{H}_\mathcal{P} := (h(\cdot, P_k))_{k\in\mathbb{N}}$ is the sequence of entropy functions with respect to the partitions $(P_k)_{k\in\mathbb{N}}$, we use the notation $u^{\mathcal{P}}$ instead of $u^{\mathcal{H}_\mathcal{P}}$.

*Remarks.*

(1) Two entropy structures are uniformly equivalent, and any candidate which is uniformly equivalent to an entropy structure is also an entropy structure. In other words, the set of entropy structures is an equivalence class for the uniform equivalence relation on the set of candidates.

(2) All entropy structures $\mathcal{H}$ converge to the metric entropy function and have the same $u^{\mathcal{H}}$: we simply denote this function by $u$.

(3) (Burguet [**2**]) If $\mathcal{H} = (h_k)_{k\in\mathbb{N}}$ is a candidate of $(X, T)$, then

$$\sup_{\mu\in M(X,T)} u^{\mathcal{H}}(\mu) = \lim_{k\to+\infty} \lim_{l\to+\infty} \sup_{\mu\in M(X,T)} (h_l - h_k)(\mu).$$

LEMMA 2.1. [**12**] *If $T$ is a homeomorphism of finite topological entropy admitting an infinite minimal factor, then there exists a refining sequence of essential partitions.*

Burguet formulated the variational principle for the tail entropy independently of the choice of an entropy structure.

THEOREM 2.1. (Tail variational principle [**2**]) *If $X$ is a compact metric space and $T : X \to X$ is a continuous function with finite topological entropy, then*

$$\sup_{\mu\in M(X,T)} u(\mu) = h^*(T).$$

*Moreover, the supremum is achieved on the closure of the ergodic measures.*

## 3. *Tail pressure and related properties*

It is possible to extend the concept of tail entropy to tail pressure associated with a potential function. Furthermore, a variational principle is also true for this tail pressure. This section first gives equivalent definitions of this new pressure, then some basic propositions are also shown.

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3.1. *Tail pressure.* Let $(X, d)$ be a compact metric space, $C(X, \mathbb{R})$ be the space of real-valued continuous functions of $X$ and $T : X \to X$ be a continuous transformation. For $f \in C(X, \mathbb{R})$ and $n \geq 1$, denote $\sum_{i=0}^{n-1} f(T^i x)$ by $(S_n f)(x)$.

Let $f \in C(X, \mathbb{R})$, $n \geq 1$, $\epsilon > 0$ and $\delta > 0$. For all $x \in X$, put

$$Q_n(T, f, x, \delta, \epsilon) = \inf\left\{ \sum_{y \in F} e^{(S_n f)(y)} \;\middle|\; F \text{ is a } (n, \delta)\text{-spanning set for } B(x, n, \epsilon) \right\}.$$

LEMMA 3.1. *As above, the following statements are true.*

(1) $Q_n(T, f, x, \delta, \epsilon) > 0$.
(2) *If* $\delta_1 < \delta_2$, *then* $Q_n(T, f, x, \delta_1, \epsilon) \geq Q_n(T, f, x, \delta_2, \epsilon)$.
(3) *If* $\epsilon_1 < \epsilon_2$, *then* $Q_n(T, f, x, \delta, \epsilon_1) \leq Q_n(T, f, x, \delta, \epsilon_2)$.
(4) $Q_n(T, 0, x, \delta, \epsilon) = r(n, \delta, B(x, n, \epsilon))$.

For $f \in C(X, \mathbb{R})$, $\epsilon > 0$ and $\delta > 0$, define

$$Q(T, f, \delta, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \sup_{x \in X} \log Q_n(T, f, x, \delta, \epsilon).$$

*Definition 3.1.* Let $T : X \to X$ be continuous and $f \in C(X, \mathbb{R})$; define the tail pressure as follows:

$$P^*(T, f) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} Q(T, f, \delta, \epsilon).$$

It is trivial that $P^*(T, 0) = h^*(T)$. Next, consider an equivalent but dual definition.

Let $f \in C(X, \mathbb{R})$, $n \geq 1$, $\epsilon > 0$ and $\delta > 0$. For $x \in X$, put

$$P_n(T, f, x, \delta, \epsilon) = \sup\left\{ \sum_{y \in E} e^{(S_n f)(y)} \;\middle|\; E \text{ is a } (n, \delta)\text{-separated subset of } B(x, n, \epsilon) \right\}.$$

LEMMA 3.2. *As above, the following statements hold.*

(5) $P_n(T, f, x, \delta, \epsilon) > 0$.
(6) $Q_n(T, f, x, \delta, \epsilon) \leq P_n(T, f, x, \delta, \epsilon)$.
(7) *If* $\delta_1 < \delta_2$, *then* $P_n(T, f, x, \delta_1, \epsilon) \geq P_n(T, f, x, \delta_2, \epsilon)$.
(8) *If* $\epsilon_1 < \epsilon_2$, *then* $P_n(T, f, x, \delta, \epsilon_1) \leq P_n(T, f, x, \delta, \epsilon_2)$.
(9) *If* $\delta_0 > 0$ *is such that* $d(x, y) \leq \delta/2$ *implies* $|f(x) - f(y)| \leq \delta_0$, *then*

$$P_n(T, f, x, \delta, \epsilon) \leq e^{n\delta_0} Q_n\left(T, f, x, \frac{\delta}{2}, \epsilon\right).$$

For $f \in C(X, \mathbb{R})$, $\epsilon > 0$ and $\delta > 0$, denote

$$P(T, f, \delta, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \sup_{x \in X} \log P_n(T, f, x, \delta, \epsilon).$$

LEMMA 3.3. *If* $f \in C(X, \mathbb{R})$, *then* $P^*(T, f) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} P(T, f, \delta, \epsilon)$.

*Proof.* The limit clearly exists. By property (6), it follows that

$$P^*(T, f) \leq \lim_{\epsilon \to 0} \lim_{\delta \to 0} P(T, f, \delta, \epsilon).$$

Property (9) shows that for any $\delta_0 > 0$, $\lim_{\epsilon \to 0} \lim_{\delta \to 0} P(T, f, \delta, \epsilon) \leq \delta_0 + P^*(T, f)$. This implies the equality. □

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Next, the definition of conditional pressure with respect to partitions is introduced. If those partitions are refining and essential, we show in Theorem 3.2 that this definition is equivalent to the definition given by spanning sets and separated sets.

If $P$ and $Q$ are two partitions with $Q$ finer than $P$, for some $A \in P^n$ and $f \in C(X, \mathbb{R})$, define

$$L_n(T, f, A, Q|P) = \sum_{B \subset A, B \in Q^n} \inf_{x \in B} e^{(S_n f)(x)}.$$

The following properties are fundamental.

(10) If $Q_2$ is finer than $Q_1$, then $L_n(T, f, A, Q_1|P) \le L_n(T, f, A, Q_2|P)$.

(11) If $P_2$ is finer than $P_1$, $A_1 \in P_1^n$, for any $A_2 \in P_2^n$ with $A_2 \subset A_1$, then $L_n(T, f, A_1, Q|P_1) \ge L_n(T, f, A_2, Q|P_2)$.

This conditional pressure can be defined by

$$L(T, f, Q|P) = \limsup_{n \to +\infty} \frac{1}{n} \sup_{A \in P^n} \log L_n(T, f, A, Q|P).$$

*Definition 3.2.* Let $f \in C(X, \mathbb{R})$ and $\mathcal{P} := (P_k)_{k \in \mathbb{N}}$ be a refining sequence of partitions; the tail pressure with respect to $\mathcal{P}$ is defined as follows:

$$P^*_{\mathcal{P}}(T, f) = \lim_{k \to +\infty} \lim_{l \to +\infty} L(T, f, P_l|P_k).$$

The limit of the last definition exists by properties (10) and (11). Furthermore, we present another equivalent definition.

If $P$ and $Q$ are two partitions with $Q$ finer than $P$, for some $A \in P^n$ and $f \in C(X, \mathbb{R})$, set

$$M_n(T, f, A, Q|P) = \sum_{B \subset A, B \in Q^n} \sup_{x \in B} e^{(S_n f)(x)}$$

and define

$$M(T, f, Q|P) = \limsup_{n \to +\infty} \frac{1}{n} \sup_{A \in P^n} \log M_n(T, f, A, Q|P).$$

THEOREM 3.1. *Let $f \in C(X, \mathbb{R})$ and $\mathcal{P} := (P_k)_{k \in \mathbb{N}}$ be a refining sequence of partitions; then*

$$P^*_{\mathcal{P}}(T, f) = \lim_{k \to +\infty} \lim_{l \to +\infty} M(T, f, P_l|P_k).$$

*Proof.* On the one hand, for some $A \in P_k^n$, it is obvious that

$$L_n(T, f, A, P_l|P_k) \le M_n(T, f, A, P_l|P_k).$$

This means that $P^*_{\mathcal{P}}(T, f) \le \lim_{k \to +\infty} \lim_{l \to +\infty} M(T, f, P_l|P_k)$.

On the other hand, if $d(x, y) \le \operatorname{diam}(P_l^n)$ implies $|f(x) - f(y)| \le \delta_0$, then

$$M_n(T, f, A, P_l|P_k) \le e^{n\delta_0} L_n(T, f, A, P_l|P_k).$$

Therefore, for any $\delta_0 > 0$,

$$\lim_{k \to +\infty} \lim_{l \to +\infty} M(T, f, P_l|P_k) \le \delta_0 + P^*_{\mathcal{P}}(T, f). \qquad \square$$

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LEMMA 3.4. [**2**] *If $P$ is an essential partition, then, for all $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{x\in X} \log \sharp\{A \in P^n \mid B(x, n, \delta) \cap A \neq \emptyset\} < \epsilon.$$

THEOREM 3.2. *If $f \in C(X, \mathbb{R})$ and $\mathcal{P} := (P_k)_{k\in\mathbb{N}}$ is a refining sequence of essential partitions, then*

$$P^*(T, f) = P^*_{\mathcal{P}}(T, f).$$

*Proof.* (1) Let $\delta_0 > 0$. According to the previous lemma, there exists $\epsilon > 0$ such that

$$\limsup_{n} \frac{1}{n} \sup_{x\in X} \log \sharp\{A \in P^n \mid B(x, n, \epsilon) \cap A \neq \emptyset\} < \delta_0.$$

For some $x \in X$, let $E$ be a $(n, \delta)$-separated subset of $B(x, n, \epsilon)$. Suppose that $\operatorname{diam}(P_l^n) < \delta$, then each element of $P_l^n$ contains at most one element of $E$. Denote

$$l := \sharp\{A \in P_k^n : B(x, n, \epsilon) \cap A \neq \varnothing\},$$
$$r := \sup_{x\in X} \sharp\{A \in P_k^n : B(x, n, \epsilon) \cap A \neq \varnothing\}$$

and

$$\mathcal{A} := \{A \in P_k^n : B(x, n, \epsilon) \cap A \neq \varnothing\}.$$

Thus,

$$\sum_{y\in E} e^{(S_n f)(y)} \le l \times \sup_{A\in\mathcal{A}} \sum_{B\subset A} \sup_{x\in B} e^{(S_n f)(x)} \le r \times \sup_{A\in P_k^n} \sum_{B\subset A} \sup_{x\in B} e^{(S_n)f(x)},$$

which implies

$$P_n(T, f, x, \delta, \epsilon) \le r \times \sup_{A\in P_k^n} M_n(T, f, P_l|P_k).$$

This yields that

$$P(T, f, \delta, \epsilon) \le M(T, f, P_l|P_k).$$

Hence,

$$P^*(T, f) \le P^*_{\mathcal{P}}(T, f).$$

(2) For some $x \in X$, consider the ball $B(x, n, \operatorname{diam}(P_k))$. Suppose that $E$ is a $(n, \delta)$-spanning set for $B(x, n, \operatorname{diam}(P_k))$.

If $\delta_1 > 0$ is such that $d(x, y) \le \delta$, this implies $|f(x) - f(y)| \le \delta_1$. Now, denote

$$r := \sup_{x\in X} \sharp\{A \in P_l^n : B(x, n, \delta) \cap A \neq \varnothing\}$$

and

$$l := \sup_{x\in E} \sharp\{A \in P_l^n : B(x, n, \delta) \cap A \neq \varnothing\}.$$

Therefore,

$$\sum_{B\subset A} \inf_{x\in B} e^{(S_n f)(x)}$$
$$\le l \times e^{n\delta_1} \times \sup_{x\in X} \left\{ \sum_{y\in E} e^{(S_n f)(y)} \;\middle|\; E \text{ is a } (n, \delta)\text{-spanning set for } B(x, n, \operatorname{diam}(P_k)) \right\}$$
$$\le r \times e^{n\delta_1} \times \sup_{x\in X} \left\{ \sum_{y\in E} e^{(S_n f)(y)} \;\middle|\; E \text{ is a } (n, \delta)\text{-spanning set for } B(x, n, \operatorname{diam}(P_k)) \right\}.$$

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By Lemma 3.1,

$$L(T,\, f,\, P_l|P_k) \le Q(T,\, f,\, \delta,\, \mathrm{diam}(P_k)) + \delta_1,$$

showing that

$$P^*(T,\, f) \ge P^*_{\mathcal{P}}(T,\, f),$$

which implies the equality holds. □

THEOREM 3.3. *If $f \in C(X, \mathbb{R})$ and $\mathcal{P} := (P_k)_{k\in\mathbb{N}}$ is a refining sequence of partitions, then*

$$\lim_{n\to+\infty} \frac{1}{n} \sup_{A\in P_k^n} \log M_n(T,\, f,\, A,\, P_l|P_k)$$

*exists and equals* $\inf_n (1/n) \sup_{A\in P_k^n} \log M_n(T,\, f,\, A,\, P_l|P_k)$.

*Proof.* For some $A \in P_k^{m+n}$, there exist $B \in P_k^n$, $C \in P_k^m$ such that $A = B \cap T^{-n}C$. It is sufficient to show that

$$M_{n+m}(T,\, f,\, A,\, P_l|P_k) \le M_n(T,\, f,\, B,\, P_l|P_k) \cdot M_m(T,\, f,\, C,\, P_l|P_k).$$

If $A' \in P_l^{m+n}$, there exist $B' \in P_l^n$, $C' \in P_l^m$ such that $A' = B' \cap T^{-n}C'$. Thus,

$$\sum_{A'\subset A} \sup_{x\in A'} e^{(S_n f)(x)} \le \left(\sum_{B'\subset B} \sup_{x\in B'} e^{(S_n f)(x)}\right)\left(\sum_{C'\subset C} \sup_{x\in C'} e^{(S_n f)(x)}\right).$$

Therefore,

$$M_{n+m}(T,\, f,\, A,\, P_l|P_k) \le M_n(T,\, f,\, B,\, P_l|P_k) \cdot M_m(T,\, f,\, C,\, P_l|P_k).$$ □

The following theorem is a result of the last two theorems.

THEOREM 3.4. *If $f \in C(X, \mathbb{R})$ and $T : X \to X$ is a continuous transformation of a compact metric space, then*

$$\begin{aligned} P^*(T,\, f) &= \lim_{\epsilon\to 0} \lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \sup_{x\in X} \log Q_n(T,\, f,\, x,\, \delta,\, \epsilon) \\ &= \lim_{\epsilon\to 0} \lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \sup_{x\in X} \log P_n(T,\, f,\, x,\, \delta,\, \epsilon). \end{aligned}$$

*Proof.* On the one hand, by Theorem 3.3,

$$\sup_{x\in X} P_n(T,\, f,\, x,\, \delta,\, \epsilon) \le r \times \sup_{A\in P_k^n} M_n(T,\, f,\, P_l|P_k).$$

Then, setting the logarithmic limit inferior in $n$, and then the limit in $l$ and $k$, and finally in $\epsilon$, we conclude that

$$\lim_{\epsilon\to 0} \lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \sup_{x\in X} \log P_n(T,\, f,\, x,\, \delta,\, \epsilon) \le P^*(T,\, f).$$

On the other hand, if $d(x,\, y) \le \mathrm{diam}(P_l^n)$ implies $|f(x) - f(y)| \le \delta_0$, then

$$e^{-n\delta_0} \sup_{A\in P_k^n} M_n(T,\, f,\, A,\, P_l|P_k) \le \sup_{A\in P_k^n} L_n(T,\, f,\, A,\, P_l|P_k)$$

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and

$$\sup_{A\in P_k^n} L_n(T, f, A, P_l|P_k) \le \sup_{x\in X} Q_n(T, f, x, \delta, \epsilon) \le \sup_{x\in X} P_n(T, f, x, \delta, \epsilon).$$

Again, setting the logarithmic limit inferior in $n$, and then the limit in $l$ and $k$, and finally in $\epsilon$, we conclude easily that

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\sup_{x\in X}\log P_n(T, f, x, \delta, \epsilon) \ge P^*(T, f),$$

which implies

$$\begin{aligned} P^*(T, f) &= \lim_{\epsilon\to 0}\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\sup_{x\in X}\log Q_n(T, f, x, \delta, \epsilon) \\ &= \lim_{\epsilon\to 0}\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\sup_{x\in X}\log P_n(T, f, x, \delta, \epsilon). \end{aligned}$$

□

3.2. *Dynamical properties.* The basic propositions of tail entropy include the power rule, the product rule and the natural extension. In the following, we study these properties of $P^*(T, f)$.

THEOREM 3.5. *Let $T : X \to X$ be a continuous transformation of a compact metric space and $f \in C(X, \mathbb{R})$. Then, the following statements are true.*

(1) *If $k > 0$, then $P^*(T^k, S_k f) = kP^*(T, f)$.*

(2) *If $T_i : X_i \to X_i$ is a continuous map of a compact metric space $(X_i, d_i)$ and $f_i \in C(X_i, \mathbb{R})$, $i = 1, 2$, then*

$$P^*(T_1 \times T_2, f_1 \times f_2) = P^*(T_1, f_1) + P^*(T_2, f_2),$$

*where $f_1 \times f_2 \in C(X_1 \times X_2, \mathbb{R})$ is defined by $(f_1 \times f_2)(x_1, x_2) = f_1(x_1) + f_2(x_2)$.*

(3) *If $(Y, \overline{T})$ is a natural extension of $(X, T)$ and $f \in C(X, \mathbb{R})$, then*

$$P^*(\overline{T}, f \circ \pi) = P^*(T, f).$$

*Proof.* (1) If $F$ is a $(nk, \delta)$-spanning set for $T$, then $F$ is a $(n, \delta)$-spanning set for $T^k$. Hence, $Q_n(T^k, S_k f, x, \delta, \epsilon) \le Q_{nk}(T, f, x, \delta, \epsilon)$ and $P^*(T^k, S_k f) \le kP^*(T, f)$. To show the opposite inequality, let $\delta_0 > 0$ and choose $\delta > 0$ such that $d(x, y) \le \delta$ implies $d_k(x, y) \le \delta_0$. If $F$ is $(n, \delta)$-spanning for $T^k$, then $F$ is $(nk, \delta_0)$-spanning for $T$. Hence, $Q_n(T^k, S_k f, x, \delta, \epsilon) \ge Q_{nk}(T, f, x, \delta_0, \epsilon)$. Thus, $P^*(T^k, S_k f) \ge kP^*(T, f)$.
(2) Consider the metric on $X_1 \times X_2$ given by

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

If $F_i$ is a $(n, \delta)$-spanning set for $B_{T_i}(x_i, n, \epsilon)$, then $F_1 \times F_2$ is a $(n, \delta)$-spanning set for $B_{T_1}(x_1, n, \epsilon) \times B_{T_2}(x_2, n, \epsilon)$ with respect to $T_1 \times T_2$. Also,

$$\begin{aligned} &\sum_{(y_1,y_2)\in F_1\times F_2} \exp\left(\sum_{i=0}^{n-1}(f_1 \times f_2)(T_1 \times T_2)^i(x_1, x_2)\right) \\ &= \left(\sum_{x_1\in F_1} \exp\left(\sum_{i=0}^{n-1} f_1(T_1^i(x_1))\right)\right)\left(\sum_{x_2\in F_2} \exp\left(\sum_{i=0}^{n-1} f_2(T_2^i(x_2))\right)\right) \end{aligned}$$

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such that

$$Q_n(T_1 \times T_1,\ f_1 \times f_2,\ x_1 \times x_2,\ \delta,\ \epsilon) \le Q_n(T_1,\ f_1,\ x_1,\ \delta,\ \epsilon) \cdot Q_n(T_2,\ f_2,\ x_2,\ \delta,\ \epsilon).$$

Therefore, $P^*(T_1 \times T_2,\ f_1 \times f_2) \le P^*(T_1,\ f_1) + P^*(T_2,\ f_2)$.

If $E_i$ is a $(n,\ \delta)$-separated set for $B_{T_i}(x_i,\ n,\ \epsilon)$, then $E_1 \times E_2$ is a $(n,\ \delta)$-separated set for $B_{T_1}(x_1,\ n,\ \epsilon) \times B_{T_2}(x_2,\ n,\ \epsilon)$ such that

$$P_n(T_1 \times T_2,\ f_1 \times f_2,\ x_1 \times x_2,\ \delta,\ \epsilon) \ge P_n(T_1,\ f_1,\ x_1,\ \delta,\ \epsilon) \cdot P_n(T_2,\ f_2,\ x_2,\ \delta,\ \epsilon),$$

since

$$\begin{aligned}&\limsup_{n\to+\infty} \frac{1}{n} \sup_{(x_1,x_2)\in X_1\times X_2} \log P_n(T_1 \times T_2,\ f_1 \times f_2,\ x_1 \times x_2,\ \delta,\ \epsilon)\\ &\quad\ge \liminf_{n\to+\infty} \frac{1}{n} \sup_{x_1\in X_1} \log P_n(T_1,\ f_1,\ x_1,\ \delta,\ \epsilon) + \limsup_{n\to+\infty} \frac{1}{n} \sup_{x_2\in X_2} \log P_n(T_2,\ f_2,\ x_2,\ \delta,\ \epsilon).\end{aligned}$$

This inequality gives $P^*(T_1 \times T_2,\ f_1 \times f_2) \ge P^*(T_1,\ f_1) + P^*(T_2,\ f_2)$.

(3) According to [**2**], we know that:

(i) any Bowen's ball for $\overline{T}$ is contained in the lift of a Bowen's ball for $T$ of the same radius and order; and

(ii) any lift of a Bowen's ball for $T$ is covered by $N$ Bowen's balls for $\overline{T}$ of the same order and of triple radius, with $N$ depending only on the radius.

First, prove $P^*(\overline{T},\ f \circ \pi) \ge P^*(T,\ f)$. Consider $x_0 \in X$. According to (ii), there exists $E \subset Y$ with $\sharp E \le N$ such that

$$B_T(x_0,\ n,\ \epsilon) \subset \pi \bigcup_{x\in E} B_{\overline{T}}(x,\ n,\ 3\epsilon).$$

Suppose that $E_1$ is a $(n,\ \delta)$-spanning set for $B_T(x_0,\ n,\ \epsilon)$, $E_2$ is a $(n,\ \delta)$-spanning set for $B_{\overline{T}}(y,\ n,\ 3\epsilon)$, for some $y \in E$ that satisfies the above relationship. The definition of $\overline{d}$ shows that if $x = (x_n)_{n\in\mathbb{Z}}$, $y = (y_n)_{n\in\mathbb{Z}}$ and $\overline{d}_n(x,\ y) \le \delta$, then $d_n(x_0,\ y_0) \le \delta$.

This yields

$$\begin{aligned}\sum_{y\in E_1} e^{(S_n f)(y)} &\le N \cdot \sup_{y\in E} \sum_{z\in E_2} e^{(S_n f)(\pi z)}\\ &= N \cdot \sup_{y\in E} \sum_{z\in E_2} e^{(S_n (f\circ\pi))(z)}\\ &\le N \cdot \sup_{y\in Y} \sum_{z\in E_2} e^{(S_n (f\circ\pi))(z)},\end{aligned}$$

which implies

$$Q_n(T,\ f,\ x_0,\ \delta,\ \epsilon) \le N \cdot \sup_{y\in Y} Q_n(\overline{T},\ f \circ \pi,\ y,\ \delta,\ 3\epsilon).$$

Therefore, $P^*(\overline{T},\ f \circ \pi) \ge P^*(T,\ f)$.

Next, prove $P^*(\overline{T},\ f \circ \pi) \le P^*(T,\ f)$.

For $\epsilon > \delta > 0$, $n \ge 1$, let $x = (x_n)_{n\in\mathbb{Z}} \in Y$ and let $E_\delta$ be a $(n,\ \delta)$-spanning set of $B_{\overline{T}}(x,\ n,\ \epsilon)$ with the cardinality $\sharp E_\delta = r(n,\ \delta,\ B_T(x_0,\ n,\ \epsilon))$. Let $E_1$ be a $(n,\ \delta)$-spanning

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set of $B_T(x_0, n, \epsilon)$ and $\sharp E_1 = \sharp E_\delta$. If $E$ is a $(n, 3\delta)$-spanning set of $B_{\overline{T}}(x, n, \epsilon)$, according to (i), then

$$B_{\overline{T}}(x, n, \epsilon) \subset \pi^{-1} B_T(x_0, n, \epsilon) \subset \bigcup_{y \in E_1} \pi^{-1} B_T(y, n, \delta),$$

and, using the second point for each $y \in E_1$, there exist $Z_y \subset Y$ with $\sharp Z_y \leq N$ such that

$$\pi^{-1} B_T(y, n, \delta) \subset \bigcup_{z \in Z_y} B_{\overline{T}}(z, n, 3\delta).$$

If $\delta > 0$ is such that $d(x, y) \leq 3\delta$ implies $|f(x) - f(y)| \leq \delta_0$, then

$$\begin{aligned} \sum_{y \in E} e^{(S_n(f \circ \pi))(y)} &\leq \sum_{y \in E_1} \sum_{z \in Z_y} e^{(S_n(f \circ \pi))(z)} \\ &\leq \sum_{y \in E_1} (N \cdot e^{(S_n f)(y) + n\delta_0}) \\ &= N \cdot e^{n\delta_0} \sum_{y \in E_1} e^{(S_n f)(y)}, \end{aligned}$$

which implies

$$Q(\overline{T}, f \circ \pi, 3\delta, \epsilon) \leq Q(T, f, \delta, \epsilon) + \delta_0.$$

Therefore, $P^*(\overline{T}, f \circ \pi) \leq P^*(T, f)$. □

## 4. *The variational principle of tail pressure*

For a continuous function $T$ of a compact metric space $(X, d)$, the well-known variational principle of topological pressure shows the relationships among pressure, entropy invariants, and potential energy from probabilistic and topological perspectives. This section provides a proof of the variational principle for the tail pressure, beginning with the following useful lemmas.

### 4.1. *Tail entropy function.*

LEMMA 4.1. [**20**] *Let $a_1, \ldots, a_k$ be real numbers. If $p_i \geq 0$ and $\sum_{i=1}^k p_i = 1$, then*

$$\sum_{i=1}^k p_i(a_i - \log p_i) \leq \log\left(\sum_{i=1}^k e^{a_i}\right)$$

*and equality holds if and only if*

$$p_i = \frac{e^{a_i}}{\sum_{i=1}^k e^{a_i}}.$$

LEMMA 4.2. [**20**] *Let $\lambda \in [0, 1]$ and $\mu, \nu$ be two probability measures. If $P, Q$ are two partitions, then*

$$H_{\lambda\mu + (1-\lambda)\nu}(P|Q) \geq \lambda H_\mu(P|Q) + (1 - \lambda) H_\nu(P|Q).$$

LEMMA 4.3. [**20**] *If $A, B, C$ and $D$ are partitions, then*

$$H_\mu(A \vee B|C \vee D) \leq H_\mu(A|C) + H_\mu(B|D).$$

Furthermore, the following lemmas give some properties of the function $u^{\mathcal{H}}(\mu) + \int f \, d\mu$, where $u^{\mathcal{H}}(\mu)$ is called the tail entropy function in [**5**] and $\int f \, d\mu$ is the potential energy.

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LEMMA 4.4.

(1) *If $\mathcal{H}$ is a candidate, then the function $u^{\mathcal{H}}(\mu) + \int f\, d\mu$ is an upper semi-continuous function. In particular, $u^{\mathcal{H}}(\mu) + \int f\, d\mu$ attains its supremum.*

(2) *If $\mathcal{H} = (h_k)_{k\in\mathbb{N}}$ is a candidate of $(X, T)$, then, for all $f \in C(X, R)$,*

$$u^{\mathcal{H}}(\mu) + \int f\, d\mu = \lim_{k\to\infty} \left( \widetilde{(h_\infty - h_k)(\mu)} + \int f\, d\mu \right).$$

(3) *If $\mathcal{H} = (h_k)_{k\in\mathbb{N}}$ is a candidate of $(X, T)$, then, for all $f \in C(X, R)$,*

$$\sup_{\mu\in M(X,T)} \left\{ u^{\mathcal{H}}(\mu) + \int f\, d\mu \right\} = \lim_{k\to\infty} \lim_{l\to\infty} \sup_{\mu\in M(X,T)} \left\{ (h_l - h_k)(\mu) + \int f\, d\mu \right\}.$$

(4) *If $\pi : (Y, S) \to (X, T)$ is an extension and $\mathcal{H} = (h_k)_{k\in\mathbb{N}}$ is a candidate of $(X, T)$, then, for all $f \in C(X, R)$,*

$$\sup_{\mu\in M(Y,S)} \left\{ u^{\pi^*\mathcal{H}}(\mu) + \int f\circ\pi\, d\mu \right\} = \sup_{\mu\in M(X,T)} \left\{ u^{\mathcal{H}}(\mu) + \int f\, d\mu \right\}.$$

(5) *If $\mathcal{H} = (h_k)_{k\in\mathbb{N}}$ is a candidate with affine and upper semi-continuous $h_k$, then the supremum of $u^{\mathcal{H}}(\mu) + \int f\, d\mu$ is achieved on the closure of the ergodic measures of $(X, T)$.*

*Proof.* The assertions (1) and (2) are obvious. Assertion (3) follows easily from [**1**, Proposition 2.4].

(4) For all $\mu \in M(Y, S)$ and $f \in C(X, R)$,

$$\begin{aligned}
&\sup_{\mu\in M(Y,S)} \left\{ u^{\pi^*\mathcal{H}}(\mu) + \int f\circ\pi\, d\mu \right\} \\
&= \lim_{k\to\infty} \lim_{l\to\infty} \sup_{\mu\in M(Y,S)} \left\{ (h_l - h_k)(\pi^*\mu) + \int f\circ\pi\, d\mu \right\} \\
&= \lim_{k\to\infty} \lim_{l\to\infty} \sup_{\mu\in M(Y,S)} \left\{ (h_l - h_k)(\pi^*\mu) + \int f\, d\pi^*\mu \right\} \\
&\le \lim_{k\to\infty} \lim_{l\to\infty} \sup_{\mu\in M(X,T)} \left\{ (h_l - h_k)(\mu) + \int f\, d\mu \right\} \\
&= \sup_{\mu\in M(X,T)} \left\{ u^{\mathcal{H}}(\mu) + \int f\, d\mu \right\}.
\end{aligned}$$

On the other hand, for each $\mu \in M(X, T)$, there exists a $\mu_1 \in M(Y, S)$ such that $\pi^*\mu_1 = \mu$. Thus, for all $f \in C(X, R)$, we obtain

$$\begin{aligned}
&\sup_{\mu\in M(X,T)} \left\{ u^{\mathcal{H}}(\mu) + \int f\, d\mu \right\} \\
&= \lim_{k\to\infty} \lim_{l\to\infty} \sup_{\mu\in M(X,T)} \left\{ (h_l - h_k)(\mu) + \int f\, d\mu \right\} \\
&= \lim_{k\to\infty} \lim_{l\to\infty} \sup_{\mu\in M(X,T)} \left\{ (h_l - h_k)(\pi^*\mu_1) + \int f\, d\pi^*\mu_1 \right\}
\end{aligned}$$

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$$\leq \lim_{k\to\infty}\lim_{l\to\infty}\sup_{\mu\in M(Y,S)}\left\{(h_l - h_k)(\pi^*\mu) + \int f\circ\pi\, d\mu\right\}$$
$$= \sup_{\mu\in M(Y,S)}\left\{u^{\pi^*\mathcal{H}}(\mu) + \int f\circ\pi\, d\mu\right\}.$$

(5) Let $\mathcal{H} = (h_k)_{k\in\mathbb{N}}$ be a candidate with affine and upper semi-continuous $h_k$. The ergodic decomposition (affine upper semi-continuous functions are harmonic) yields

$$\sup_{\mu\in M(X,T)}\left\{(h_\infty - h_k)(\mu) + \int f\, d\mu\right\} = \sup_{\mu\in M^e(X,T)}\left\{(h_\infty - h_k)(\mu) + \int f\, d\mu\right\},$$

where $M^e(X, T)$ denotes the set of ergodic measures. Moreover, according to [**1**, Proposition 2.4], this can be applied to the non-increasing sequence of upper semi-continuous functions $\widetilde{(h_\infty - h_k)(\mu)} + \int f\, d\mu$ defined on the closure of the ergodic measures. Thus,

$$\sup_{\mu\in\overline{M^e(X,T)}}\left\{u^{\mathcal{H}}(\mu) + \int f\, d\mu\right\} = \lim_{k\to\infty}\sup_{\overline{M^e(X,T)}}\left(\widetilde{(h_\infty - h_k)(\mu)} + \int f\, d\mu\right)$$

and, finally,

$$\sup_{\mu\in M(X,T)}\left\{u^{\mathcal{H}}(\mu) + \int f\, d\mu\right\} = \lim_{k\to\infty}\sup_{M(X,T)}\left(\widetilde{(h_\infty - h_k)(\mu)} + \int f\, d\mu\right)$$
$$= \sup_{\mu\in\overline{M^e(X,T)}}\left\{u^{\mathcal{H}}(\mu) + \int f\, d\mu\right\}. \qquad \square$$

*Remarks.* There exist entropy structures $\mathcal{H} = (h_k)_{k\in\mathbb{N}}$ with affine and upper semi-continuous $h_k$ (for example $\mathcal{H}^{fun}$; see [**5**]). In particular, the supremum of $u(\mu) + \int f\, d\mu$ is achieved on the closure of the ergodic measures.

4.2. *Relationships.*

THEOREM 4.1. (Variational principle of tail pressure) *If $T : X \to X$ is a continuous map of a compact metric space $(X, d)$ with finite topological entropy and $f \in C(X, \mathbb{R})$, then*

$$P^*(T, f) = \sup\left\{u(\mu) + \int f\, d\mu \;\middle|\; \mu \in M(X, T)\right\}.$$

*Moreover, the supremum is achieved on the closure of the ergodic measures.*

*Proof.* First of all, we will prove the variational principle of tail pressure, in which $(X, T)$ admits a refining sequence of essential partitions.

(1) Let $\mu \in M(X, T)$; first show that

$$u(\mu) + \int f\, d\mu \leq P^*(T, f).$$

Let $\mathcal{P} := (P_k)_{k\in\mathbb{N}}$ be a refining essential sequence of partitions of $X$. If $C \in P_l^n$, let $\alpha(C)$ denote $\sup\{(S_n f)(x) \mid x \in C\}$. Then,

$$H_\mu(P_l^n|P_k^n) + \int S_n f\, d\mu = \sum_{A\in P_k^n}\mu(A)\left(H(\mu_A, P_l^n) + \int_A S_n f\, d\mu_A\right)$$
$$\leq \sum_{A\in P_k^n}\mu(A)\sum_{C\in P_l^n}\mu_A(C)[-\log\mu_A(C) + \alpha(C)]$$

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$$\begin{aligned}&\leq \sum_{A\in P_k^n} \mu(A) \sup_{A\in P_k^n} \log \sum_{C\in P_l^n} e^{\alpha(C)}\\ &= \sup_{A\in P_k^n} \log \sum_{C\in P_l^n} e^{\alpha(C)}\\ &= \sup_{A\in P_k^n} \log \sum_{C\subset A} \sup_{x\in C} e^{(S_n f)(x)}.\end{aligned}$$

Thus, we have

$$h(\mu, P_l|P_k) + \int f\, d\mu \leq M(T, f, P_l|P_k),$$

and then $u(\mu) + \int f\, d\mu \leq P^*(T, f)$.

(2) Next, find $\mu \in M(X, T)$ with $u(\mu) + \int f\, d\mu \geq P^*(T, f)$. Let $E$ be a $(n, \delta)$-separated set and let $\mathcal{P} := (P_k)_{k\in\mathbb{N}}$ be a refining essential sequence of partitions of $X$. Let $\sigma_n \in M(X)$ be the atomic measure by the formula

$$\sigma_n = \frac{\sum_{y\in E} e^{(S_n f)(y)}\delta_y}{\sum_{z\in E} e^{(S_n f)(z)}}.$$

Fix $A \in P_k^n$, and such that $A \cap B(x, n, \epsilon) \neq \emptyset$. Without loss of generality, we assume that $\operatorname{diam}(P_l^n) \leq \delta$. Let $\mu_n \in M(X)$ be defined by

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} (\sigma_n \lfloor A) \circ T^{-i}.$$

Since $M(X)$ is compact, choose a subsequence $\{n_j\}$ of the natural numbers such that $\{\mu_{n_j}\}$ converges in $M(X)$ to some $\mu \in M(X, T)$.

Burguet [**2**] shows that, for $0 \leq j < m < n$, it is possible to cut the interval $(0, n)$ into $[n/m] - 2$ intervals of length $m : (j, j + mk - 1), \ldots, (j + km, j + (k + 1)m - 1), \ldots,$ with at most $3m$ remaining terms. Since each element of $P_l^n$ contains at most one element of $E$, according to Lemma 4.1, we have

$$\begin{aligned}H_{\sigma_n\lfloor A}(P_l^n|P_k^n) + \int S_n f\, d(\sigma_n\lfloor A) &= H_{\sigma_n\lfloor A}(P_l^n) + \int_A S_n f\, d(\sigma_n\lfloor A)\\ &= \log \sum_{y\in E\cap A} e^{(S_n f)(y)},\end{aligned}$$

which implies that

$$\begin{aligned}&\log \sum_{y\in E\cap A} e^{(S_n f)(y)} = H_{\sigma_n\lfloor A}(P_l^n|P_k^n) + \int S_n f\, d(\sigma_n\lfloor A)\\ &\leq \sum_{k=0}^{[n/m]-2} H_{\sigma_n\lfloor A}\Bigg(\bigvee_{i=j+km}^{j+(k+1)m-1} T^{-i}P_l \Bigg| \bigvee_{i=j+km}^{j+(k+1)m-1} T^{-i}P_k\Bigg) + 3m\log(\sharp P_l)\\ &\quad + \int S_n f\, d(\sigma_n\lfloor A)\\ &\leq \sum_{k=0}^{[n/m]-2} H_{(\sigma_n\lfloor A)\circ T^{-j+km}}(P_l^m|P_k^m) + 3m\log(\sharp P_l) + \int S_n f\, d(\sigma_n\lfloor A).\end{aligned}$$

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Summing for $0 \le j < m$ leads to

$$m \log \sum_{y \in E \cap A} e^{(S_n f)(y)} \le \sum_{k=0}^{n-1} H_{(\sigma_n \lfloor A) \circ T^{-k}}(P_l^m | P_k^m) + 3m^2 \log(\sharp P_l) + m \int S_n f \, d(\sigma_n \lfloor A)$$
$$\le n H_{\mu_n}(P_l^m | P_k^m) + 3m^2 \log(\sharp P_l) + m \int S_n f \, d(\sigma_n \lfloor A).$$

Dividing by $n$ yields

$$\frac{m}{n} \log \sum_{y \in E \cap A} e^{(S_n f)(y)} \le H_{\mu_n}(P_l^m | P_k^m) + \frac{3m^2}{n} \log(\sharp P_l) + m \int f \, d\mu_n.$$

On the other hand,

$$\frac{m}{n} \log \sum_{B \subset A} \inf_{x \in B} e^{(S_n f)(x)} \le \frac{m}{n} \log \sum_{y \in E \cap A} e^{(S_n f)(y)},$$

where $B \in P_l^n$, $B \subset A$. Since the partitions $P_l$ and $P_k$ are essential, the function $\mu \to H_\mu(P_l^m | P_k^m)$ is continuous. Thus, taking the limit and replacing $n$ by $n_j$ yields

$$mL(T, f, P_l | P_k) \le H_\mu(P_l^m | P_k^m) + m \int f \, d\mu,$$

and then

$$L(T, f, P_l | P_k) \le \frac{1}{m} H_\mu(P_l^m | P_k^m) + \int f \, d\mu.$$

Thus, $u(\mu) + \int f \, d\mu \ge P^*(T, f)$.

In the general case, Lemma 2.1 shows that, for an irrational $\theta$, the product of the natural extension of $(X, T)$ with the rotation of angle $\theta$ on the circle, $(\overline{X} \times \mathbb{S}^1, \overline{T} \times R_\theta)$, admits a refining sequence of essential partitions. Therefore, according to the previous proof,

$$P^*(\overline{T} \times R_\theta, (f \circ \pi) \times g) = \sup \left\{ u(\mu) + \int (f \circ \pi) \times g \, d\mu \;\middle|\; \mu \in M(\overline{X} \times \mathbb{S}^1, \overline{T} \times R_\theta) \right\},$$

where $g \in C(\mathbb{S}^1, R_\theta)$, $f \in C(X, R)$ and $f \times g$ is defined by $(f \times g)(x_1, x_2) = f(x_1) + g(x_2)$. Suppose that $\pi_{\overline{X}} : (\overline{X} \times \mathbb{S}^1, \overline{T} \times R_\theta) \to (\overline{X}, \overline{T})$ is defined by $\pi_{\overline{X}}(x, y) = x$, for all $(x, y) \in \overline{X} \times \mathbb{S}^1$. Therefore, $(\overline{X} \times \mathbb{S}^1, \overline{T} \times R_\theta)$ is an extension of $(\overline{X}, \overline{T})$.

If $g = 0$, Theorem 3.5 shows that $P^*(\overline{T} \times R_\theta, (f \circ \pi) \times g) = P^*(T, f)$. Moreover, according Lemma 4.4, we obtain

$$\sup \left\{ u(\mu) + \int (f \circ \pi) \times g \, d\mu \;\middle|\; \mu \in M(\overline{X} \times \mathbb{S}^1, \overline{T} \times R_\theta) \right\}$$
$$= \sup \left\{ u(\mu) + \int (f \circ \pi) \circ \pi_{\overline{X}} \, d\mu \;\middle|\; \mu \in M(\overline{X} \times \mathbb{S}^1, \overline{T} \times R_\theta) \right\}$$
$$= \sup \left\{ u(\mu) + \int (f \circ \pi) \, d\mu \;\middle|\; \mu \in M(\overline{X}, \overline{T}) \right\}$$
$$= \sup \left\{ u(\mu) + \int f \, d\mu \;\middle|\; \mu \in M(X, T) \right\}.$$

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Finally,

$$P^*(T, f) = \sup\left\{u(\mu) + \int f \, d\mu \;\middle|\; \mu \in M(X, T)\right\}. \qquad \square$$

A point $x \in X$ is said to be a non-wandering point of $T$ if, for each neighborhood $U$ of $x$, there exists $n \in N$ such that $T^n(U) \cap U \neq \emptyset$. Let $\Omega(T)$ denote the non-wandering set of $T$. It is well known that $\mu(\Omega(T)) = 1$ for each $\mu \in M(X, T)$.

COROLLARY 1. *Let $T : X \to X$ be a continuous map of a compact metric space and $f \in C(X, \mathbb{R})$. Then:*

(1) $P^*(T, f) = P^*(T|_{\Omega(T)}, f|_{\Omega(T)})$*; and*

(2) $P^*(T, f) = P^*(T|_{\bigcap_{n=0}^{\infty} T^n X}, f|_{\bigcap_{n=0}^{\infty} T^n X})$.

COROLLARY 2. *If $T : X \to X$ is uniquely ergodic and $M(X, T) = \{\mu\}$, then*

$$P^*(T, f) = u(\mu) + \int f \, d\mu.$$

## 5. *Tail pressure determines invariant measures*

This section shows how $P^*(T, \cdot)$ determines the invariant measure of $T$ when $T : X \to X$ is a continuous map of a compact metric space $X$. Recall that a finite signed measure on $X$ is a map $\mu : \mathcal{B}(X) \to \mathbb{R}$, which is countably additive.

LEMMA 5.1. *Let $T : X \to X$ be a continuous transformation of a compact metric space $X$. If $f, g \in C(X, \mathbb{R})$, $\epsilon > 0$, $\delta > 0$, and $c \in \mathbb{R}$, then the following are true.*

(1) $P^*(T, 0) = h^*(T)$.

(2) *If $f \leq g$, then $P^*(T, f) \leq P^*(T, g)$. In particular,*

$$h^*(T) + \inf f \leq P^*(T, f) \leq h^*(T) + \sup f.$$

(3) $P^*(T, \cdot)$ *is either finite valued or constantly $\infty$.*

(4) $|P^*(T, f) - P^*(T, g)| \leq \|f - g\|$.

(5) $P^*(T, f + c) = P^*(T, f) + c$.

(6) $P^*(T, f + g \circ T - g) = P^*(T, f)$.

(7) $P^*(T, f + g) \leq P^*(T, f) + P^*(T, g)$.

(8) $P^*(T, \cdot, \delta, \epsilon)$ *is convex, so if $P^*(T, \cdot) < \infty$, then $P^*(T, \cdot)$ is convex.*

(9) $P^*(T, cf) \leq cP^*(T, f)$ *if $c \geq 1$, and $P^*(T, cf) \geq cP^*(T, f)$ if $c \leq 1$.*

(10) $|P^*(T, f)| \leq P^*(T, |f|)$.

*Proof.* The proof follows [**20**] and is omitted. $\square$

LEMMA 5.2. (Tail pressure determines invariant measures) *Let $T : X \to X$ be a continuous transformation of a compact metric space with finite topological entropy. If $\mu : \mathcal{B}(X) \to \mathbb{R}$ is a finite signed measure, then $\mu \in M(X, T)$ if and only if $\int f \, d\mu \leq P^*(T, f)$, for all $f \in C(X, \mathbb{R})$.*

*Proof.* The proof follows [**20**] and is omitted. $\square$

The definition of $u^{\mathcal{H}}$ shows it is an upper semi-continuous function. In particular, $u^{\mathcal{H}}$ attains its supremum.

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LEMMA 5.3. *Let $T : X \to X$ be a continuous map of a compact metric space with finite topological entropy and for all $\mu_0 \in M(X, T)$. Thus,*

$$u(\mu_0) = \inf\left\{ P^*(T, f) - \int f \, d\mu_0 \;\middle|\; f \in C(X, \mathbb{R}) \right\}.$$

*Proof.* The proof follows [**20**] and is omitted. □

## 6. *Tail equilibrium states*

Classical thermodynamic theory deals with dynamic equilibrium states. The local state of a system at thermodynamic equilibrium is determined by the values of its intensive parameters, such as pressure or temperature. Specifically, thermodynamic equilibrium is characterized by the minimum of a thermodynamic potential. This section gives some applications of tail pressure $P^*(T, \cdot)$ to equilibrium states.

*Definition 6.1.* Let $T : X \to X$ be a continuous map of a compact metric space $X$ and let $f \in C(X, \mathbb{R})$. A member $\mu$ of $M(X, T)$ is called a tail equilibrium state for $f$ if $P^*(T, f) = u(\mu) + \int f \, d\mu$. Let $M_f^*(X, T)$ denote the collection of all tail equilibrium states for $f$.

*Definition 6.2.* Let $T : X \to X$ be a continuous map of a compact metrizable space with finite topological entropy and let $f \in C(X, \mathbb{R})$. A tangent functional to $P^*(T, \cdot)$ at $f$ is a finite signed measure $\mu : \mathcal{B}(X) \to \mathbb{R}$ such that $P^*(T, f + g) - P^*(T, f) \geq \int g \, d\mu$, for all $g \in C(X, \mathbb{R})$. Let $t_f^*(X, T)$ denote the collection of all tangent functionals to $P^*(T, f)$ at $f$.

LEMMA 6.1. *Let $T : X \to X$ be a continuous map of a compact metric space with finite topological entropy and $f \in C(X, \mathbb{R})$. Then, $M_f^*(X, T) = t_f^*(X, T) \subset M(X, T)$.*

*Proof.* The proof follows [**20**] and is omitted. □

LEMMA 6.2. *Let $T : X \to X$ be a continuous map of a compact metric space with finite topological entropy and $f \in C(X, \mathbb{R})$. Then:*

(1) *$M_f^*(X, T)$ is compact and non-empty; and*
(2) *if $f, g \in C(X, \mathbb{R})$, and there exists $c \in \mathbb{R}$ such that $f - g - c$ belongs to the closure of the set $\{h \circ T - h \mid h \in C(X, \mathbb{R})\}$ in $C(X, \mathbb{R})$, then $M_f^*(X, T) = M_g^*(X, T)$.*

*Proof.* (1) It follows that the function $u$ is an upper semi-continuous function and, in particular, $u$ attains its supremum. The variational principle can show that $M_f^*(X, T)$ is compact.

(2) Notice that for all $\mu \in M(X, T)$, $\int f \, d\mu = \int g \, d\mu + c$. Therefore,

$$u(\mu) + \int f \, d\mu = u(\mu) + \int g \, d\mu + c \quad \text{and} \quad P^*(T, f) = P^*(T, g) + c.$$

Hence, $M_f^*(X, T) = M_g^*(X, T)$. □

LEMMA 6.3. *If $T$ has finite topological entropy, then $T$ is uniquely ergodic if and only if $P^*(T, \cdot) : C(X, R) \to R$ is differentiable at each point of $C(X, R)$.*

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*Proof.* Lemmas 6.1 and 6.2 imply that $M_f^*(X, T) = t_f^*(X, T) \neq \emptyset$. Thus, the conclusion follows. □

*Remark.* For a transformation with zero tail entropy (asymptotically $h$-expansive in Bowen terminology), a tail equilibrium state for a real-valued continuous function $f$ is an invariant probability which maximizes the integral of $f$ with respect to invariant probabilities. This kind of problem has been studied for many years by several authors. A good discussion of these properties and applications appears in [**3, 4, 14, 15**].

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