Combinatorics, Probability and Computing (1997) 7, 493–496. Printed in the United Kingdom
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A Note on Distributional Equality in the Cyclic Tour Property for Markov Chains

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Received 25 February 1996; revised 5 June 1996

The cyclic tour property has previously been an equality for the expected time to complete a tour, compared with that for the reverse tour, for reversible Markov chains. We give a simple bijection to show that the equality can be extended to the distributions involved. The bijection is based on rotation of circular words.

We follow the notation used in Aldous [1, Chapter 3.2]. Given a sequence of $m+1$ states $i_0 = a, i_1, i_2, \ldots, i_m$, the Markov chain $X_0, X_1, \ldots$ starting from $X_0 = a$ completes a cyclic tour at time $T = k$ if k is the smallest value such that $X_0, X_1, \ldots, X_k$ contains, as a subsequence allowing deletions, the word $i_0 i_1 i_2 \cdots i_m i_0$. Similarly, the time T^* for a reversed tour is the smallest k such that $X_0, X_1, \ldots, X_k$ contains $i_0 i_m i_{m-1} \cdots i_2 i_1 i_0$ as a subsequence. The cyclic tour property, from Aldous [1], Coppersmith, Tetali and Winkler [2] and Tetali [3], is that for a reversible Markov chain starting from state $a = i_0, \mathbf{E}T = \mathbf{E}T^*$. We will prove a stronger result, that for the chain starting at $a = i_0$, T and T^* have the same distribution. This is equivalent to showing, for all $k > 0$,

$$\mathbf{P}_a(T = k) = \mathbf{P}_a(T^* = k). \tag{1}$$

In the notes on Chapter 3 from Aldous [1], the author writes, with respect to the cyclic tour property at the level of expectations: "Though probabilists would regard the 'cyclic tour' Lemma 1 as obvious, László Lovász pointed out a complication, that with a careful definition of starts and ends of tours these times are not invariant under time-reversal. The sophisticated fix is to use doubly-infinite stationary chains and observe that tours in reversed time just interleave tours in forward time, so by ergodicity their asymptotic rates are equal. Alternatively one can verify the original assertion by plugging in the explicit formula for $E_i T_j$ ... and using symmetry ..."

https://doi.org/10.1017/S0963548397003076 Published online by Cambridge University Press

An example should keep things clear. We assume that a, b, c, d are distinct states and $m = 3, i_0 = a, i_1 = b, i_2 = c, i_3 = d$. If the sample path is

$$X_0 X_1 \cdots X_{10} \cdots = a\ d\ a\ b\ a\ c\ b\ a\ d\ c\ a\ \cdots$$

then $T = 10$, with the first b at time 3, the next c at time 5, the next d at time 8, and the next a at time 10. If we read in reverse, from time 10 backwards, i.e. if the sample path were

$$X_0 X_1 \cdots X_{10} \cdots = a\ c\ d\ a\ b\ c\ a\ b\ a\ d\ a\ \cdots$$

then $T^* = 8$, with the first d at time 2, the next c at time 5, the next b at time 7, and the next a at time 8.

Our proof of (1) is based on a bijection, involving the rotation and flipping of words of length k. Since we consider words as circular, identifying sites 0 and k, we will use $0, 1, \ldots, k-1$ as sites. Our notation for a word $\mathbf{w}$, its rotation $r(\mathbf{w})$, and its flip or reverse $f(\mathbf{w})$ are

$$\begin{aligned} \mathbf{w} &= w_0 w_1 w_2 \cdots w_{k-2} w_{k-1}, \\ r(\mathbf{w}) &= w_1 w_2 \cdots w_{k-2} w_{k-1} w_0, \\ f(\mathbf{w}) &= w_0 w_{k-1} w_{k-2} \cdots w_2 w_1. \end{aligned}$$

Recall from dihedral groups that $f \circ r^j = r^{-j} \circ f$.

Consider the sets of circular words of length k which begin with a and complete the tour in time *exactly* k [respectively, *at most* k]:

$$A = \{\mathbf{w} : w_0 = a, \text{ and if } X_0 X_1 \cdots X_{k-1} = \mathbf{w} \text{ and } X_k = a, \text{ then } T = k\},$$

$$B = \{\mathbf{w} : w_0 = a, \text{ and if } X_0 X_1 \cdots X_{k-1} = \mathbf{w} \text{ and } X_k = a, \text{ then } T \leqslant k\}.$$

Note that $A \subset B$. For the reverse tour we have the corresponding notation

$$A^* = \{\mathbf{w} : w_0 = a, \text{ and if } X_0 X_1 \cdots X_{k-1} = \mathbf{w} \text{ and } X_k = a, \text{ then } T^* = k\},$$

$$B^* = \{\mathbf{w} : w_0 = a, \text{ and if } X_0 X_1 \cdots X_{k-1} = \mathbf{w} \text{ and } X_k = a, \text{ then } T^* \leqslant k\}.$$

The only trivial relation involving time reversal is that $\mathbf{w} \in B$ if and only if $f(\mathbf{w}) \in B^*$. Our example above corresponds to $k = 10, \mathbf{w} = adabacbadc \in A, f(\mathbf{w}) = acdabcabad \in B^* \setminus A^*$.

We define $J : B \to \{0, 1, \ldots, k-1\}$ by

$$\text{for } \mathbf{w} \in B, \quad J(\mathbf{w}) = \max\{j < k : r^j(\mathbf{w}) \in B\}.$$

Note that $j = 0$ is always in the set whose maximum is taken. Since B requires that a word starts with a, the only values j which need be examined are those with $w_j = a$. In our example, $\mathbf{w} = adabacbadc$, these are $j = 0, 2, 4$, and 7. Here $r^7(\mathbf{w}) = adcadabacb \notin B$, and $r^4(\mathbf{w}) = acbadcadab \in B$, so $j = 4$. Note that in this example, $r^4(\mathbf{w})$ completes the tour at time 8, so $r^4(\mathbf{w}) \notin A$.

For $\mathbf{w} \in B$, let $J = J(\mathbf{w})$ so that $r^J(\mathbf{w}) \in B$, and hence $f(r^J(\mathbf{w})) \in B^*$. Thus we define a map

$$g = f \circ r^J : B \to B^*,$$

with $g(\mathbf{w}) = f(r^{J(\mathbf{w})}(\mathbf{w}))$. Using the reverse tour in place of the original tour, we define

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maps J^* and g^* on B^*. In detail, for $\mathbf{w} \in B^*, J^*(\mathbf{w}) = \max\{j < k : r^j(\mathbf{w}) \in B^*\}$, and $g^*(\mathbf{w}) = f(r^{J^*(\mathbf{w})}(\mathbf{w})) \in B$.

Lemma 1. *For $\mathbf{w} \in B$, let $j = J(\mathbf{w}), \mathbf{w}^* = g(\mathbf{w})$, and $j^* = J^*(\mathbf{w}^*)$. Then $j^* = j$, and hence g and g^* are inverse to each other.*

Proof. First, since $j = J(\mathbf{w})$, we have $r^j(\mathbf{w}^*) = r^j \circ f \circ r^j(\mathbf{w}) = f(\mathbf{w}) \in B^*$, and hence $j^* \geqslant j$. Now assume, to establish a contradiction, that $j < j^* < k$. Since $j^* = J^*(\mathbf{w}^*)$, we have $r^{j^*}(\mathbf{w}^*) \in B^*$, hence B contains $f \circ r^{j^*}(\mathbf{w}^*) = f \circ r^{j^*} \circ f \circ r^j(\mathbf{w}) = r^{j-j^*}(\mathbf{w}) = r^{k+j-j^*}(\mathbf{w})$. Since $j < k + j - j^* < k$, this contradicts the maximality involved in $j = J(\mathbf{w})$. Thus it is not the case that $j < j^* < k$, and since always $j^* < k$, we conclude $j = j^*$. Thus $g^*(\mathbf{w}^*) = f \circ r^{j^*}(\mathbf{w}^*) = f \circ r^j \circ f \circ r^j(\mathbf{w}) = \mathbf{w}$, i.e. $g^* \circ g$ is the identity on B. The same result, applied to the reverse tour, is that $g \circ g^*$ is the identity on B^*. □

Lemma 2. *If $\mathbf{w} \in A$, then $g(\mathbf{w}) \in A^*$. Hence g is a bijection between A and A^*.*

Proof. Assume $\mathbf{w} \in A$ and let $j = J(\mathbf{w}), \mathbf{w}^* = g(\mathbf{w})$. Assume, in order to derive a contradiction, that $\mathbf{w}^* \in B^* \setminus A^*$. The reverse tour can be completed in $\mathbf{w}^*$ in time $0 < l < k$. There are two cases, and it helps to draw a picture for each. If $l \leqslant j$, then $w_j w_{j-1} \cdots w_0$ contains $w_j w_{j-1} \cdots w_{j-l}$ which contains a reverse tour, so that $w_0 w_1 \cdots w_j$ contains a tour, implying $\mathbf{w} \in B \setminus A$. If $l > j$, then $w_j w_{j-1} \cdots w_{j-l}$ contains a reverse tour, so $w_{j-l} \cdots w_0 w_1 \cdots w_j$ contains a tour, hence $r^{k-(l-j)}(\mathbf{w}) \in B$, and $j < k - (l - j) < k$ contradicts $j = J(\mathbf{w})$. This shows that $g : A \to A^*$. The same result, applied to the reverse tour, states that $g^* : A^* \to A$. □

The cycle tour property is sometimes stated in terms of not necessarily reversible Markov chains. If the Markov chain $X_0, X_1, \ldots$ has stationary probabilities π_i and transition probabilities p_{ij}, then the reversed chain $X_0^*, X_1^*, \ldots$ has the same stationary probabilities, and transition probabilities p_{ij}^* such that $\pi_i p_{ij} = \pi_j p_{ji}^*$. In this setup, T^* is defined as the time for $X_0^*, X_1^*, \ldots$ to complete a reversed tour.

Theorem 1. *For any specified tour $i_0 = a, i_1, \ldots, i_m, i_0$, with both the original chain and the reversed chain starting from state a, T and T^* have the same distribution.*

Proof. Fix $k > 0$. For $\mathbf{w} \in B, \mathbf{P}_a(X_0 X_1 \cdots X_{k-1} = \mathbf{w},\ X_k = a) = \mathbf{P}_a(X_0^* X_1^* \cdots X_{k-1}^* = g(\mathbf{w}), X_k^* = a)$. Summing over all $\mathbf{w} \in A$ yields $\mathbf{P}_a(T = k) = \mathbf{P}_a(T^* = k)$. □

We observe that if we had argued using only B and B^*, without Lemma 2, the result deduced would not be $\mathbf{P}_a(T \leqslant k) = \mathbf{P}_a(T^* \leqslant k)$, but rather $\mathbf{P}_a(T \leqslant k, X_k = a) = \mathbf{P}_a(T^* \leqslant k, X_k^* = a)$.

Acknowledgements

Work supported by National Science Foundation grant DMS 90-05833. We thank Jim Propp for helpful comments on an earlier draft.

https://doi.org/10.1017/S0963548397003076 Published online by Cambridge University Press

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