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ASYMPTOTIC BEHAVIOR OF EXTREMAL EVENTS FOR AGGREGATE DEPENDENT RANDOM VARIABLES

DIE CHEN
Department of Statistics and Finance
School of Management
University of Science and Technology of China
Hefei, Anhui 230026, China

Department of School of Mathematical Sciences
University of Electronic Science and Technology of China
Chengdu, Sichuan 611731, China
E-mail: chendie@mail.ustc.edu.cn

TIANTIAN MAO and TAIZHONG HU
Department of Statistics and Finance,
School of Management
University of Science and Technology of China
Hefei, Anhui 230026, China
E-mails: tmao@ustc.edu.cn; thu@ustc.edu.cn

Consider a portfolio of n identically distributed risks $X_1, \ldots, X_n$ with dependence structure modelled by an Archimedean survival copula. It is known that the probability of a large aggregate loss of $\sum_{i=1}^{n} X_i$ is in proportion to the probability of a large individual loss of X_1. The proportionality factor depends on the dependence strength and the tail behavior of the individual risk. In this paper, we establish analogous results for an aggregate loss of the form $g(X_1, \ldots, X_n)$ under the more general model in which the X_i's have different but tail-equivalent distributions and the copula remains unchanged, where g is a homogeneous function of order 1. Properties of these factors are studied, and asymptotic Value-at-Risk behaviors of functions of dependent risks are also given. The main results generalize those in Wüthrich [16], Alink, Löwe, and Wüthrich [2], Barbe, Fougères, and Genest [4], and Embrechts, Nešlehová, and Wüthrich [9].

1. INTRODUCTION

Consider a portfolio of n identically distributed dependent risks $X_1, \ldots, X_n$ with a common marginal distribution function F. The dependence structure of $\mathbf{X} = (X_1, \ldots, X_n)$ is modelled by copulas. Copulas are one approach that give a detailed picture of dependence structures. We assume that $-\mathbf{X}$ has an Archimedean copula with generator ψ, which is regularly varying at zero with index $-\alpha < 0$. Wüthrich [16] and Alink, Löwe and Wüthrich [2] proved that the probability of a large aggregate loss of $\sum_{i=1}^{n} X_i$ scales like the probability of

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a large individual loss of X_1, times a proportionality factor. This factor, also termed as limiting constant, is different for the cases where the tail behavior lies in the maximum domain of attraction of the Fréchet, the Weibull, or the Gumbel distribution. The corresponding limiting constants are denoted by q_n^{F}, q_n^{W}, and q_n^{G}, which depend on the dependence strength and the tail behavior of the individual risk. For example, in the Fréchet case,

$$\mathsf{P}\left(\sum_{i=1}^{n} X_i > u\right) \sim q_n^{\mathrm{F}} \cdot \mathsf{P}(X_1 > u), \quad \text{as } u \to \infty.$$

For $n > 2$, Barbe, Fougères and Genest [4] and Embrechts, Nešlehová and Wüthrich [9] gave the properties of the limiting constant q_n^{F} for the Fréchet case, and Chen et al. [6] investigate properties of the limiting constants q_n^{W} and q_n^{G} for the Weibull and Gumbel cases.

In this paper, we establish analogous results for an aggregate loss of the form $g(X_1, \ldots, X_n)$ under the more general model in which the X_i's have different but tail-equivalent distributions (that is, satisfying (3.8)) and the copula remains unchanged, where g is an increasing and homogeneous function of order 1. The corresponding limiting constants are denoted by q_g^{F}, q_g^{W}, and q_g^{G} according to whether the tail behavior belongs to the maximum domain of attraction of the Fréchet, the Weibull, or the Gumbel distribution, respectively. Properties of these limiting constants are studied, and asymptotic Value-at-Risk behaviors of functions of dependent risks are also given. The main results generalize those in Wüthrich [16], Alink et al. [2], Barbe et al. [4], and Embrechts et al. [9].

This paper is organized as follows. Section 2 recalls the concept of regular variation, the notion of Archimedean copula, the definition of the supermodular order, and the maximum domains of attraction. Section 3 gives our main results concerning the existence and expressions of the limiting constants q_g^{F}, q_g^{W} and q_g^{G}. Section 4 investigates properties of these limiting constants. In Section 5, we discuss asymptotic Value-at-Risk behavior of functions of dependent risks. Section 6 is an appendix where we prove our results.

Throughout, the terms "increasing"and "decreasing"mean "non-decreasing"and "non-increasing", respectively. $x_+ = \max\{x, 0\}$ for any $x \in \mathbb{R}$. For any random variable X with distribution function F, denote by ω_F the right endpoint of the support of X. Here $\omega_F \leq +\infty$.

2. PRELIMINARIES

2.1. Regular Variation

We recall the notion of regular variation. Standard references on regular variation are Bingham, Goldie, and Teugels [5], de Haan and Ferreira [7], and Resnick [14].

A measurable function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be regularly varying at infinity with index $\alpha \in \mathbb{R}\backslash\{0\}$, written as $h \in \mathrm{RV}_\alpha$, if, for any $x > 0$,

$$\lim_{t\to\infty} \frac{h(tx)}{h(t)} = x^\alpha. \tag{2.1}$$

If Eq. (2.1) holds with $\alpha = 0$ for any $x > 0$, then h is said to be slowly varying at infinity and written as $h \in \mathrm{RV}_0$. If Eq. (2.1) holds with $\alpha = -\infty$ for any $x > 0$, then h is said to be rapidly varying at infinity and written as $h \in \mathrm{RV}_{-\infty}$.

Similarly, one can define regular variation at 0^+ replacing $t \to \infty$ in (2.1) by $t \to 0^+$. If h is regularly varying at 0^+ with index $\alpha \in \mathbb{R}\backslash\{0\}$ [resp. slowly varying at 0^+, rapidly varying at 0^+], denote it by $h \in \mathrm{RV}_\alpha(0^+)$ [resp. $\mathrm{RV}_0(0^+)$, $\mathrm{RV}_{-\infty}(0^+)$].

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2.2. Copulas

A copula is a multivariate distribution function with all univariate uniform $(0,1)$ margins. The copula of a multivariate distribution function is the dependence function obtained by getting rid of all the information from the margins, which contains dependence information of the margins concerned. For more on copulas, see Joe [10] and Nelsen [13].

An n-dimensional Archimedean copula is often defined as a copula of the form

$$C^{\psi}(\mathbf{u}) = \psi^{-1}\left(\sum_{i=1}^{n}\psi(u_i)\right), \quad \mathbf{u}\in[0,1]^n,$$

where $\psi : [0,1] \to [0,\infty]$ is a strictly decreasing, convex function such that $\psi(1) = 0$. An extra condition is needed to ensure that C^{ψ} is indeed a distribution function for $n \geq 3$; see McNeil and Nešlehová [11]. The function ψ is called a generator of C^{ψ}.

A sufficient condition for C^{ψ} to be a copula is that the function ψ^{-1}, the inverse of ψ, is the Laplace transform of some positive random variable. For example, $\psi_{\alpha}(t) = t^{-\alpha} - 1$ with $\alpha > 0$ is a generator of the following Clayton copula with parameter α:

$$C_{\alpha}^{\mathrm{Cl}}(\mathbf{u}) = \left(\sum_{i=1}^{n} u_i^{-\alpha} - (n-1)\right)^{-1/\alpha}, \quad \mathbf{u}\in(0,1)^n.$$

Here, $\psi_{\alpha} \in \mathrm{RV}_{-\alpha}(0^+)$, and $\psi_{\alpha}^{-1}(u) = (1+u)^{-1/\alpha}$ is the Laplace transform of the Gamma distribution with scale parameter 1 and shape parameter $1/\alpha$. The Clayton copula will be used in the proof of Theorem 4.1.

2.3. The Supermodular Order

A real-valued function $h : \mathbb{R}^n \to \mathbb{R}$ is said to be supermodular if

$$h(\mathbf{x}\vee\mathbf{y}) + h(\mathbf{x}\wedge\mathbf{y}) \geq h(\mathbf{x}) + h(\mathbf{y}), \quad \forall\,\mathbf{x},\mathbf{y}\in\mathbb{R}^n.$$

Here $\vee$ and $\wedge$ denote, respectively, the componentwise maximum and the componentwise minimum. A function $h(x_1,\ldots,x_n)$ is supermodular if and only if $h(\ldots,x_i,\ldots,x_j,\ldots)$ is supermodular in (x_i,x_j) for any $i \neq j$ with the other variables held fixed. If h has continuous second partial derivatives, then the supermodularity of h is equivalent to $(\partial^2/\partial x_i\partial x_j)h(\mathbf{x}) \geq 0$ for all $1 \leq i \neq j \leq n$ and all $\mathbf{x}\in\mathbb{R}^n$. h is said to be submodular if $-h$ is supermodular.

Definition 2.1: *Let* $\mathbf{X} = (X_1,\ldots,X_n)$ *and* $\mathbf{Y} = (Y_1,\ldots,Y_n)$ *be two random vectors.* $\mathbf{X}$ *is said to be less supermodularly dependent than* $\mathbf{Y}$*, denoted by* $\mathbf{X} \leq_{\mathrm{sm}} \mathbf{Y}$*, if* $\mathsf{E}[h(\mathbf{X})] \leq \mathsf{E}[h(\mathbf{Y})]$ *holds for all supermodular functions* $h : \mathbb{R}^n \to \mathbb{R}$ *for which the expectations exist.*

For more on the supermodular order, one can refer to Müller and Stoyan (2002, Section 3.9), Shaked and Shanthikumar (2007, Section 9.A), and references therein. The following lemma will be useful in the proof of Theorem 4.1.

Lemma 2.2: (Embrechts et al. [9]) *Assume that* $-\mathbf{X}$ *and* $-\mathbf{Y}$ *have a Clayton copula with respective parameters* $\alpha_X > 0$ *and* $\alpha_Y > 0$*. If* $\alpha_X < \alpha_Y$*, then* $\mathbf{X} \leq_{\mathrm{sm}} \mathbf{Y}$*.*

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2.4. Maximum Domains of Attraction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with common distribution function F. Consider the sequence of partial maxima

$$M_n = \max\{X_1, \ldots, X_n\}, \quad n \geq 1.$$

The Fisher–Tippett theorem gives the only possible limit laws for maxima M_n when properly normalized and centered (see Theorem 3.2.3 in Embrechts, Klüppelberg, and Mikosch [8]): If there exist constants $c_n > 0$ and $d_n \in \mathbb{R}$ and some non-degenerate distribution function H such that

$$\frac{M_n - d_n}{c_n} \xrightarrow{\mathrm{d}} H,$$

where $\xrightarrow{\mathrm{d}}$ means convergence in distribution, then H belongs to the type of one of the following three distribution functions

$$\text{Fréchet}: \quad \Phi_\alpha(x) = \begin{cases} 0, & x \leq 0, \\ \exp\{-x^{-\alpha}\}, & x > 0, \end{cases} \quad \alpha > 0,$$

$$\text{Weibull}: \quad \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\}, & x \leq 0, \\ 1, & x > 0, \end{cases} \quad \alpha > 0,$$

$$\text{Gumbel}: \quad \Lambda(x) = \exp\{-e^{-x}\}, \quad x \in \mathbb{R}.$$

The distribution function F is said to belong to the maximum domain of attraction of the extreme distribution H, denoted by $F \in \mathrm{MDA}(H)$.

The next proposition gives characterizations of the MDAs of extreme value distributions.

Proposition 2.3 (Embrechts et al. [8], Section 3.3):

(i) $F \in \mathrm{MDA}(\Phi_\alpha) \iff \overline{F} \in \mathrm{RV}_{-\alpha}$.

(ii) $F \in \mathrm{MDA}(\Psi_\alpha) \iff \omega_F < \infty,\ \overline{F}_*(x) = \overline{F}(\omega_F - 1/x) \in \mathrm{RV}_{-\alpha}$.

(iii) $F \in \mathrm{MDA}(\Lambda)$ *if and only if there exists some positive function* $a(u)$ *such that*

$$\lim_{u \uparrow \omega_F} \frac{\overline{F}(u + a(u)t)}{\overline{F}(u)} = e^{-t}, \quad t \in \mathbb{R}. \tag{2.2}$$

One possible choice of the auxiliary function $a(t)$ *is* $m(t)$*, the mean residual life function of* X *with distribution* F*, given by* $m(t) = \mathsf{E}[X - t | X > t]$ *for* $t < \omega_F$.

3. ASYMPTOTIC BEHAVIOR OF TAIL PROBABILITIES

Model 3.1: *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *be a random vector which satisfies that*

- X_i *has a continuous distribution function* $F_i(x)$ *for each* i*;*
- $\mathbf{X}$ *has an Archimedean survival copula with generator* ψ*, that is,* $-\mathbf{X}$ *has an Archimedean copula with generator* ψ*;*
- $\psi \in \mathrm{RV}_{-\alpha}(0^+)$ *for* $\alpha > 0$.

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Recall that a function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous of order 1 if $g(\lambda \mathbf{x}) = \lambda g(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}_+$.

THEOREM 3.2: *Let $g : \mathbb{R}^n \to \mathbb{R}$ be an increasing homogeneous function of order 1 such that $g(\mathbf{1}) > 0$, where $\mathbf{1} = (1, \ldots, 1)$, and let F be a baseline distribution function satisfying*

$$\lim_{x \uparrow \omega_F} \frac{\overline{F}_i(x)}{\overline{F}(x)} = 1 \quad \text{for } i = 1, \ldots, n. \tag{3.1}$$

Under Model 3.1 with $n \geq 2$, there exist positive constants $q_g^{\mathrm{F}}(\alpha, \beta)$, $q_g^{\mathrm{W}}(\alpha, \beta)$ and $q_g^{\mathrm{G}}(\alpha)$ such that following holds true:

(i) (The Fréchet case) *If $F \in \mathrm{MDA}(\Phi_\beta)$ with $\beta > 0$, then*

$$lim_{u \to \infty} \frac{1}{\overline{F}(u)} \, \mathsf{P}\big(g(\mathbf{X}) > u\big) = q_g^{\mathrm{F}}(\alpha, \beta) \tag{3.2}$$

with

$$q_g^{\mathrm{F}}(\alpha, \beta) = \int_{\mathbb{R}_+^n} 1_{\left\{g(1/x_1, \ldots, 1/x_n) > 1\right\}} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \left(\sum_{i=1}^n x_i^{-\alpha\beta} \right)^{-1/\alpha} dx_1 \cdots dx_n. \tag{3.3}$$

(ii) (The Weibull case) *If $F \in \mathrm{MDA}(\Psi_\beta)$ with $\beta > 0$, then*

$$\lim_{u \to \infty} \frac{1}{\overline{F}(\omega_F - 1/u)} \, \mathsf{P}\left(g(\omega_F \mathbf{1} - \mathbf{X}) \leq \frac{1}{u} \right) = q_g^{\mathrm{W}}(\alpha, \beta) \tag{3.4}$$

with

$$q_g^{\mathrm{W}}(\alpha, \beta) = \int_{\mathbb{R}_+^n} 1_{\left\{g(x_1, \ldots, x_n) \leq 1\right\}} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \left(\sum_{i=1}^n x_i^{-\alpha\beta} \right)^{-1/\alpha} dx_1 \cdots dx_n. \tag{3.5}$$

(iii) (The Gumbel case) *If $F \in \mathrm{MDA}(\Lambda)$ with an auxiliary function $a(u)$, then*

$$\lim_{u \uparrow \omega_F} \frac{1}{\overline{F}(u)} \, \mathsf{P}\big(g(u\mathbf{1} - \mathbf{X}) \leq a(u)\big) = q_g^{\mathrm{G}}(\alpha) \tag{3.6}$$

with

$$q_g^{\mathrm{G}}(\alpha) = \int_{\mathbb{R}^n} 1_{\left\{g(x_1, \ldots, x_n) \leq 1\right\}} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \left(\sum_{i=1}^n e^{-\alpha x_i} \right)^{-1/\alpha} dx_1 \cdots dx_n. \tag{3.7}$$

The proof of Theorem 3.2 is given in the appendix. Properties of limiting constants $q_g^{\mathrm{F}}(\alpha, \beta)$, $q_g^{\mathrm{W}}(\alpha, \beta)$ and $q_g^{\mathrm{G}}(\alpha)$ will be studied in Section 4. Wüthrich [16] and Alink et al. [2] considered the special case of Theorem 3.2 with $g(\mathbf{x}) = \sum_{i=1}^n x_i$ under the additional assumption that all the X_i's are identically distributed.

There are various examples of increasing homogeneous functions of order 1. The following are some elementary examples:

- $g_1(\mathbf{x}) = \sum_{i=1}^n c_i x_i$, where $c_i > 0$ for each i;
- $g_2(\mathbf{x}) = \min\{x_1, \ldots, x_n\}$;

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- $g_3(\mathbf{x}) = \max\{x_1, \ldots, x_n\}$;
- $g_4(\mathbf{x}) = \prod_{i=1}^n (x_i \vee 0)^{\alpha_i}$, where $\alpha_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$.

For example, choosing $g(\mathbf{x}) = \sum_{i=1}^n c_i x_i$ in Theorem 3.2, we have the following result.

COROLLARY 3.3: *Let $\mathbf{c} = (c_1, \ldots, c_n)$ be a vector with $c_i > 0$ for each i. Under the conditions of Theorem 3.2, the following holds true:*

(i) (The Fréchet case) *If $F \in \mathrm{MDA}(\Phi_\beta)$ with $\beta > 0$, then*

$$\lim_{u\to\infty} \frac{1}{\overline{F}(u)} \mathsf{P}\left(\sum_{i=1}^n c_i X_i > u\right) = q_{\mathbf{c}}^{\mathrm{F}}(\alpha, \beta)$$

with

$$q_{\mathbf{c}}^{\mathrm{F}}(\alpha, \beta) = \int_{\mathbb{R}_+^n} 1_{\left\{\sum_{i=1}^n c_i/x_i > 1\right\}} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \left(\sum_{i=1}^n x_i^{-\alpha\beta}\right)^{-1/\alpha} d\mathbf{x}.$$

(ii) (The Weibull case) *If $F \in \mathrm{MDA}(\Psi_\beta)$ with $\beta > 0$, then*

$$\lim_{u\to\infty} \frac{1}{\overline{F}(\omega_F - 1/u)} \mathsf{P}\left(\sum_{i=1}^n c_i X_i > \sigma_{\mathbf{c}}\, \omega_F - \frac{1}{u}\right) = q_{\mathbf{c}}^{\mathrm{W}}(\alpha, \beta)$$

with $\sigma_{\mathbf{c}} = \sum_{i=1}^n c_i$ and

$$q_{\mathbf{c}}^{\mathrm{W}}(\alpha, \beta) = \int_{\mathbb{R}_+^n} 1_{\left\{\sum_{i=1}^n c_i x_i \le 1\right\}} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \left(\sum_{i=1}^n x_i^{-\alpha\beta}\right)^{-1/\alpha} d\mathbf{x}.$$

(iii) (The Gumbel case) *If $F \in \mathrm{MDA}(\Lambda)$, then*

$$\lim_{u\uparrow\omega_F} \frac{1}{\overline{F}(u)} \mathsf{P}\left(\sum_{i=1}^n c_i X_i > \sigma_{\mathbf{c}} u\right) = q_{\mathbf{c}}^{\mathrm{G}}(\alpha) \cdot e^{-1/\sigma_{\mathbf{c}}}$$

with

$$q_{\mathbf{c}}^{\mathrm{G}}(\alpha) = \int_{\mathbb{R}^n} 1_{\left\{\sum_{i=1}^n c_i x_i \le 1\right\}} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \left(\sum_{i=1}^n e^{-\alpha x_i}\right)^{-1/\alpha} d\mathbf{x}.$$

The first two parts of the following corollary are special consequences of Theorem 3.2 (i) and (ii) by choosing appropriate homogeneous functions g, respectively. The first part is due to Alink [1]. The third part can not be obtained from Theorem 3.2(iii) directly. However, it can be proved by using a similar argument to that in the proof of Theorem 3.2(iii) by using Lemma A.2 instead of Lemma A.1(iii). For completeness, we state this result here.

COROLLARY 3.4: *Under Model 3.1 with $n \ge 2$, assume that there exists a baseline distribution function F satisfying*

$$\lim_{x\uparrow\omega_F} \frac{\overline{F}_i(x)}{\overline{F}(x)} = c_i \in (0, \infty) \quad \textit{for } i = 1, \ldots, n. \tag{3.8}$$

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(i) (The Fréchet case) *If* $F \in \mathrm{MDA}(\Phi_\beta)$ *with* $\beta > 0$, *then*

$$\lim_{u\to\infty} \frac{1}{\overline{F}(u)} \mathsf{P}\left(\sum_{i=1}^{n} X_i > u\right) = q^{\mathrm{F}}_{\mathbf{c}*}(\alpha,\beta) \tag{3.9}$$

with $\mathbf{c} = (c_1, \ldots, c_n)$ *and*

$$q^{\mathrm{F}}_{\mathbf{c}*}(\alpha,\beta) = \int_{\mathbb{R}^n_+} 1_{\left\{\sum_{i=1}^n 1/x_i > 1\right\}} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \left(\sum_{i=1}^{n} x_i^{-\alpha\beta} c_i^{-\alpha}\right)^{-1/\alpha} d\mathbf{x}.$$

(ii) (The Weibull case) *If* $F \in \mathrm{MDA}(\Psi_\beta)$ *with* $\beta > 0$, *then*

$$\lim_{u\to\infty} \frac{1}{\overline{F}(\omega_F - 1/u)} \mathsf{P}\left(\sum_{i=1}^{n} X_i > n\omega_F - \frac{1}{u}\right) = q^{\mathrm{W}}_{\mathbf{c}*}(\alpha,\beta) \tag{3.10}$$

with

$$q^{\mathrm{W}}_{\mathbf{c}*}(\alpha,\beta) = \int_{\mathbb{R}^n_+} 1_{\left\{\sum_{i=1}^n x_i \le 1\right\}} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \left(\sum_{i=1}^{n} x_i^{-\alpha\beta} c_i^{-\alpha}\right)^{-1/\alpha} d\mathbf{x}.$$

(iii) (The Gumbel case) *If* $F \in \mathrm{MDA}(\Lambda)$, *then*

$$\lim_{u\uparrow\omega_F} \frac{1}{\overline{F}(u)} \mathsf{P}\left(\sum_{i=1}^{n} X_i > nu\right) = q^{\mathrm{G}}_{\mathbf{c}*}(\alpha) \cdot e^{-1/n}$$

with

$$q^{\mathrm{G}}_{\mathbf{c}*}(\alpha) = \int_{\mathbb{R}^n} 1_{\left\{\sum_{i=1}^n x_i \le 1\right\}} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \left(\sum_{i=1}^{n} c_i^{-\alpha} e^{-\alpha x_i}\right)^{-1/\alpha} d\mathbf{x}.$$

Equation (3.8) defines a tail-equivalence relation on the set of all distributions. If (3.8) holds, then F_i and F have the same right endpoint. For example, F is the unit exponential distribution, and F_i is the logistic distribution function defined by $F_i(x) = 1/(1+e^{-x})$ for $x \in \mathbb{R}$. Then $\overline{F}_i(x)/\overline{F}(x) \to 1$ as $x \to \infty$.

4. PROPERTIES OF LIMITING CONSTANTS

Under Model 3.1 with $n \ge 2$, Barbe et al. [4], Embrechts et al. [9] and Chen et al. [6] have obtained the monotonicities and boundary values of the limiting constants $q^{\mathrm{F}}_g(\alpha,\beta)$, $q^{\mathrm{W}}_g(\alpha,\beta)$ and $q^{\mathrm{G}}_g(\alpha)$ when $g(\mathbf{x}) = \sum_{i=1}^n x_i$ and the X_i's are identically distributed. In this section, we will investigate properties of these limiting constants for general homogeneous functions g.

4.1. Monotonicities

THEOREM 4.1: *Let* $q^{\mathrm{F}}_g(\alpha,\beta)$ *and* $q^{\mathrm{W}}_g(\alpha,\beta)$ *be as defined by* (3.3) *and* (3.5), *respectively, with* $\alpha > 0$ *and* $\beta > 0$. *Under the conditions of Theorem* 3.2, *we have the following.*

(i) The Fréchet case

- *For* $\beta > 1$, $q^{\mathrm{F}}_g(\alpha,\beta)$ *is increasing in* α *if* g *is supermodular;*

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- *For $\beta < 1$, $q_g^{\mathrm{F}}(\alpha, \beta)$ is decreasing in α if g is submodular;*
- *$q_g^{\mathrm{F}}(\alpha, \beta)$ is increasing in β for each α if $g(x_1^{1/\beta}, \ldots, x_n^{1/\beta}) = 1$ on $\mathbb{R}_+^n$ implies $x_i \le 1$ for all i;*
- *$q_g^{\mathrm{F}}(\alpha, \beta)$ is decreasing in β for each α if $g(x_1^{1/\beta}, \ldots, x_n^{1/\beta}) = 1$ on $\mathbb{R}_+^n$ implies $x_i \ge 1$ for all i.*

(ii) The Weibull case

- *$q_g^{\mathrm{W}}(\alpha, \beta)$ is increasing in α for each β if g is submodular;*
- *$q_g^{\mathrm{W}}(\alpha, \beta)$ is increasing in β for each α if $g(x_1^{-1/\beta}, \ldots, x_n^{-1/\beta}) = 1$ on $\mathbb{R}_+^n$ implies $x_i \le 1$ for all i;*
- *$q_g^{\mathrm{W}}(\alpha, \beta)$ is decreasing in β for each α if $g(x_1^{-1/\beta}, \ldots, x_n^{-1/\beta}) = 1$ on $\mathbb{R}_+^n$ implies $x_i \ge 1$ for all i.*

It is still unknown whether there is monotonicity of the limiting constant $q_g^{\mathrm{G}}(\alpha)$ for the the Gumbel case under some condition on g.

Remark 4.2: Monotonicities of the limiting constants q_g^{F} and q_g^{W} for some special functions g are as follows:

(1) Choose $g_1(\mathbf{x}) = \sum_{i=1}^n c_i x_i$ with $c_i \ge 1$ for each i. Then $g_1(\mathbf{x})$ is both supermodular and submodular. Moreover,

$$g_1(x_1^{1/\beta}, \ldots, x_n^{1/\beta}) = 1,\ \mathbf{x} \in \mathbb{R}_+^n \Longrightarrow x_i \le 1,\ i = 1, \ldots, n,$$

and

$$g_1(x_1^{-1/\beta}, \ldots, x_n^{-1/\beta}) = 1,\ \mathbf{x} \in \mathbb{R}_+^n \Longrightarrow x_i \ge 1,\ i = 1, \ldots, n.$$

By Theorem 4.1, we have

- $q_{g_1}^{\mathrm{F}}(\alpha, \beta)$ is increasing in α for $\beta > 1$, decreasing in α for $\beta < 1$, and increasing in β for each α;
- $q_{g_1}^{\mathrm{W}}(\alpha, \beta)$ is increasing in α for each β, and decreasing in β for each α.

(2) Choose $g_2(\mathbf{x}) = \min\{x_1, \ldots, x_n\}$. Since $g_2(\mathbf{x})$ is supermodular, by Theorem 4.1, we have

- $q_{g_2}^{\mathrm{F}}(\alpha, \beta)$ is increasing in α for $\beta > 1$, and decreasing in β for each α;
- $q_{g_2}^{\mathrm{W}}(\alpha, \beta)$ is increasing in β for each α.

(3) Choose $g_3(\mathbf{x}) = \max\{x_1, \ldots, x_n\}$. Since $g_3(\mathbf{x})$ is submodular, by Theorem 4.1, we have

- $q_{g_3}^{\mathrm{F}}(\alpha, \beta)$ is decreasing in α for $\beta < 1$, and increasing in β for each α;
- $q_{g_3}^{\mathrm{W}}(\alpha, \beta)$ is increasing in α for each β, and decreasing in β for each α.

(4) In Theorem 4.1, sufficient conditions are given under which $q_g^{\mathrm{F}}(\alpha, \beta)$ and $q_g^{\mathrm{W}}(\alpha, \beta)$ possess monotonicities with respect to β. However, these conditions are not necessary. For example, choose $g_4(\mathbf{x}) = \prod_{i=1}^n (x_i \vee 0)^{\alpha_i}$, where $\alpha_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$. Note that

$$g_4\left(x_1^{1/\beta}, \ldots, x_n^{1/\beta}\right) \ge 1 \Longleftrightarrow g_4\left(x_1, \ldots, x_n\right) \ge 1$$

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and

$$g_4\left(x_1^{-1/\beta},\dots,x_n^{-1/\beta}\right)\le 1 \Longleftrightarrow g_4\left(x_1^{-1},\dots,x_n^{-1}\right)\le 1.$$

Therefore, from (A.6) and (A.10), it follows that both $q^{\mathrm{F}}_{g_4}(\alpha,\beta)$ and $q^{\mathrm{W}}_{g_4}(\alpha,\beta)$ are constant in β. However, the conditions in Theorem 4.1 are not satisfied.

4.2. Boundary Values

By the polar coordinate transformation on $\mathbb{R}^n_+$,

$$\mathbf{x}\longmapsto\left(|\mathbf{x}|,\frac{\mathbf{x}}{|\mathbf{x}|}\right)\quad\text{with } |\mathbf{x}|=\sum_{i=1}^n x_i,$$

$q^{\mathrm{F}}_g(\alpha,\beta)$ in (A.6) can be written as

$$\begin{aligned}q^{\mathrm{F}}_g(\alpha,\beta)&=\int_{\mathbb{R}_+\times\mathbb{S}_{n-1}} 1_{\left\{\left[g(w_1^{1/\beta},\dots,w_n^{1/\beta})\right]^{-\beta}\le r\right\}}\,h_\alpha(\mathbf{w})\,r^{-2}dr\,d\mathbf{w}\\&=\int_{\mathbb{S}_{n-1}}\left(g(w_1^{1/\beta},\dots,w_n^{1/\beta})\right)^{\beta}h_\alpha(\mathbf{w})\,d\mathbf{w},\end{aligned}\tag{4.1}$$

where $\mathbb{S}_{n-1}=\{\mathbf{w}\in\mathbb{R}^n_+:\ |\mathbf{w}|=1\}$ is the unit simplex, and $h_\alpha(\mathbf{w})$ is defined in (A.7). Similarly, we can rewrite $q^{\mathrm{W}}_g(\alpha,\beta)$ in (A.10) and $q^{G}_g(\alpha)$ as follows:

$$q^{\mathrm{W}}_g(\alpha,\beta)=\int_{\mathbb{S}_{n-1}}\left(g(w_1^{-1/\beta},\dots,w_n^{-1/\beta})\right)^{-\beta}h_\alpha(\mathbf{w})d\mathbf{w},\tag{4.2}$$

and

$$\begin{aligned}q^{\mathrm{G}}_g(\alpha)&=\int_{\mathbb{R}^n_+}1_{\left\{g(-\log x_1,\dots,-\log x_n)\le 1\right\}}h_\alpha(\mathbf{x})d\mathbf{x}\\&=\int_{\mathbb{R}_+\times\mathbb{S}_{n-1}}1_{\left\{g(-\log w_1-\log r,\dots,-\log w_n-\log r)\le 1\right\}}h_\alpha(\mathbf{w})\,r^{-2}dr\,d\mathbf{w}.\end{aligned}$$

Since $g(\mathbf{x})$ is increasing, there exists $r_0=r_0(\mathbf{x})$ such that

$$g(-\log x_1-\log r,\dots,-\log x_n-\log r)\le 1\Longleftrightarrow r\ge r_0(\mathbf{x})\tag{4.3}$$

for any fixed $\mathbf{x}\in\mathbb{R}^n_+$. Under most circumstances, for any fixed $\mathbf{x}\in\mathbb{R}^n_+$, r_0 is a solution to the equation

$$g(-\log x_1-\log r,\dots,-\log x_n-\log r)=1.\tag{4.4}$$

Without loss of generality, we assume r_0 is the smallest one, that is,

$$r_0(\mathbf{x})=\inf\left\{r\in\mathbb{R}_+:\ g(-\log x_1-\log r,\dots,-\log x_n-\log r)=1\right\}.$$

It is easy to see that $r_0(\mathbf{x})>0$ for any $\mathbf{x}\in\mathbb{R}^n_+$. Define

$$\eta_g(\mathbf{x})=\frac{1}{r_0(\mathbf{x})}.\tag{4.5}$$

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Then

$$q_g^{\mathrm{G}}(\alpha) = \int_{\mathbb{R}_+ \times \mathbb{S}_{n-1}} 1_{\left\{r_0(\mathbf{w}) \le r\right\}} h_\alpha(\mathbf{w})\, r^{-2} dr d\mathbf{w}$$

$$= \int_{\mathbb{S}_{n-1}} \eta_g(\mathbf{w}) h_\alpha(\mathbf{w}) d\mathbf{w}. \tag{4.6}$$

Moreover, if $g(\mathbf{x})$ is homogeneous of order 1, then $\eta_g(\mathbf{x})$ is also homogeneous of order 1 as can be seem from (4.4).

With these representations (4.1), (4.2), and (4.6), we obtain the boundary values of the three limiting constants.

THEOREM 4.3: *Under the assumptions in Theorem 3.2, we have the following.*

(i) For the Fréchet case,

$$\lim_{\alpha \to 0} q_g^{\mathrm{F}}(\alpha, \beta) = \sum_{i=1}^{n} [g(\boldsymbol{e}_i)]^\beta, \quad \lim_{\alpha \to \infty} q_g^{\mathrm{F}}(\alpha, \beta) = [g(\boldsymbol{1})]^\beta,$$

where $\boldsymbol{e}_i$ is the vector with ith component 1 and other components 0.

(ii) For the Weibull case,

$$\lim_{\alpha \to 0} q_g^{\mathrm{W}}(\alpha, \beta) = \sum_{i=1}^{n} [g(\kappa_i)]^{-\beta}, \quad \lim_{\alpha \to \infty} q_g^{\mathrm{W}}(\alpha, \beta) = [g(\boldsymbol{1})]^{-\beta},$$

where κ_i is the vector with ith component 1 and other components $+\infty$.

(iii) For the Gumbel case,

$$\lim_{\alpha \to 0} q_g^{\mathrm{G}}(\alpha) = \sum_{i=1}^{n} \eta_g(\boldsymbol{e}_i), \quad \lim_{\alpha \to \infty} q_g^{\mathrm{G}}(\alpha) = \eta_g(\boldsymbol{1}),$$

where $\eta_g(\mathbf{x})$ is defined by (4.5).

PROOF: From Barbe et al. [4], it is known that $h_\alpha(\mathbf{w})/n$ is a probability density function on $\mathbb{S}_{n-1}$ for all $\alpha > 0$. Denote by H_α the corresponding distribution function. It is known from Barbe et al. [4] and Embrechts et al. [9] that

$$H_\alpha \xrightarrow{\mathrm{d}} H_\infty \ \ (\alpha \to \infty) \ \text{ and } \ H_\alpha \xrightarrow{\mathrm{d}} H_0 \ \ (\alpha \to 0),$$

where H_∞ is the degenerate distribution of a random vector concentrating all its probability mass 1 on the point $(1/n, \ldots, 1/n) \in \mathbb{S}_{n-1}$, and H_0 is the distribution of a random vector giving probability mass $1/n$ to each point in C_n, where C_n is the set of the corners of $\mathbb{S}_{n-1}$, given by $C_n = \{(1, 0, \ldots, 0),\ (0, 1, 0, \ldots, 0),\ \ldots,\ (0, \ldots, 0, 1)\}$. Therefore, from (4.1),

$$\lim_{\alpha \to 0} q_g^{\mathrm{F}}(\alpha, \beta) = \lim_{\alpha \to 0} n \int_{\mathbb{S}_{n-1}} \left(g(w_1^{1/\beta}, \ldots, w_n^{1/\beta})\right)^\beta \, dH_\alpha(\mathbf{w})$$

$$= n \int_{\mathbb{S}_{n-1}} \left(g(w_1^{1/\beta}, \ldots, w_n^{1/\beta})\right)^\beta \, dH_0(\mathbf{w}) = \sum_{i=1}^{n} [g(\boldsymbol{e}_i)]^\beta,$$

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$$\lim_{\alpha\to\infty} q_g^{\mathrm{F}}(\alpha,\beta) = n\int_{\mathbb{S}_{n-1}} \left(g(w_1^{1/\beta},\ldots,w_n^{1/\beta})\right)^{\beta}\, dH_\infty(\mathbf{w}) = [g(\mathbf{1})]^{\beta}.$$

Similarly, from (4.2) and (4.6), the desired results for the Weibull and the Gumbel cases follow. This completes the proof. ■

Remark 4.4: From (4.3) and (4.4), we can derive $\eta_g(\mathbf{x})$ or its specific values for some function g as follows:

- For $g_1(\mathbf{x}) = \sum_{i=1}^n c_i x_i$ with $c_i > 0$ for each i,

$$\eta_{g_1}(\mathbf{x}) = \exp\left\{\frac{1}{\sigma_{\mathbf{c}}}\left[1 + \sum_{i=1}^n c_i \log x_i\right]\right\}, \quad \mathbf{x}\in\mathbb{R}_+^n,$$

  where $\sigma_{\mathbf{c}} = \sum_{i=1}^n c_i$;
- For $g_2(\mathbf{x}) = \min\{x_1,\ldots,x_n\}$, $\eta_{g_2}(\mathbf{x}) = e\cdot\max\{x_1,\ldots,x_n\}$;
- For $g_3(\mathbf{x}) = \max\{x_1,\ldots,x_n\}$, $\eta_{g_3}(\mathbf{x}) = e\cdot\min\{x_1,\ldots,x_n\}$;
- For $g_4(\mathbf{x}) = \prod_{i=1}^n (x_i\vee 0)^{\alpha_i}$ with $\alpha_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$, $\eta_{g_4}(\mathbf{1}) = e$ and $\eta_{g_4}(\boldsymbol{e}_i) = 0$ for each i. It should be noted that $\eta_{g_4}(\boldsymbol{e}_i) = 0$ is obtained from (4.3) rather than (4.4).

Remark 4.5: Boundary values of the limiting constants with respect to α for some specific functions g are as follows:

- For $g_1(\mathbf{x}) = \sum_{i=1}^n c_i x_i$ with $c_i > 0$ for each i,

$$\lim_{\alpha\to 0} q_{g_1}^{\mathrm{F}}(\alpha,\beta) = \sum_{i=1}^n c_i^{\beta}, \quad \lim_{\alpha\to\infty} q_{g_1}^{\mathrm{F}}(\alpha,\beta) = \sigma_{\mathbf{c}}^{\beta},$$

$$\lim_{\alpha\to 0} q_{g_1}^{\mathrm{W}}(\alpha,\beta) = 0, \quad \lim_{\alpha\to\infty} q_{g_1}^{\mathrm{W}}(\alpha,\beta) = \sigma_{\mathbf{c}}^{-\beta},$$

  and

$$\lim_{\alpha\to 0} q_{g_1}^{\mathrm{G}}(\alpha) = 0, \quad \lim_{\alpha\to\infty} q_{g_1}^{\mathrm{G}}(\alpha) = e^{1/\sigma_{\mathbf{c}}}.$$

- For $g_2(\mathbf{x}) = \min\{x_1,\ldots,x_n\}$,

$$\lim_{\alpha\to 0} q_{g_2}^{\mathrm{F}}(\alpha,\beta) = 0, \quad \lim_{\alpha\to\infty} q_{g_2}^{\mathrm{F}}(\alpha,\beta) = 1,$$

$$\lim_{\alpha\to 0} q_{g_2}^{\mathrm{W}}(\alpha,\beta) = n, \quad \lim_{\alpha\to\infty} q_{g_2}^{\mathrm{W}}(\alpha,\beta) = 1,$$

  and

$$\lim_{\alpha\to 0} q_{g_2}^{\mathrm{G}}(\alpha) = ne, \quad \lim_{\alpha\to\infty} q_{g_2}^{\mathrm{G}}(\alpha) = e.$$

- For $g_3(\mathbf{x}) = \max\{x_1,\ldots,x_n\}$,

$$\lim_{\alpha\to 0} q_{g_3}^{\mathrm{F}}(\alpha,\beta) = n, \quad \lim_{\alpha\to\infty} q_{g_3}^{\mathrm{F}}(\alpha,\beta) = 1,$$

$$\lim_{\alpha\to 0} q_{g_3}^{\mathrm{W}}(\alpha,\beta) = 0, \quad \lim_{\alpha\to\infty} q_{g_3}^{\mathrm{W}}(\alpha,\beta) = 1,$$

  and

$$\lim_{\alpha\to 0} q_{g_3}^{\mathrm{G}}(\alpha) = 0, \quad \lim_{\alpha\to\infty} q_{g_3}^{\mathrm{G}}(\alpha) = e.$$

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- For $g_4(\mathbf{x}) = \prod_{i=1}^n (x_i \vee 0)^{\alpha_i}$ with $\alpha_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$,

$$\lim_{\alpha\to 0} q_{g_4}^{\mathrm{F}}(\alpha,\beta) = 0, \quad \lim_{\alpha\to\infty} q_{g_4}^{\mathrm{F}}(\alpha,\beta) = 1,$$
$$\lim_{\alpha\to 0} q_{g_4}^{\mathrm{W}}(\alpha,\beta) = 0, \quad \lim_{\alpha\to\infty} q_{g_4}^{\mathrm{W}}(\alpha,\beta) = 1,$$

and

$$\lim_{\alpha\to 0} q_{g_4}^{\mathrm{G}}(\alpha) = 0, \quad \lim_{\alpha\to\infty} q_{g_4}^{\mathrm{G}}(\alpha) = e.$$

Remark 4.6: Boundary values of the limiting constants with respect to β for some specific functions g are as follows:

- For $g_1(\mathbf{x}) = \sum_{i=1}^n c_i x_i$ with $c_i > 0$ for each i,

$$\lim_{\beta\to 0} q_{g_1}^{\mathrm{F}}(\alpha,\beta) = \int_{\mathbb{S}_{n-1}} \max\{w_1,\dots,w_n\}\, h_\alpha(\boldsymbol{w})\, d\mathbf{w},$$

$$\lim_{\beta\to\infty} \frac{q_{g_1}^{\mathrm{F}}(\alpha,\beta)}{n^\beta} = \begin{cases} +\infty, & \text{if } \sigma_{\mathbf{c}} > n, \\ 0, & \text{if } \sigma_{\mathbf{c}} < n, \\ \displaystyle\int_{\mathbb{S}_{n-1}} \left(\prod_{i=1}^n w_i^{c_i}\right)^{1/n} h_\alpha(\boldsymbol{w})\, d\mathbf{w}, & \text{if } \sigma_{\mathbf{c}} = n. \end{cases}$$

To see this, for fixed $\mathbf{w} \in \mathbb{S}_{n-1}$, define

$$\lambda_1(\beta) = \left(\sum_{i=1}^n c_i w_i^{1/\beta}\right)^\beta.$$

It can be checked that $\lim_{\beta\to 0} \lambda_1(\beta) = \max\{w_1,\dots,w_n\}$, and

$$\lim_{\beta\to\infty} \frac{\lambda_1(\beta)}{n^\beta} = \begin{cases} +\infty, & \text{if } \sigma_{\mathbf{c}} > n, \\ 0, & \text{if } \sigma_{\mathbf{c}} < n, \\ \left(\prod_{i=1}^n w_i^{c_i}\right)^{1/n}, & \text{if } \sigma_{\mathbf{c}} = n. \end{cases}$$

Similarly,

$$\lim_{\beta\to 0} q_{g_1}^{\mathrm{W}}(\alpha,\beta) = \int_{\mathbb{S}_{n-1}} \min\{w_1,\dots,w_n\}\, h_\alpha(\boldsymbol{w})\, d\mathbf{w},$$

$$\lim_{\beta\to\infty} \frac{q_{g_1}^{\mathrm{W}}(\alpha,\beta)}{n^{-\beta}} = \lim_{\beta\to\infty} \frac{q_{g_1}^{\mathrm{F}}(\alpha,\beta)}{n^\beta}.$$

- For $g_2(\mathbf{x}) = \min\{x_1,\dots,x_n\}$,

$$q_{g_2}^{\mathrm{F}}(\alpha,\beta) = \int_{\mathbb{S}_{n-1}} \min\{w_1,\dots,w_n\}\, h_\alpha(\boldsymbol{w})\, d\mathbf{w},$$

$$q_{g_2}^{\mathrm{W}}(\alpha,\beta) = \int_{\mathbb{S}_{n-1}} \max\{w_1,\dots,w_n\}\, h_\alpha(\boldsymbol{w})\, d\mathbf{w}$$

do not depend on β.

- For $g_3(\mathbf{x}) = \max\{x_1,\dots,x_n\}$, $q_{g_3}^{\mathrm{F}}(\alpha,\beta) = q_{g_2}^{\mathrm{W}}(\alpha,\beta)$ and $q_{g_3}^{\mathrm{W}}(\alpha,\beta) = q_{g_2}^{\mathrm{F}}(\alpha,\beta)$.
- For $g_4(\mathbf{x}) = \prod_{i=1}^n (x_i \vee 0)^{\alpha_i}$ with $\alpha_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$, both $q_{g_4}^{\mathrm{F}}(\alpha,\beta)$ and $q_{g_4}^{\mathrm{W}}(\alpha,\beta)$ are constant in β. This has been pointed out in Remark 4.2(4).

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5. PROPERTIES FOR VALUE-AT-RISK

Let X be a random variable with distribution function F. The Value-at-Risk (VaR) with respect to the level $p \in (0,1)$ is defined as the generalized inverse of F:

$$\mathrm{VaR}_p[X] = F^{-1}(p) = \inf\{t \in \mathbb{R} : \ F(t) \geq \alpha\}.$$

In this section, we investigate properties of Value-at-Risk of $g(\mathbf{X})$ and $g(\omega_F \mathbf{1} - \mathbf{X})$ for the Fréchet and the Weibull cases, respectively, where $g(\mathbf{x})$ is a homogeneous function of order 1.

THEOREM 5.1: *Denote by $\mathscr{G}$ the class of the special homogeneous function g of order* 1 *which depends on only one variable, that is,*

$$\mathscr{G} = \big\{g : \ g(\mathbf{x}) = cx_i \ \textit{for some } c \in \mathbb{R} \textit{ and } 1 \leq i \leq n\big\}.$$

Under the assumptions in Theorem 3.2*, we have*

(i) For the Fréchet case,

$$\lim_{p \to 1^-} \frac{\mathrm{VaR}_p[g(\mathbf{X})]}{g\big(\mathrm{VaR}_p[X_1], \ldots, \mathrm{VaR}_p[X_n]\big)} = \frac{[q_g^{\mathrm{F}}(\alpha,\beta)]^{1/\beta}}{g(\boldsymbol{1})}.$$

Moreover, if $\beta > 1$, $g(\mathbf{x})$ is supermodular on $\mathbb{R}_+^n$ and $g \notin \mathscr{G}$, there exists $p_0 > 0$ such that for all $p_0 < p < 1$,

$$\mathrm{VaR}_p[g(\mathbf{X})] < g\left(\mathrm{VaR}_p[X_1], \ldots, \mathrm{VaR}_p[X_n]\right); \tag{5.1}$$

and if $\beta < 1$, $g(\mathbf{x})$ is submodular on $\mathbb{R}_+^n$ and $g \notin \mathscr{G}$, there exists $p_1 > 0$ such that for all $p_1 < p < 1$,

$$\mathrm{VaR}_p[g(\mathbf{X})] > g\left(\mathrm{VaR}_p[X_1], \ldots, \mathrm{VaR}_p[X_n]\right). \tag{5.2}$$

(ii) For the Weibull case,

$$\lim_{p \to 0^+} \frac{\mathrm{VaR}_p[g(\omega_F \mathbf{1} - \mathbf{X})]}{g\big(\mathrm{VaR}_p[\omega_F - X_1], \ldots, \mathrm{VaR}_p[\omega_F - X_n]\big)} = \frac{[q_g^{\mathrm{W}}(\alpha,\beta)]^{-1/\beta}}{g(\mathbf{1})}.$$

Moreover, if $g(\mathbf{x})$ is submodular on $\mathbb{R}_+^n$ and $g \notin \mathscr{G}$, there exists $p_2 < 1$ such that for all $0 < p < p_2$,

$$\mathrm{VaR}_p[g(\omega_F \mathbf{1} - \mathbf{X})] > g\big(\mathrm{VaR}_p[\omega_F - X_1], \ldots, \mathrm{VaR}_p[\omega_F - X_n]\big). \tag{5.3}$$

Theorem 5.1 can be applied to several homogeneous functions g. The special case of Theorem 5.1(i) with $g(\mathbf{x}) = \sum_{i=1}^n x_i$ is due to Embrechts et al. [9].

APPENDIX

A.1. Proof of Theorem 3.2

By making some modifications of the proof of Theorem 2.2 in Alink et al. [2], we can give a proof of Theorem 3.2.

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LEMMA A.1: *Under the conditions of Theorem 3.2, we have*

(i) (The Fréchet case) *For* $\epsilon \in (0,1)$ *and* $\mathbf{x} \in (0, 1/\epsilon] \times \mathbb{R}_+^{n-1}$,

$$\lim_{u\to\infty} \mathsf{P}\left(X_i \geq \frac{u}{x_i},\ i = 1, \ldots, n \middle| X_1 \geq \epsilon u \right) = \left(\sum_{i=1}^{n} x_i^{-\alpha\beta} \right)^{-1/\alpha} \epsilon^{\beta}.$$

(ii) (The Weibull case) *For* $\epsilon \in (0,1)$ *and* $\mathbf{x} \in (0, 1/\epsilon] \times \mathbb{R}_+^{n-1}$,

$$\lim_{u\to\infty} \mathsf{P}\left(X_i \geq \omega_F - \frac{x_i}{u},\ i = 1, \ldots, n \middle| X_1 \geq \omega_F - \frac{1}{\epsilon u} \right) = \left(\sum_{i=1}^{n} x_i^{-\alpha\beta} \right)^{-1/\alpha} \epsilon^{\beta}.$$

(iii) (The Gumbel case) *For* $\epsilon \in (0,1)$ *and* $\mathbf{x} \in (-\infty, 1/\epsilon] \times \mathbb{R}^{n-1}$,

$$\lim_{u\uparrow\omega_F} \mathsf{P}\left(X_i \geq u - x_i a(u),\ i = 1, \ldots, n \middle| X_1 \geq u - \frac{a(u)}{\epsilon} \right) = e^{-1/\epsilon} \left(\sum_{i=1}^{n} e^{-\alpha x_i} \right)^{-1/\alpha}.$$

PROOF: We give the proof of part (i); the proofs for parts (ii) and (iii) are similar. Fix $\epsilon \in (0,1)$, $x_1 \in (0, 1/\epsilon]$ and $(x_2, \ldots, x_n) \in \mathbb{R}_+^{n-1}$. Since $\overline{F} \in \mathrm{RV}_{-\beta}$, we have

$$\lim_{u\to\infty} \frac{\overline{F}_i(u/x_i)}{\overline{F}(u)} = \lim_{u\to\infty} \frac{\overline{F}_i(u/x_i)}{\overline{F}(u/x_i)} \frac{\overline{F}(u/x_i)}{\overline{F}(u)} = x_i^{\beta}.$$

Choose $\delta > 0$ arbitrarily. Thus, for u large enough,

$$\overline{F}_i\left(\frac{u}{x_i}\right) \leq (x_i + \delta)^{\beta} \overline{F}(u) \quad \text{and} \quad (\epsilon + \delta)^{-\beta} \overline{F}(u) \leq \overline{F}_1(\epsilon u),$$

$$\psi\left((x_i + \delta)^{\beta} \overline{F}(u) \right) \geq \left((x_i + \delta)^{\beta} + \delta \right)^{-\alpha} \psi\left(\overline{F}(u) \right)$$

and

$$\sum_{i=1}^{n} \left((x_i + \delta)^{\beta} + \delta \right)^{-\alpha} \psi(\overline{F}(u)) \geq \psi\left(\left(\sum_{i=1}^{n} ((x_i + \delta)^{\beta} + \delta)^{-\alpha} - \delta \right)_+^{-1/\alpha} \overline{F}(u) \right).$$

So, we have

$$\begin{aligned}
&\limsup_{u\to\infty} \mathsf{P}\left(X_i \geq \frac{u}{x_i},\ i = 1, \ldots, n \middle| X_1 \geq \epsilon u \right) \\
&= \limsup_{u\to\infty} \frac{1}{\overline{F}_1(\epsilon u)} \psi^{-1}\left(\sum_{i=1}^{n} \psi\left(\overline{F}_i\left(\frac{u}{x_i} \right) \right) \right) \\
&\leq \limsup_{u\to\infty} \frac{1}{(\epsilon + \delta)^{-\beta} \overline{F}(u)} \psi^{-1}\left(\sum_{i=1}^{n} \psi\left((x_i + \delta)^{\beta} \overline{F}(u) \right) \right) \\
&\leq \limsup_{u\to\infty} \frac{\psi^{-1}\left(\sum_{i=1}^{n} ((x_i + \delta)^{\beta} + \delta)^{-\alpha} \psi(\overline{F}(u)) \right)}{(\epsilon + \delta)^{-\beta} \overline{F}(u)} \\
&\leq \frac{\left(\sum_{i=1}^{n} ((x_i + \delta)^{\beta} + \delta)^{-\alpha} - \delta \right)_+^{-1/\alpha}}{(\epsilon + \delta)^{-\beta}}.
\end{aligned}$$

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Letting $\delta \to 0$ yields

$$\limsup_{u\to\infty} \mathsf{P}\left(X_i \ge \frac{u}{x_i},\ i=1,\dots,n \middle| X_1 \ge \epsilon u \right) \le \left(\sum_{i=1}^n x_i^{-\alpha\beta} \right)^{-1/\alpha} \epsilon^\beta.$$

Similarly,

$$\liminf_{u\to\infty} \mathsf{P}\left(X_i \ge \frac{u}{x_i},\ i=1,\dots,n \middle| X_1 \ge \epsilon u \right) \ge \left(\sum_{i=1}^n x_i^{-\alpha\beta} \right)^{-1/\alpha} \epsilon^\beta.$$

This proves part (i). ■

For $\epsilon > 0$, $\alpha > 0$ and $\beta > 0$, define

$$H_{\epsilon,\alpha,\beta}(\mathbf{x}) = \left(\sum_{i=1}^n x_i^{-\alpha\beta} \right)^{-1/\alpha} \epsilon^\beta, \quad \mathbf{x} \in \mathbb{R}_+^n,$$

and

$$H_{\epsilon,\alpha}(\mathbf{x}) = \left(\sum_{i=1}^n e^{-\alpha x_i} \right)^{-1/\alpha} e^{-1/\epsilon}.$$

It is easy to see that $H_{\epsilon,\alpha,\beta}(\mathbf{x})$ is a distribution functions with support $(0, 1/\epsilon) \times \mathbb{R}_+^{n-1}$, while $H_{\epsilon,\alpha}(\mathbf{x})$ is a distribution function with support $(-\infty, 1/\epsilon) \times \mathbb{R}^{n-1}$. Denote by $h_{\epsilon,\alpha,\beta}$ and $h_{\epsilon,\alpha}$ the density functions of $H_{\epsilon,\alpha,\beta}$ and $H_{\epsilon,\alpha}$, respectively, and define

$$\begin{aligned} H^{\mathrm{F}}(\epsilon) &= \epsilon^{-\beta} \int_{\mathbb{R}_+^n} 1_{\left\{ g(1/x_1,\dots,1/x_n) \ge 1,\ x_1 \le 1/\epsilon \right\}} \cdot h_{\epsilon,\alpha,\beta}(\mathbf{x}) d\mathbf{x} \\ &= \int_{\mathbb{R}_+^n} 1_{\left\{ g(1/x_1,\dots,1/x_n) \ge 1,\ x_1 \le 1/\epsilon \right\}} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \left(\sum_{i=1}^n x_i^{-\alpha\beta} \right)^{-1/\alpha} d\mathbf{x} \end{aligned}$$

and

$$\begin{aligned} H^{\mathrm{G}}(\epsilon) &= e^{1/\epsilon} \int_{\mathbb{R}^n} 1_{\left\{ g(x_1,\dots,x_n) \le 1,\ x_1 \le 1/\epsilon \right\}} \cdot h_{\epsilon,\alpha}(\mathbf{x}) d\mathbf{x} \\ &= \int_{\mathbb{R}^n} 1_{\left\{ g(x_1,\dots,x_n) \le 1,\ x_1 \le 1/\epsilon \right\}} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \left(\sum_{i=1}^n e^{-\alpha x_i} \right)^{-1/\alpha} d\mathbf{x}. \end{aligned}$$

Since $H^{\mathrm{F}}(\epsilon)$ and $H^{\mathrm{G}}(\epsilon)$ are both decreasing in ϵ, we have

$$0 < q_g^{\mathrm{F}}(\alpha,\beta) = \lim_{\epsilon \downarrow 0} H^{\mathrm{F}}(\epsilon) \le \infty, \quad 0 < q_g^{\mathrm{G}}(\alpha) = \lim_{\epsilon \downarrow 0} H^{\mathrm{G}}(\epsilon) \le \infty.$$

Proof of Theorem 3.2(i): For any $u > 0$, let $\mathbf{Y}^{(u)} = \left(Y_1^{(u)}, \dots, Y_n^{(u)} \right)$ be a random vector whose distribution is the same as the conditional distribution of $(u/X_1, \dots, u/X_n)$ given $X_1 \ge \epsilon u$, that is,

$$\mathbf{Y}^{(u)} \stackrel{\mathrm{d}}{=} \left[\left(\frac{u}{X_1}, \dots, \frac{u}{X_n} \right) \middle| X_1 \ge \epsilon u \right].$$

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Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be a random vector with distribution function $H_{\epsilon,\alpha,\beta}$, and denote by $H^{(u)}$ the distribution function of $\mathbf{Y}^{(u)}$. From Lemma A.1(i), it follows that, for each $i \neq 1$,

$$\lim_{u\to\infty} \mathsf{P}(X_i \leq 0 | X_1 \geq \epsilon u) = 0.$$

Hence

$$\lim_{u\to\infty} H^{(u)}(\mathbf{x}) = 0 \text{ for some } x_i \leq 0,$$

and

$$\begin{aligned}
\lim_{u\to\infty} H^{(u)}(\mathbf{x}) &= \lim_{u\to\infty} \mathsf{P}\left[\left(\frac{u}{X_1} \leq x_1, \ldots, \frac{u}{X_n} \leq x_n \right) \middle| X_1 \geq \epsilon u \right] \\
&= \lim_{u\to\infty} \mathsf{P}\left[\left(\frac{u}{X_1} \leq x_1, X_1 > 0; \ldots; \frac{u}{X_n} \leq x_n, X_n > 0 \right) \middle| X_1 \geq \epsilon u \right] \\
&= \lim_{u\to\infty} \mathsf{P}\left(X_i \geq \frac{u}{x_i},\ i = 1, \ldots, n \middle| X_1 \geq \epsilon u \right) \\
&= H_{\epsilon,\alpha,\beta}(\mathbf{x})
\end{aligned}$$

for $\mathbf{x} \in (0, 1/\epsilon] \times \mathbb{R}_+^{n-1}$. This means that $\mathbf{Y}^{(u)}$ converges to $\mathbf{Y}$ in distribution as u tends to infinity. Thus

$$\mathsf{P}\left(g\left(\frac{1}{Y_1^{(u)}}, \ldots, \frac{1}{Y_n^{(u)}} \right) \geq 1 \right) = \mathsf{P}\left(g(\mathbf{X}) \geq u | X_1 \geq \epsilon u \right)$$

converges to

$$\mathsf{P}\left(g\left(\frac{1}{Y_1}, \ldots, \frac{1}{Y_n} \right) \geq 1 \right) = \epsilon^\beta H^{\mathrm{F}}(\epsilon).$$

For the lower bound of (3.2), we have

$$\liminf_{u\to\infty} \frac{1}{\overline{F}(u)} \mathsf{P}\left(g(\mathbf{X}) \geq u \right) \geq \liminf_{u\to\infty} \frac{\overline{F}_1(\epsilon u)}{\overline{F}(u)} \mathsf{P}\big(g(\mathbf{X}) \geq u | X_1 \geq \epsilon u \big) = H^{\mathrm{F}}(\epsilon).$$

Since $\epsilon > 0$ is arbitrary,

$$\liminf_{u\to\infty} \frac{1}{\overline{F}(u)} \mathsf{P}\left(g(\mathbf{X}) \geq u \right) \geq q_g^{\mathrm{F}}(\alpha, \beta). \tag{A.1}$$

For the upper bound of (3.2), note that

$$\begin{aligned}
\limsup_{u\to\infty} \frac{1}{\overline{F}(u)} \mathsf{P}\left(g(\mathbf{X}) \geq u \right) \leq & \limsup_{u\to\infty} \frac{1}{\overline{F}(u)} \mathsf{P}\left(g(\mathbf{X}) \geq u,\ X_1 \geq \epsilon u \right) \\
& + \limsup_{u\to\infty} \frac{1}{\overline{F}(u)} \mathsf{P}\left(g(\mathbf{X}) \geq u,\ X_1 < \epsilon u \right).
\end{aligned}$$

For the first term, we have

$$\limsup_{u\to\infty} \frac{1}{\overline{F}(u)} \mathsf{P}\left(g(\mathbf{X}) \geq u,\ X_1 \geq \epsilon u \right) = q_g^{\mathrm{F}}(\alpha, \beta).$$

For the second term, choose $0 < \epsilon < 1/g(\mathbf{1})$ and denote $X_{n:n} = \max\{X_1, \ldots, X_n\}$. Then

$$\begin{aligned}
& \limsup_{u\to\infty} \frac{1}{\overline{F}(u)} \mathsf{P}\big(g(\mathbf{X}) \geq u,\ X_1 < \epsilon u \big) \\
& \leq \limsup_{u\to\infty} \frac{1}{\overline{F}(u)} \mathsf{P}\left(g(X_{n:n}\mathbf{1}) \geq u,\ X_{n:n} > 0,\ X_1 < \epsilon u \right)
\end{aligned}$$

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$$
\begin{aligned}
&= \limsup_{u\to\infty} \frac{1}{\overline{F}(u)}\ \mathsf{P}\left(X_{n:n} g(\mathbf{1}) \geq u,\ X_1 < \epsilon u\right) \\
&\leq \limsup_{u\to\infty} \frac{1}{\overline{F}(u)} \sum_{i=2}^{n} \mathsf{P}\left(X_i \geq \frac{u}{g(\mathbf{1})},\ X_1 < \epsilon u\right) \\
&= \limsup_{u\to\infty} \frac{1}{\overline{F}(u)} \sum_{i=2}^{n} \left(\mathsf{P}\left(X_i \geq \frac{u}{g(\mathbf{1})}\right) - \mathsf{P}\left(X_i \geq \frac{u}{g(\mathbf{1})},\ X_1 \geq \epsilon u\right)\right) \\
&= \sum_{i=2}^{n} \left([g(\mathbf{1})]^{\beta} - \left([g(\mathbf{1})]^{-\alpha\beta} + \epsilon^{\alpha\beta}\right)^{-1/\alpha}\right),
\end{aligned}
$$

where the first inequality follows from the monotonicity of g and the fact that $g(\mathbf{0}) = 0$, and the last equality follows from (3.1) and Lemma A.1(i). Hence,

$$
\limsup_{u\to\infty} \frac{1}{\overline{F}(u)} \mathsf{P}\left(g(\mathbf{X}) \geq u\right) \leq q_g^{\mathrm{F}}(\alpha,\beta) + \sum_{i=2}^{n} \left([g(\mathbf{1})]^{\beta} - \left([g(\mathbf{1})]^{-\alpha\beta} + \epsilon^{\alpha\beta}\right)^{-1/\alpha}\right).
$$

Since $\epsilon > 0$ is arbitrary, setting $\epsilon \to 0^+$ yields

$$
\limsup_{u\to\infty} \frac{1}{\overline{F}(u)}\ \mathsf{P}\left(g(\mathbf{X}) \geq u\right) \leq q_g^{\mathrm{F}}(\alpha,\beta). \tag{A.2}
$$

Now, (3.2) follows from (A.1) and (A.2). This completes the proof. ■

Proof of Theorem 3.2(ii): The proof is similar to that of part (i) with modification of the definition of the random vector $\mathbf{Y}^{(u)}$. $F \in \mathrm{MDA}(\Psi_\beta)$ implies $\omega_F < \infty$. For any $u > 0$, $\mathbf{Y}^{(u)}$ is redefined by

$$
\mathbf{Y}^{(u)} \stackrel{\mathrm{d}}{=} \left[\left(u(\omega_F - X_1), \ldots, u(\omega_F - X_n)\right) \middle| X_1 \geq \omega_F - \frac{1}{\epsilon u}\right].
$$

By Lemma A.1(ii), $\mathbf{Y}^{(u)}$ converges to $\mathbf{Y}$ in distribution as u tends to infinity. The rest of the proof is omitted. ■

Proof of Theorem 3.2(iii): For any $u > 0$, let $\mathbf{Y}^{(u)}$ be a random vector satisfying

$$
\mathbf{Y}^{(u)} \stackrel{\mathrm{d}}{=} \left[\left(\frac{u - X_1}{a(u)}, \ldots, \frac{u - X_n}{a(u)}\right) \middle| \frac{u - X_1}{a(u)} \leq \frac{1}{\epsilon}\right],
$$

and let $\mathbf{Y}$ be a random vector with distribution function $H_{\epsilon,\alpha}$. From Lemma A.1(iii), it follows that $\mathbf{Y}^{(u)}$ converges to $\mathbf{Y}$ in distribution as u tends to ω_F from below. Thus

$$
\mathsf{P}\left(g(\mathbf{Y}^{(u)}) \leq 1\right) = \mathsf{P}\left(g(u\mathbf{1} - \mathbf{X}) \leq a(u) \middle| X_1 \geq u - \frac{a(u)}{\epsilon}\right)
$$

converges to

$$
\mathsf{P}\left(g(\mathbf{Y}) \leq 1\right) = e^{-1/\epsilon} H^{\mathrm{G}}(\epsilon).
$$

For the lower bound of (3.6), we have

$$
\begin{aligned}
&\liminf_{u\uparrow\omega_F} \frac{1}{\overline{F}(u)} \mathsf{P}\left(g(u\mathbf{1} - \mathbf{X}) \leq a(u)\right) \\
&\geq \liminf_{u\uparrow\omega_F} \frac{\overline{F}_1(u - a(u)/\epsilon)}{\overline{F}(u)}\ \mathsf{P}\left(g(u\mathbf{1} - \mathbf{X}) \leq a(u) \middle| X_1 \geq u - \frac{a(u)}{\epsilon}\right) = H^{\mathrm{G}}(\epsilon).
\end{aligned}
$$

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Since $\epsilon > 0$ is arbitrary,

$$\liminf_{u\uparrow\omega_F} \frac{1}{\overline{F}(u)} \mathsf{P}\left(g(u\mathbf{1}-\mathbf{X}) \le a(u)\right) \ge q_g^{\mathrm{G}}(\alpha). \tag{A.3}$$

For the upper bound of (3.6), note that

$$\begin{aligned}
&\limsup_{u\uparrow\omega_F} \frac{1}{\overline{F}(u)} \mathsf{P}\left(g(u\mathbf{1}-\mathbf{X}) \le a(u)\right) \\
&\quad \le \limsup_{u\uparrow\omega_F} \frac{1}{\overline{F}(u)} \mathsf{P}\left(g(u\mathbf{1}-\mathbf{X}) \le a(u),\ X_1 \ge u - \frac{a(u)}{\epsilon}\right) \\
&\qquad + \limsup_{u\uparrow\omega_F} \frac{1}{\overline{F}(u)} \mathsf{P}\left(g(u\mathbf{1}-\mathbf{X}) \le a(u),\ X_1 < u - \frac{a(u)}{\epsilon}\right).
\end{aligned}$$

For the first term, we have

$$\limsup_{u\uparrow\omega_F} \frac{1}{\overline{F}(u)} \mathsf{P}\left(g(u\mathbf{1}-\mathbf{X}) \le a(u),\ X_1 \ge u - \frac{a(u)}{\epsilon}\right) = q_g^{\mathrm{G}}(\alpha).$$

For the second term, choose $0 < \epsilon < g(\mathbf{1})$. Then

$$\begin{aligned}
&\limsup_{u\uparrow\omega_F} \frac{1}{\overline{F}(u)} \mathsf{P}\left(g(u\mathbf{1}-\mathbf{X}) \le a(u),\ X_1 < u - \frac{a(u)}{\epsilon}\right) \\
&\le \limsup_{u\uparrow\omega_F} \frac{1}{\overline{F}(u)} \mathsf{P}\left(g((u-X_{n:n})\mathbf{1}) \le a(u),\ X_1 < u - \frac{a(u)}{\epsilon}\right) \\
&\le \limsup_{u\uparrow\omega_F} \frac{1}{\overline{F}(u)} \left[\mathsf{P}\left(X_{n:n} \ge u,\ X_1 < u - \frac{a(u)}{\epsilon}\right)\right. \\
&\quad \left. + \mathsf{P}\left(u > X_{n:n} \ge u - \frac{a(u)}{g(\mathbf{1})},\ X_1 < u - \frac{a(u)}{\epsilon}\right)\right] \\
&= \limsup_{u\uparrow\omega_F} \frac{1}{\overline{F}(u)} \mathsf{P}\left(X_{n:n} \ge u - \frac{a(u)}{g(\mathbf{1})},\ X_1 < u - \frac{a(u)}{\epsilon}\right) \\
&\le \limsup_{u\uparrow\omega_F} \frac{1}{\overline{F}(u)} \sum_{i=2}^{n} \mathsf{P}\left(X_i \ge u - \frac{a(u)}{g(\mathbf{1})},\ X_1 < u - \frac{a(u)}{\epsilon}\right) \\
&= \limsup_{u\uparrow\omega_F} \frac{1}{\overline{F}(u)} \sum_{i=2}^{n} \left(\mathsf{P}\left(X_i \ge u - \frac{a(u)}{g(\mathbf{1})}\right) - \mathsf{P}\left(X_i \ge u - \frac{a(u)}{g(\mathbf{1})},\ X_1 \ge u - \frac{a(u)}{\epsilon}\right)\right) \\
&= \sum_{i=2}^{n} \left(e^{1/g(\mathbf{1})} - \left[e^{-\alpha/\epsilon} + e^{-\alpha/g(\mathbf{1})}\right]^{-1/\alpha}\right),
\end{aligned}$$

where the last equality follows from (3.1), $F \in \mathrm{MDA}(\Lambda)$ and Lemma A.1(i). Since $\epsilon > 0$ is arbitrary, setting $\epsilon \to 0^+$ yields

$$\limsup_{u\uparrow\omega_F} \frac{1}{\overline{F}(u)} \mathsf{P}\left(g(u\mathbf{1}-\mathbf{X}) \le a(u)\right) \le q_g^{\mathrm{G}}(\alpha). \tag{A.4}$$

Now, (3.6) follows from (A.3) and (A.4). This completes the proof. ■

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A.2. Proof of Corollary 3.4

Proof of Corollary 3.4: (i) Define $Y_i = c_i^{-1/\beta} X_i$ for each i. Then the survival function of Y_i is given by $\overline{G}_i(x) = \overline{F}_i(c_i^{1/\beta} x)$, satisfying

$$\lim_{u\to\infty} \frac{\overline{G}_i(u)}{\overline{F}(u)} = \lim_{u\to\infty} \frac{\overline{F}_i\left(c_i^{1/\beta} u\right)}{\overline{F}\left(c_i^{1/\beta} u\right)} \cdot \frac{\overline{F}\left(c_i^{1/\beta} u\right)}{\overline{F}(u)} = 1$$

because of (3.8) and $\overline{F} \in \mathrm{RV}_{-\beta}$. Note that $\mathbf{Y}$ and $\mathbf{X}$ have the same survival copula. Choosing $g(\mathbf{x}) = \sum_{i=1}^n c_i^{1/\beta} x_i$ and applying Theorem 3.2(i) to $\mathbf{Y}$, we have (3.9) with

$$q^{\mathrm{F}}_{\mathbf{c}*}(\alpha,\beta) = \int_{\mathbb{R}_+^n} 1_{\left\{\sum_{i=1}^n c_i^{1/\beta}/x_i \ge 1\right\}} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \left(\sum_{i=1}^n x_i^{-\alpha\beta}\right)^{-1/\alpha} d\mathbf{x}$$

$$= \int_{\mathbb{R}_+^n} 1_{\left\{\sum_{i=1}^n 1/t_i \ge 1\right\}} \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \left(\sum_{i=1}^n t_i^{-\alpha\beta} c_i^{-\alpha}\right)^{-1/\alpha} d\mathbf{t}.$$

This proves part (i).

(ii) By Proposition 2.3 (ii), $F \in \mathrm{MDA}(\Psi_\beta)$ implies that $\omega_F < +\infty$ and $\overline{F}_*(x) = \overline{F}(\omega_F - 1/x) \in \mathrm{RV}_{-\beta}$. Define

$$Y_i = \omega_F - c_i^{1/\beta}(\omega_F - X_i), \quad i = 1, \ldots, n,$$

and denote by G_i the distribution function of Y_i. Then

$$\lim_{x\uparrow\omega_F} \frac{\overline{G}_i(x)}{\overline{F}(x)} = \lim_{x\uparrow\omega_F} \frac{\overline{F}_i\left(\omega_F - c_i^{-1/\beta}(\omega_F - x)\right)}{\overline{F}(\omega_F - (\omega_F - x))} = \lim_{u\to\infty} \frac{\overline{F}_i\left(\omega_F - c_i^{-1/\beta}/u\right)}{\overline{F}(\omega_F - 1/u)}$$

$$= \lim_{u\to\infty} \frac{\overline{F}_i\left(\omega_F - c_i^{-1/\beta}/u\right)}{\overline{F}\left(\omega_F - c_i^{-1/\beta}/u\right)} \cdot \frac{\overline{F}\left(\omega_F - c_i^{-1/\beta}/u\right)}{\overline{F}(\omega_F - 1/u)} = 1.$$

Note that $\mathbf{Y}$ and $\mathbf{X}$ have the same survival copula. Choosing $g(\mathbf{x}) = \sum_{i=1}^n c_i^{-1/\beta} x_i$ and applying Theorem 3.2(ii) to $\mathbf{Y}$, we have (3.10) with

$$q^{\mathrm{W}}_{\mathbf{c}*}(\alpha,\beta) = \int_{\mathbb{R}_+^n} 1_{\left\{\sum_{i=1}^n c_i^{-1/\beta} x_i \le 1\right\}} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \left(\sum_{i=1}^n x_i^{-\alpha\beta}\right)^{-1/\alpha} d\mathbf{x}$$

$$= \int_{\mathbb{R}_+^n} 1_{\left\{\sum_{i=1}^n t_i \le 1\right\}} \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \left(\sum_{i=1}^n t_i^{-\alpha\beta} c_i^{-\alpha}\right)^{-1/\alpha} d\mathbf{t}.$$

This proves part (ii).

(iii) The key step is Lemma A.2 below, whose proof is similar to that of Lemma A.1(iii). The rest of the proof is omitted. ■

LEMMA A.2: *Under the conditions of Corollary 3.4, if $F \in \mathrm{MDA}(\Lambda)$ with an auxiliary function $a(\cdot)$, then for $\epsilon \in (0,1)$ and $\mathbf{x} \in (-\infty, 1/\epsilon] \times \mathbb{R}^{n-1}$,*

$$\lim_{u\uparrow\omega_F} \mathsf{P}\left(X_i \ge u - x_i a(u),\ i = 1, \ldots, n \,\middle|\, X_1 \ge u - \frac{a(u)}{\epsilon}\right) = c_1^{-1} e^{-1/\epsilon} \left(\sum_{i=1}^n c_i^{-\alpha} e^{-\alpha x_i}\right)^{-1/\alpha}.$$

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A.3. Proof of Theorem 4.1

From the representations of (3.3), (3.5), and (3.7), it is known that the three limiting constants just depend on the dependence strength and marginal tail behavior. So, to study their properties, it suffices for us to choose one specific model that satisfies the conditions of Theorem 3.2.

Proof of Theorem 4.1(i): The proof proceeds along similar lines as in Embrechts et al. [9]. Choose two random vectors $\mathbf{X}$ and $\mathbf{Y}$ such that $\mathbf{X}$ and $\mathbf{Y}$ have a Clayton survival copulas with parameters $0 < \alpha_1 < \alpha_2 < \infty$, respectively, and X_i and Y_i have the same marginal distribution function F_i for each i. By Lemma 2.2, we have $\mathbf{X} \leq_{\mathrm{sm}} \mathbf{Y}$.

First, consider the case $\beta > 1$. Suppose that $g(\mathbf{x})$ is supermodular. Then the function $(g(\mathbf{x}) - u)_+$ is also supermodular in $\mathbf{x}$ for any fixed positive u. Thus, $\mathsf{E}\left[(g(\boldsymbol{X}) - u)_+\right] \leq \mathsf{E}\left[(g(\boldsymbol{Y}) - u)_+\right]$ or

$$\int_0^\infty \mathsf{P}(g(\mathbf{X}) > u + y)dy \leq \int_0^\infty \mathsf{P}(g(\mathbf{Y}) > u + y)dy,$$

which can be written as

$$\int_0^\infty \frac{\mathsf{P}(g(\mathbf{X}) > u + y)}{\overline{F}(u + y)} \cdot \frac{\overline{F}(u + y)}{u\overline{F}(u)}dy \leq \int_0^\infty \frac{\mathsf{P}(g(\mathbf{Y}) > u + y)}{\overline{F}(u + y)} \cdot \frac{\overline{F}(u + y)}{u\overline{F}(u)}dy.$$

By Theorem 3.2(i) and Karamata's theorem (see Proposition 1.5.10 in Bingham et al. [5]), letting $u \to \infty$ yields $q_g^{\mathrm{F}}(\alpha_1, \beta) \leq q_g^{\mathrm{F}}(\alpha_2, \beta)$. This means that $q_g^{\mathrm{F}}(\alpha, \beta)$ is increasing in α for $\beta > 1$.

Next, consider the case $\beta < 1$. Suppose that $g(\mathbf{x})$ is submodular. For any fixed $u > 0$ and $v > 0$, the function $\mathbf{x} \longmapsto (u + v) \wedge g(\mathbf{x})$ is submodular in $\mathbf{x}$. By Lemma 2.2, we have

$$\mathsf{E}\left[(u + v) \wedge g(\mathbf{X})\right] \geq \mathsf{E}\left[(u + v) \wedge g(\mathbf{Y})\right].$$

A similar argument to that of the proof of Theorem 2.3(c) in Embrechts et al. [9] yields that

$$\begin{aligned} 0 \leq\ & \mathsf{E}\left[u \wedge g(\mathbf{X})\right] - \mathsf{E}\left[u \wedge g(\mathbf{Y})\right] \\ & + \int_0^v \mathsf{P}(g(\mathbf{X}) > u + y)dy - \int_0^v \mathsf{P}(g(\mathbf{Y}) > u + y)dy. \end{aligned} \tag{A.5}$$

Note that for u large enough,

$$\begin{aligned} & \int_0^v \mathsf{P}(g(\mathbf{X}) > u + y)dy - \int_0^v \mathsf{P}(g(\mathbf{Y}) > u + y)dy \\ &= \int_0^v \frac{\mathsf{P}(g(\mathbf{X}) > u + y)}{\overline{F}(u + y)} \cdot \overline{F}(u + y)dy - \int_0^v \frac{\mathsf{P}(g(\mathbf{Y}) > u + y)}{\overline{F}(u + y)} \cdot \overline{F}(u + y)dy \\ &\leq \left[(1 + \epsilon)q_g^{\mathrm{F}}(\alpha_1, \beta) - (1 - \epsilon)q_g^{\mathrm{F}}(\alpha_2, \beta)\right] \int_0^v \overline{F}(u + y)dy \end{aligned}$$

and that $\int_0^v \overline{F}(u + y)dy \to +\infty$ as $v \to +\infty$ since $\overline{F} \in \mathrm{RV}_{-\beta}$ with $0 < \beta < 1$. In terms of these, we conclude from (A.5) that $(1 + \epsilon)q_g^{\mathrm{F}}(\alpha_1, \beta) - (1 - \epsilon)q_g^{\mathrm{F}}(\alpha_2, \beta) \geq 0$ and, hence $q_g^{\mathrm{F}}(\alpha_1, \beta) \geq q_g^{\mathrm{F}}(\alpha_2, \beta)$. This means that $q_g^{\mathrm{F}}(\alpha, \beta)$ is decreasing in α.

To establish the monotonicity of $q_g^{\mathrm{F}}(\alpha, \beta)$ with respect to β, we write $q_g^{\mathrm{F}}(\alpha, \beta)$ as

$$q_g^{\mathrm{F}}(\alpha, \beta) = \int_{\mathbb{R}_+^n} 1_{\left\{g\left(x_1^{1/\beta}, \ldots, x_n^{1/\beta}\right) \geq 1\right\}}\ h_\alpha(\mathbf{x})\mathbf{dx} \tag{A.6}$$

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by using the transformation $x_i \longmapsto x_i^{-1/\beta}$, where

$$h_\alpha(\mathbf{x}) = (-1)^n \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \left(\sum_{i=1}^n x_i^\alpha \right)^{-1/\alpha}$$
$$= \prod_{i=0}^{n-1} (1 + i\alpha) \left(\sum_{i=1}^n x_i^\alpha \right)^{-1/\alpha - n} \left(\prod_{i=1}^n x_i \right)^{\alpha - 1}, \quad \mathbf{x} \in \mathbb{R}_+^n. \tag{A.7}$$

If $g(x_1^{1/\beta}, \ldots, x_n^{1/\beta}) = 1$ implies $x_i \le 1$ for all i, then

$$g(x_1^{1/\beta_1}, \ldots, x_n^{1/\beta_1}) = 1 \Longrightarrow g(x_1^{1/\beta_2}, \ldots, x_n^{1/\beta_2}) \ge 1 \text{ for any } 0 < \beta_1 < \beta_2$$

since g is increasing. This implies that the set $\{g(x_1^{1/\beta}, \ldots, x_n^{1/\beta}) \ge 1\}$ is increasing in β and, hence, $q_g^{\mathrm{F}}(\alpha, \beta)$ is increasing in β from (A.6). On the other hand, if $g(x_1^{1/\beta}, \ldots, x_n^{1/\beta}) = 1$ implies $x_i \ge 1$ for all i, then

$$g(x_1^{1/\beta_1}, \ldots, x_n^{1/\beta_1}) = 1 \Longrightarrow g(x_1^{1/\beta_2}, \ldots, x_n^{1/\beta_2}) \le 1 \text{ for any } 0 < \beta_1 < \beta_2.$$

So, the set $\{g(x_1^{1/\beta}, \ldots, x_n^{1/\beta}) \ge 1\}$ is decreasing in β. Again, from (A.6), it follows that $q_g^{\mathrm{F}}(\alpha, \beta)$ is decreasing in β. ■

Proof of Theorem 4.1(ii): For the Weibull case, suppose that g is submodular. Let $\mathbf{X}$ and $\mathbf{Y}$ be as defined in part (i). By Theorem 3.2(ii),

$$q_g^{\mathrm{W}}(\alpha_1, \beta) = \lim_{u \to \infty} \frac{1}{\overline{F}(\omega_F - 1/u)} \, \mathsf{P}\left(g(\omega_F \mathbf{1} - \mathbf{X}) \le \frac{1}{u} \right)$$

and

$$q_g^{\mathrm{W}}(\alpha_2, \beta) = \lim_{u \to \infty} \frac{1}{\overline{F}(\omega_F - 1/u)} \, \mathsf{P}\left(g(\omega_F \mathbf{1} - \mathbf{Y}) \le \frac{1}{u} \right)$$

Choose $u > 0$. Since the function $\mathbf{x} \longmapsto (1/u - g(\omega_F \mathbf{1} - \mathbf{x}))_+$ is supermodular, by Lemma 2.2, we have

$$\mathsf{E}\left(\frac{1}{u} - g(\omega_F \mathbf{1} - \mathbf{X}) \right)_+ \le \mathsf{E}\left(\frac{1}{u} - g(\omega_F \mathbf{1} - \mathbf{Y}) \right)_+. \tag{A.8}$$

Note that

$$\mathsf{E}\left(\frac{1}{u} - g(\omega_F \mathbf{1} - \mathbf{X}) \right)_+ = \int_0^\infty \mathsf{P}\left(0 \le g(\omega_F \mathbf{1} - \mathbf{X}) < \frac{1}{u} - x \right) dx$$
$$= \int_0^{1/u} \mathsf{P}\left(g(\omega_F \mathbf{1} - \mathbf{X}) < \frac{1}{u} - x \right) dx$$
$$= \int_1^\infty \mathsf{P}\left(g(\omega_F \mathbf{1} - \mathbf{X}) < \frac{1}{uy} \right) \cdot \frac{1}{uy^2} \, dy,$$

where the last equality follows by variable substitution $1/u - x = 1/(yu)$. From (A.8), we have

$$\int_1^\infty \mathsf{P}\left(g(\omega_F \mathbf{1} - \mathbf{X}) < \frac{1}{uy} \right) \cdot \frac{1}{y^2} \, dy \le \int_1^\infty \mathsf{P}\left(g(\omega_F \mathbf{1} - \mathbf{Y}) < \frac{1}{uy} \right) \cdot \frac{1}{y^2} \, dy,$$

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which can be written as

$$\int_1^\infty \frac{\mathsf{P}\left(g(\omega_F\mathbf{1}-\mathbf{X})<\frac{1}{uy}\right)}{\overline{F}\left(\omega_F-\frac{1}{uy}\right)}\cdot\frac{\overline{F}\left(\omega_F-\frac{1}{uy}\right)}{\overline{F}\left(\omega_F-\frac{1}{u}\right)}\cdot\frac{1}{y^2}\,dy$$

$$\le\int_1^\infty \frac{\mathsf{P}\left(g(\omega_F\mathbf{1}-\mathbf{Y})<\frac{1}{uy}\right)}{\overline{F}\left(\omega_F-\frac{1}{uy}\right)}\cdot\frac{\overline{F}\left(\omega_F-\frac{1}{uy}\right)}{\overline{F}\left(\omega_F-\frac{1}{u}\right)}\cdot\frac{1}{y^2}\,dy. \tag{A.9}$$

Note that, for $y\ge 1$, the first two fractions in the integrands of both sides in (A.9) are less than 1. By the dominated convergence theorem, setting $u\to\infty$ in both sides of (A.9) yields

$$\int_1^\infty q_g^{\mathrm{W}}(\alpha_1,\beta)\,y^{-\beta-2}dy\le\int_1^\infty q_g^{\mathrm{W}}(\alpha_2,\beta)\,y^{-\beta-2}dy,$$

implying $q_g^{\mathrm{W}}(\alpha_1,\beta)\le q_g^{\mathrm{W}}(\alpha_2,\beta)$. That is, $q_g^{\mathrm{W}}(\alpha,\beta)$ is increasing in α for each $\beta>0$.

The monotonicity of $q_g^{\mathrm{W}}(\alpha,\beta)$ with respect to β can be proved by using a similar argument to that of part (i) by observing that

$$q_g^{\mathrm{W}}(\alpha,\beta)=\int_{\mathbb{R}_+^n}1_{\left\{g\left(x_1^{-1/\beta},\ldots,x_n^{-1/\beta}\right)\le 1\right\}}\,h_\alpha(\mathbf{x})\mathbf{dx}. \tag{A.10}$$

This completes the proof. ■

A.4. Proof of Theorem 5.1

Proof of Theorem 5.1(i): From Theorem 3.2(i), we have

$$\lim_{u\to\infty}\frac{1}{\overline{F}(u)}\,\mathsf{P}(g(\mathbf{X})>u)=q_g^{\mathrm{F}}(\alpha,\beta)$$

and

$$\lim_{u\to\infty}\frac{1}{\overline{F}(u)}\,\mathsf{P}\left(X_i>[q_g^{\mathrm{F}}(\alpha,\beta)]^{-1/\beta}u\right)=q_g^{\mathrm{F}}(\alpha,\beta).$$

Then

$$\lim_{u\to\infty}\frac{\mathsf{P}(g(\mathbf{X})>u)}{\mathsf{P}\left(X_i>[q_g^F(\alpha,\beta)]^{-1/\beta}u\right)}=1,$$

which implies

$$\lim_{p\to1^-}\frac{\mathrm{VaR}_p[g(\mathbf{X})]}{\mathrm{VaR}_p[X_i]}=[q_g^{\mathrm{F}}(\alpha,\beta)]^{1/\beta}.$$

Since $g(\mathbf{x})$ is homogeneous of order 1, it is easy to see that $g(\mathbf{x})$ is continuous at point $\mathbf{1}$. Therefore,

$$\lim_{p\to1^-}\frac{\mathrm{VaR}_p[g(\mathbf{X})]}{g\big(\mathrm{VaR}_p[X_1],\ldots,\mathrm{VaR}_p[X_n]\big)}=\lim_{p\to1^-}\frac{\mathrm{VaR}_p[g(\mathbf{X})]}{\mathrm{VaR}_p[X]\cdot g\left(\frac{\mathrm{VaR}_p[X_1]}{\mathrm{VaR}_p[X]},\ldots,\frac{\mathrm{VaR}_p[X_n]}{\mathrm{VaR}_p[X]}\right)}$$

$$=\frac{[q_g^{\mathrm{F}}(\alpha,\beta)]^{1/\beta}}{g(\mathbf{1})},$$

since $\mathrm{VaR}_p[X_i]/\mathrm{VaR}_p[X]\to1$ for each i as p tends to 1 from below.

Now, consider the case $\beta>1$. Suppose that $g(\mathbf{x})$ is supermodular on $\mathbb{R}_+^n$ and $g\notin\mathscr{G}$. Then $g^*(\mathbf{x})\equiv\left[g\big(\mathbf{x}^{1/\beta}\big)\right]^\beta$ is supermodular, homogeneous of order 1, and nonlinear in $\mathbf{x}\in\mathbb{R}_+^n$ (It can be

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shown that $g^*(\mathbf{x})$ is linear in $\mathbf{x} \in \mathbb{R}_+^n$ if and only if $g \in \mathscr{G}$). Here and henceforth, we use the notation $\mathbf{x}^\gamma = (x_1^\gamma, \ldots, x_n^\gamma)$ for any γ and $\mathbf{x} \in \mathbb{R}_+^n$. We state two assertions as follows:

- For any $\mathbf{w} \in \mathbb{S}_{n-1}$, we have

$$\sum_{\pi \in \Pi} \left[g\left(\mathbf{w}_\pi^{1/\beta} \right) \right]^\beta \le (n-1)! \sum_{i=1}^n \left[g\left(w_i^{1/\beta} \mathbf{1} \right) \right]^\beta \tag{A.11}$$

$$= (n-1)! \sum_{i=1}^n w_i \left[g(\mathbf{1}) \right]^\beta = (n-1)! \left[g(\mathbf{1}) \right]^\beta,$$

  where $\{\mathbf{w}_\pi, \pi = (\pi_1, \ldots, \pi_n) \in \Pi\}$ is the set of all permutations of $\mathbf{w}$.
- The set of the points $\mathbf{w} \in \mathbb{S}_{n-1}$ such that the strict inequality in (A.11) holds has a positive measure (with respect to the Lebesgue measure on $\mathscr{B}(\mathbb{R}^{n-1})$).

The first assertion (A.11) can be proved by induction on n and using the following inequalities: for any $\mathbf{x} \in \mathbb{R}_+^n$, any disjoint proper subsets $L, J, K \subset \{1, \ldots, n\}$ with $L \cup J \cup K = \{1, \ldots, n\}$ and any fixed $j_0 \in J$ and $k_0 \in K$,

$$\begin{aligned} &g^*\left(x_\ell, \ell \in L; x_{j_0}, j \in J; x_{k_0}, k \in K\right) + g^*\left(x_\ell, \ell \in L; x_{k_0}, j \in J; x_{j_0}, k \in K\right) \\ &\quad \le g^*\left(x_\ell, \ell \in L; x_{j_0}, j \in J \cup K\right) + g^*\left(x_\ell, \ell \in L; x_{k_0}, k \in J \cup K\right), \end{aligned}$$

where $g^*(x_\ell, \ell \in L; x_{j_0}, j \in J; x_{k_0}, k \in K)$ is the value of function g^* at the point with ℓth ($\ell \in L$), jth ($j \in J$) and kth ($k \in K$) components being x_ℓ, x_{j_0} and x_{k_0}, respectively. The other notations can be interpreted similarly. The second assertion can be proved by contradiction. Assume that the equality in (A.11) holds for $\mathbf{w} \in \mathbb{S}_{n-1}$ almost everywhere. Choose any two points $\mathbf{x}, \mathbf{y} \in \mathbb{S}_{n-1}$. Then

$$\begin{aligned} 2(n-1)! g^*(\mathbf{1}) &= \sum_{\pi \in \Pi} g^*(\mathbf{x}_\pi) + \sum_{\pi \in \Pi} g^*(\mathbf{y}_\pi) \\ &= g^*(\mathbf{x}) + g^*(\mathbf{y}) + \sum_{\pi \in \Pi \setminus \{\boldsymbol{I}\}} \left[g^*(\mathbf{x}_\pi) + g^*(\mathbf{y}_\pi) \right] \\ &\le g^*(\mathbf{x} \vee \mathbf{y}) + g^*(\mathbf{x} \wedge \mathbf{y}) + \sum_{\pi \in \Pi \setminus \{\boldsymbol{I}\}} \left[g^*(\mathbf{x}_\pi) + g^*(\mathbf{y}_\pi) \right] \\ &\le g^*(\mathbf{x} \vee \mathbf{y}) + g^*(\mathbf{x} \wedge \mathbf{y}) + \sum_{\pi \in \Pi \setminus \{\boldsymbol{I}\}} \left[g^*((\mathbf{x} \vee \mathbf{y})_\pi) + g^*((\mathbf{x} \wedge \mathbf{y})_\pi) \right] \\ &= 2(n-1)! g^*(\mathbf{1}), \end{aligned}$$

where $\boldsymbol{I} \in \Pi$ is the identity permutation. This means that

$$g^*(\mathbf{x}) + g^*(\mathbf{y}) = g^*(\mathbf{x} \vee \mathbf{y}) + g^*(\mathbf{x} \wedge \mathbf{y}), \quad \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{S}_{n-1},$$

which contradicts the nonlinearity of g^*.

Note that $h_\alpha(\mathbf{w})/n$ is a symmetric density function with support $\mathbb{S}_{n-1}$. Thus, from (4.1) and the two assertions above, we conclude that

$$q_g^{\mathrm{F}}(\alpha, \beta) < \left[g(\mathbf{1}) \right]^\beta.$$

So, (5.1) holds for p large enough.

For $\beta < 1$ and $g(\mathbf{x})$ is submodular on $\mathbb{R}_+$, (5.2) can be proved similarly by observing inequality (A.11) is reversed. ■

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Proof of Theorem 5.1(ii): Similarly, from Theorem 3.2(ii), we have

$$\lim_{u\to\infty}\frac{1}{\overline{F}(\omega_F-1/u)}\,\mathsf{P}\left(g(\omega_F\mathbf{1}-\mathbf{X})\le\frac{1}{u}\right)=q_g^{\mathrm{W}}(\alpha,\beta)$$

and

$$\lim_{u\to\infty}\frac{1}{\overline{F}(\omega_F-1/u)}\,\mathsf{P}\left(\omega_F-X_i<\frac{[q_g^{\mathrm{W}}(\alpha,\beta)]^{1/\beta}}{u}\right)=q_g^{\mathrm{W}}(\alpha,\beta),$$

which imply

$$\lim_{u\to\infty}\frac{\mathsf{P}(g(\omega_F\mathbf{1}-\mathbf{X})\le 1/u)}{\mathsf{P}\left(\omega_F-X_i<[q_g^{\mathrm{W}}(\alpha,\beta)]^{1/\beta}/u\right)}=1,\quad i=1,\dots,n.$$

Then,

$$\lim_{p\to0^+}\frac{\mathrm{VaR}_p[g(\omega_F\mathbf{1}-\mathbf{X})]}{\mathrm{VaR}_p[\omega_F-X_i]}=[q_g^{\mathrm{W}}(\alpha,\beta)]^{-1/\beta},\quad i=1,\dots,n.$$

and hence

$$\begin{aligned}&\lim_{p\to0^+}\frac{\mathrm{VaR}_p[g(\omega_F\mathbf{1}-\mathbf{X})]}{g\big(\mathrm{VaR}_p[\omega_F-X_1],\dots,\mathrm{VaR}_p[\omega_F-X_n]\big)}\\&=\lim_{p\to0^+}\frac{\mathrm{VaR}_p[g(\omega_F\mathbf{1}-\mathbf{X})]}{\mathrm{VaR}_p[\omega_F-X]\cdot g\left(\frac{\mathrm{VaR}_p[\omega_F-X_1]}{\mathrm{VaR}_p[\omega_F-X]},\dots,\frac{\mathrm{VaR}_p[\omega_F-X_n]}{\mathrm{VaR}_p[\omega_F-X]}\right)}\\&=\frac{[q_g^{\mathrm{W}}(\alpha,\beta)]^{-1/\beta}}{g(\mathbf{1})},\end{aligned}$$

where we use the fact that $\mathrm{VaR}_p[\omega_F-X_i]/\mathrm{VaR}_p[\omega_F-X]\to1$ for each i as $p\to0^+$.

If $g(\mathbf{x})$ is submodular on $\mathbb{R}_+^n$ and and $g\notin\mathscr{G}$, then $\left[g\left(\mathbf{x}^{-1/\beta}\right)\right]^{-\beta}$ is supermodular, homogeneous of order 1, and nonlinear in $\mathbf{x}\in\mathbb{R}_+^n$. A similar argument to the above in part (i) yields that, for any $\mathbf{w}\in\mathbb{S}_{n-1}$,

$$\sum_{\pi\in\Pi}\left[g\left(\mathbf{w}_\pi^{-1/\beta}\right)\right]^{-\beta}\le(n-1)!\sum_{i=1}^n\left[g\left(w_i^{-1/\beta}\mathbf{1}\right)\right]^{-\beta}=(n-1)!\,[g(\mathbf{1})]^{-\beta}.\tag{A.12}$$

Moreover, the set of the points $\mathbf{w}\in\mathbb{S}_{n-1}$ such that the strict inequality in (A.12) holds has a positive measure (with respect to the Lebesgue measure on $\mathscr{B}(\mathbb{R}^{n-1})$). Observing this, it follows from (4.2) that $q_g^{\mathrm{W}}(\alpha,\beta)<[g(\mathbf{1})]^{-\beta}$. So, for $p>0$ small enough, (5.3) holds. This completes the proof. ■

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