Combinatorics, Probability and Computing (2002) 11, 129–133. © 2002 Cambridge University Press
DOI: 10.1017/S0963548301005004 Printed in the United Kingdom

Multi-Coloured Hamilton Cycles in Random Edge-Coloured Graphs

COLIN COOPER[2] and ALAN FRIEZE[1]†

[1] School of Mathematical and Computing Sciences, Goldsmiths College, University of London, London SE14 6NW, UK
(e-mail: c.cooper@gold.ac.uk)

[2] Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA15213, USA
(e-mail: alan@random.math.cmu.edu)

Received 5 March 2001; revised 11 June 2001

We define a space of random edge-coloured graphs $\mathscr{G}_{n,m,\kappa}$ which correspond naturally to edge κ-colourings of $G_{n,m}$. We show that there exist constants K_0, $K_1 \leqslant 21$ such that, provided $m \geqslant K_0 n \log n$ and $\kappa \geqslant K_1 n$, then a random edge-coloured graph contains a multi-coloured Hamilton cycle with probability tending to 1 as the number of vertices n tends to infinity.

1. Introduction

Let $\mathscr{G}_{n,m,\kappa}$ denote the space of random edge-coloured graphs, defined as follows. Each G in $\mathscr{G}_{n,m,\kappa}$ has vertex set $[n]$, edge set $E_m(G)$ of size m, and each edge is coloured with a label from $[\kappa]$. Thus $|\mathscr{G}_{n,m,\kappa}| = \binom{N}{m}\kappa^m$ where $N = \binom{n}{2}$. The elements of $\mathscr{G}_{n,m,\kappa}$ are given the uniform measure. A random edge-coloured graph $G_{n,m,\kappa}$ is a graph G sampled uniformly at random from $\mathscr{G}_{n,m,\kappa}$.

Given $G \in \mathscr{G}_{n,m,\kappa}$, a subset S of the edges of G is *multi-coloured* if no two edges of S have the same colour. We are interested in conditions on n, m, κ which imply that w.h.p.‡ $G_{n,m,\kappa}$ contains a multi-coloured Hamilton cycle. We also consider the corresponding randomly arc-coloured random digraph $D_{n,m,\kappa}$, which is defined in the analogous way. We prove the following.

Theorem 1.1. *There exist constants $K_0, K_1 > 0$ such that, if $m \geqslant K_0 n \log n$ and $\kappa \geqslant K_1 n$, then w.h.p. $G_{n,m,\kappa}$ (resp. $D_{n,m,\kappa}$) contains a multi-coloured Hamilton cycle.*

† Supported in part by NSF grant CCR-9818411.

‡ A sequence of events $\mathscr{E}_n$ is said to occur *with high probability* (w.h.p.) if $\lim_{n\to\infty} \mathbf{Pr}(\mathscr{E}_n) = 1$.

https://doi.org/10.1017/S0963548301005004 Published online by Cambridge University Press

We note that Theorem 1.1 is best possible up to a constant factor, for we need at least n distinct colours and the underlying graph (resp. digraph) does not contain a Hamilton cycle w.h.p. until $m > n \log n/2$. The values of the constants ($K_0, K_1 = 21$) we use in the proof of Theorem 1.1 are, however, not the best possible values.

There are two general types of results on multi-coloured structures: w.h.p. existence under random colouring and guaranteed existence under adversarial colouring. When considering adversarial (worst-case) colouring, the guaranteed existence of multi-coloured structures, is called an anti-Ramsey property.

Erdős, Nešetřil and Rödl [5], Hahn and Thomassen [8] and Albert, Frieze and Reed [1] (correction in Rue [10]) considered colourings of the edges of the complete graph K_n where no colour is used more than k times. It was shown in [1] that, if $k \leqslant n/64$, then there must be a multi-coloured Hamilton cycle. Cooper and Frieze [3] proved a random graph threshold for this property to hold in almost every graph in the space studied.

With respect to random colouring, Janson and Wormald [9] gave conditions for the existence of a multi-coloured Hamilton cycle in a random regular graph. We also mention that Frieze and McKay [7] found a tight threshold for the existence of a multi-coloured spanning tree.

2. A sequence of random graphs

Because we are concerned with monotone properties, we can work entirely with the independent model $G_{n,p,\kappa}$, where $p = m/N$ and the underlying uncoloured graph is $G_{n,p}$. Let p_1 satisfy $1 - p = (1 - p_1)^2$. Let D_{n,p_1} be the random digraph where each arc occurs independently with probability p_1. Suppose now that we randomly colour the arcs of D_{n,p_1} with κ colours to obtain the random coloured graph $D_{n,p_1,\kappa}$. Ignoring orientation gives us the random graph $G_{n,p,\kappa}$, provided we make a random choice from the two possible colours when coalescing the edges of directed 2-cycles.

Next, let $\mathscr{D}_{d-\text{out},\kappa}$ denote the following set of arc-coloured *digraphs*. Each $D \in \mathscr{D}_{d-\text{out},\kappa}$ has vertex set $[n]$, each vertex has out-degree d and the arcs of D are multi-coloured by $[\kappa]$, *i.e.*, no colour is used more than once. Thus $|\mathscr{D}_{d-\text{out},\kappa}| = \binom{n-1}{d}^n \binom{\kappa}{dn}(dn)!$. The arc-coloured digraph $D_{d-\text{out},\kappa}$ is chosen uniformly at random from $\mathscr{D}_{d-\text{out},\kappa}$. In this paper we will be concerned with $d = O(1)$; in particular, we assume that $d = 5$ from now on.

The central idea of this paper is to use a network flow algorithm to take $D_{n,p_1,\kappa}$ and, conditional on an event of probability $1 - o(1)$, return as output a multi-coloured sub-digraph D. The distribution of D will be that of $D_{d-\text{out},\kappa}$. If we ignore orientation in $D_{d-\text{out},\kappa}$ and *delete* parallel edges, then we obtain the random multi-coloured graph $G_{d-\text{out},\kappa}$. Ignoring colours now gives us the random graph $G_{d-\text{out}}$. If it is known w.h.p. that $G_{d-\text{out}}$ is Hamiltonian, then we will have proved that w.h.p. $G_{n,p,\kappa}$ contains the required multi-coloured H. To prove Theorem 1.1 for $G_{n,m,\kappa}$ we only have to do this for $d \geqslant 5$, and then apply the result of Frieze and Łuczak [6] which states that such a graph is Hamiltonian w.h.p. There is a technical point here. In the usual construction of $G_{d-\text{out}}$ we coalesce rather than delete parallel edges. It is not difficult to see that the proof of [6] is easily modifiable to handle this. On the other hand the result of Cooper and Frieze [4] that G_{4-out} is Hamiltonian w.h.p. seems to run into difficulty.

https://doi.org/10.1017/S0963548301005004 Published online by Cambridge University Press

2.1. Network flow construction

We define a flow network $\mathscr{N}$ as follows. $\mathscr{N}$ has source s and sink t. The vertex set W consists of s,t, the set of colours $C=[\kappa]$ and the set $V=[n]$ of vertices of the $D_{n,p_1,\kappa}$ under consideration. For each colour $x\in C$ there is an arc (s,x) in $\mathscr{N}$ of capacity 1. There is an arc (x,v) in $\mathscr{N}$ of infinite capacity for every $v\in V$ for which there is an arc (v,w) in $D_{n,p_1,\kappa}$ with tail v and colour x. Finally, for each vertex $v\in V$ there is an arc (v,t) of capacity d.

For $S\subseteq C$, let $N(S)=\{v:\ x\in S,\ v\in V,\ (x,v)\in\mathscr{N}\}$ be the out-neighbour set of S in $\mathscr{N}$. A cut of finite capacity can be obtained from a set $S\subseteq C$ and $N(S)\subseteq V$. Let $T=N(S)$, $W=\{s\}\cup S\cup T$, and let $\overline{W}=(C\setminus S)\cup(V\setminus T)\cup\{t\}$. The capacity of the cut $(W:\overline{W})$ is $\kappa-|S|+d|T|$. Applying the max-flow min-cut theorem we see that $\mathscr{N}$ admits a flow of value dn if and only if, for all $S\subseteq C$,

$$\kappa-|S|+d|N(S)|\geqslant dn. \tag{2.1}$$

We estimate the probability that (2.1) is not true because, for some set S, $|N(S)|<n-(\kappa-|S|)/d$. In other words, there exists a set of colours S of size s and a set of vertices $\overline{T}$ of size $|\overline{T}|>(\kappa-s)/d$ such that every arc of D whose tail is in $\overline{T}$ has a colour in $C\setminus S$.

Now, p_1 satisfies $1-p=(1-p_1)^2$ and so $p_1\geqslant p/2$, for $1-\sqrt{1-p}\geqslant p/2$ for $p\geqslant 0$. We see therefore that $np_1\geqslant K_0\log n$.

Let $\mathscr{E}$ denote the subset of D_{n,p_1} for which $\delta^+(D_{n,p_1})>np_1/2$.

We first estimate $\mathbf{Pr}(\overline{\mathscr{E}})$. By the Chernoff inequality,

$$\mathbf{Pr}(\delta^+(D_{n,p_1})\leqslant np_1/2)\leqslant ne^{-np_1/8}=O(n^{1-K_0/8}), \tag{2.2}$$

which is $O(n^{-13/8})$ for $K_0\geqslant 21$. Thus $\mathbf{Pr}(\overline{\mathscr{E}})=o(1)$.

Let

$$L(s)=2\binom{\kappa}{s}\binom{n}{\lceil(\kappa-s)/d\rceil}\left(\frac{\kappa-s}{\kappa}\right)^{(\kappa-s)np_1/(2d)}$$

be an upper bound on the probability that some set of size s does not satisfy (2.1) conditional on $\mathscr{E}$. The range of s we need to consider is between $\kappa-dn+1$ and $\kappa-1$. For, if $|S|<\kappa-dn$ then (2.1) is true with $N(S)=\emptyset$, and if $s=\kappa$ then as $\delta^+(D)\geqslant np_1/2$, $\overline{T}=\emptyset$.

The probability that (2.1) is not satisfied is bounded by Θ, where

$$\Theta=\mathbf{Pr}(\overline{\mathscr{E}})+\sum_{s=\kappa-dn+1}^{\kappa-1}L(s). \tag{2.3}$$

As $\mathbf{Pr}(\overline{\mathscr{E}})=o(1)$, we can concentrate on the summation term in (2.3).

https://doi.org/10.1017/S0963548301005004 Published online by Cambridge University Press

Now, choosing $\kappa \geqslant 21n$, and putting $\lceil(\kappa - s)/d\rceil = (\kappa - s)/d + f_s$, $0 \leqslant f_s < 1$,

$$\begin{aligned}
\sum_{s=\kappa-dn+1}^{\kappa-1} L(s) &\leqslant 2ned \sum_{s=\kappa-dn+1}^{\kappa-1} \left(\frac{\kappa e}{\kappa - s} \left(\frac{ned}{\kappa - s} \right)^{1/d} \left(\frac{\kappa - s}{\kappa} \right)^{(K_0 \log n)/(2d)} \right)^{\kappa - s} \\
&\leqslant 2ned \sum_{s=\kappa-dn+1}^{\kappa-1} \left(3 \left(\frac{\kappa - s}{\kappa} \right)^{K_0 \log n/(2d) - 1/d - 1} \right)^{\kappa - s} \\
&\leqslant 2ned \sum_{s=\kappa-dn+1}^{\kappa-1} \left(3 \left(\frac{dn - 1}{edn} \right)^{(K_0 \log n)/(2d) - 1/d - 1} \right)^{\kappa - s} \qquad (2.4) \\
&\leqslant 2ed^2 n^2 \exp\left\{ -\frac{K_0 \log n}{2d} + \frac{1}{d} + 3 \right\}, \qquad (2.5)
\end{aligned}$$

which is $O(n^{-1/10})$ for $K_0 \geqslant 21$ when $d = 5$.

Thus w.h.p. $\mathcal{N}$ contains a flow of value nd. The capacities of $\mathcal{N}$ are integral and so we can assume this flow is integral. It decomposes into nd (s, t)-paths, each of which assigns a colour x to a vertex v. By construction a colour can be assigned at most once to an edge and each vertex is assigned d colours. For each assignment of a colour x to a vertex v, we choose (randomly from D) an arc of colour x which has tail v. We thus obtain a multi-coloured member of $\mathscr{D}_{d-\text{out},\kappa}$. It is easy to argue that the underlying uncoloured digraph is distributed as $D_{d-\text{out},\kappa}$. Indeed we could start with $D_{n,p_1,\kappa}$ and then replace each arc (v, w) by $(v, \pi_v(w))$, where the π_v, $v \in V$ are independent permutations of $V \setminus \{v\}$. After this transformation the digraph is still distributed as $D_{n,p_1,\kappa}$. We run the network flow algorithm and w.h.p. we obtain a multi-coloured member of $\mathscr{D}_{d-\text{out},\kappa}$. By replacing each arc (v, w) by $(v, \pi_v^{-1}(w))$, we obtain a subgraph of the original $D_{n,p_1,\kappa}$ which is distributed as $D_{d-\text{out},\kappa}$. For those cases where both (v, w) and (w, v) are selected by the algorithm to be edges of $D_{n,p_1,\kappa}$, we simply delete this edge. We have to do this because of the possibility that the network algorithm chooses a different colour for $\{v, w\}$ to the one chosen in going from $D_{n,p_1,\kappa}$ to $G_{n,p,\kappa}$.

In summary, w.h.p. $D_{n,p_1,\kappa}$ contains a multi-coloured subgraph which is distributed as $D_{d-\text{out},\kappa}$. Ignoring orientation, we obtain a graph which w.h.p. contains a Hamilton cycle. This verifies Theorem 1.1 for the case of undirected graphs.

Consider now the directed case, that is, we start with $D_{n,p,\kappa}$. We first split this into two independent copies D_1, D_2 of $D_{n,p_1,\kappa}$. We then use a slightly modified network. Now we have vertices s, t, C and two copies V_1, V_2 of V. The s, C edges are as before and there are V_1, t and V_2, t edges of capacity d (now we can take $d = 3$). We join $x \in C$ to $v \in V_1$ by an infinite capacity arc if V_1 contains an arc of colour x and tail v. We join $x \in C$ to $v \in V_2$ by an infinite capacity arc if V_1 contains an arc of colour x and head v. The network flow algorithm constructs a random multi-coloured 3-in, 3-out digraph, which w.h.p. has a Hamilton cycle, by the result of [4], even after removing parallel arcs. This is why we take $d = 3$ and appeal to the proof of the result in [2], that a random 3-in, 3-out digraph is Hamiltonian w.h.p. The proof there will survive the deletion of parallel arcs.

https://doi.org/10.1017/S0963548301005004 Published online by Cambridge University Press

As a final remark, we have not really made arguments about Hamiltonicity, but only about constructing a random subgraph which is distributed as $D_{d-\mathrm{out},\kappa}$. Clearly, other monotone graph properties can be treated in this manner.

Finally, we mention two natural related problems. Suppose we fix K_0 at the threshold value $\frac{1}{2} + o(1)$. What is the least value of $K_1 = K_1(n)$ for which $G_{n,p,\kappa}$ contains a multi-coloured Hamilton cycle w.h.p.? Similarly, if we fix $K_1 = 1$, what is the least value of $K_0 = K_0(n)$ for which $G_{n,p,\kappa}$ contains a multi-coloured Hamilton cycle w.h.p.? It is prudent to observe that we must take $K_0 \geqslant 1 + o(1)$ so that w.h.p. each colour occurs at least once.

References

[1] Albert, M., Frieze, A. M. and Reed, B. (1995) Multicoloured Hamilton cycles. *Electron. J. Combin.* **2** R10.

[2] Cooper, C. and Frieze, A. M. (1994) Hamilton cycles in a class of random directed graphs. *J. Combin. Theory Ser. B* **62** 151–163

[3] Cooper, C. and Frieze, A. M. (1995) Multicoloured Hamilton cycles in random graphs: An anti-Ramsey threshold. *Electron. J. Combin.* **2** R19.

[4] Cooper, C. and Frieze, A. M. (2000) Hamilton cycles in random graphs and directed graphs. *Random Struct. Alg.* **16** 369–401.

[5] Erdős, P., Nešetřil, J. and Rödl, V. (1983) Some problems related to partitions of edges of a graph. In *Graphs and Other Combinatorial Topics*, Teubner, Leipzig, 54–63.

[6] Frieze, A. M. and Łuczak, T. (1990) Hamiltonian cycles in a class of random graphs: One step further. In *Proceedings of Random Graphs '87* (M. Karoński, J. Jaworski and A. Ruciński, eds), Wiley, 53–59.

[7] Frieze, A. M. and McKay, B. D. (1994) Multicoloured trees in random graphs. *Random Struct. Alg.* **5** 45–56.

[8] Hahn, G. and Thomassen, C. (1986) Path and cycle sub-Ramsey numbers, and an edge colouring conjecture. *Discrete Math.* **62** 29–33

[9] Janson, S. and Wormald, N. C. Rainbow Hamilton cycles in random regular graphs. To appear.

[10] Rue, R. (1995) Comment on [1]. *Electron. J. Combin.* **2** R10.

https://doi.org/10.1017/S0963548301005004 Published online by Cambridge University Press