A STATISTICAL METHODOLOGY FOR ASSESSING THE MAXIMAL STRENGTH OF TAIL DEPENDENCE

BY

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ABSTRACT

Several diagonal-based tail dependence indices have been suggested in the literature to quantify tail dependence. They have well-developed statistical inference theories but tend to underestimate tail dependence. For those problems when assessing the maximal strength of dependence is important (e.g., co-movements of financial instruments), the maximal tail dependence index was introduced, but it has so far lacked empirical estimators and statistical inference results, thus hindering its practical use. In the present paper, we suggest an empirical estimator for the index, explore its statistical properties, and illustrate its performance on simulated data.

KEYWORDS

Copula, maximal tail dependence, statistical estimation, Kendall's distribution.

1. INTRODUCTION

Extreme co-movements manifest in a variety of problems associated with risk management and are typically measured using tail dependence indices (e.g., Joe, 1993; Coles *et al.*, 1999; Hua and Joe, 2011; Joe, 2014, and references therein). Definitions of these indices rely on the identity

$$C(u, u) = \ell(u)u^{\kappa} \quad \text{when } u \downarrow 0, \tag{1.1}$$

where $C : [0, 1]^2 \to [0, 1]$ is the bivariate copula of underlying dependent risks, ℓ is a slowly varying at 0 function (e.g., Bingham *et al.*, 1987), and $\kappa \in [1, \infty)$ is the lower tail order, which we pedantically call the lower tail order of *diagonal* dependence. The corresponding upper tail order is obtained by replacing C in definition (1.1) by the corresponding survival copula, and for this reason we do not discuss this analogous "upper" case. Hence, throughout the rest of this

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paper, we drop the qualifier "lower" and simply call κ the tail order of diagonal dependence (TODD).

Equation (1.1) describes asymptotic behavior of the copula C when its arguments shrink to the origin $(0, 0)$ along the *diagonal* path $(u, u)_{0\le u\le 1}$, which may or may not reflect the maximal strength of tail dependence carried by the copula, as illustrated by Furman *et al.* (2015). Hence, it becomes natural to seek a path (e.g., Figure 1), called path of maximal tail dependence (MTD), that gives rise to the tail order of *maximal* dependence, henceforth shorthanded as TOMD and denoted by κ^*. When the diagonal path is not an MTD path, as is the case for many copulas (Furman *et al.*, 2015) including the Marshall–Olkin (M-O) copula, the classical measures of tail dependence based on equation (1.1) inevitably underestimate the actual (i.e., maximal) tail dependence. It therefore becomes natural and important (e.g., when assessing the strength of co-movements of financial instruments) to aim at assessing the strength of tail dependence based on an MTD path, which may or may not be unique, may or may not be diagonal.

Having said this, we ought to make it clear that the classical *diagonal*-based indices of tail dependence, such as κ noted above, are enormously useful by conveying a wealth of information, as elucidated in many scholarly writings, such as the monographs by Nelsen (2006), Joe (2014), Durante and Sempi (2015), and McNeil *et al.* (2015); see also the references therein. We also note in this regard the recently introduced notion of p-concentration, which is a diagonal-based measure of dependence that plays a pivotal role in characterizing the risk measure expected shortfall (Wang and Zitikis, 2020).

The aforementioned κ as well as other tail dependence indices associated with equation (1.1) have empirical estimators and well-established statistical inference theories (e.g., Draisma *et al.*, 2004; Frahm *et al.*, 2005; Krupskii and Joe, 2019, and references therein). The tail order κ^*, as far as we are aware of, does not yet have an empirical estimator and thus statistical inference theory. This hinders practical use of κ^* when assessing the strength of tail dependence. The purpose of the present paper is to construct a consistent empirical estimator of the tail order κ^* and discuss its practical implementation.

The main idea behind the construction hinges on the realization that it is possible to circumvent the need for having closed-form expressions for MTD paths. Explaining how this can be done makes up the contents of Section 2, where we gradually introduce and discuss the estimator. In Section 3, we explore statistical properties of the estimator, including its consistency, bias and its reduction. In the same section, we also test the performance of the estimator on simulated independent and identically distributed (iid) pairs that arise from the Marshall–Olkin copula. In Section 4, we explain how to mitigate the influence of slowly varying functions, such as ℓ in equation (1.1), on the TOMD estimator, where we also illustrate the technique in a simulation study based on the generalized Clayton copula. Section 5 concludes the paper with a summary of main contributions and suggestions for future research. Proofs and other technicalities with illustrative graphs are given in the Appendix.

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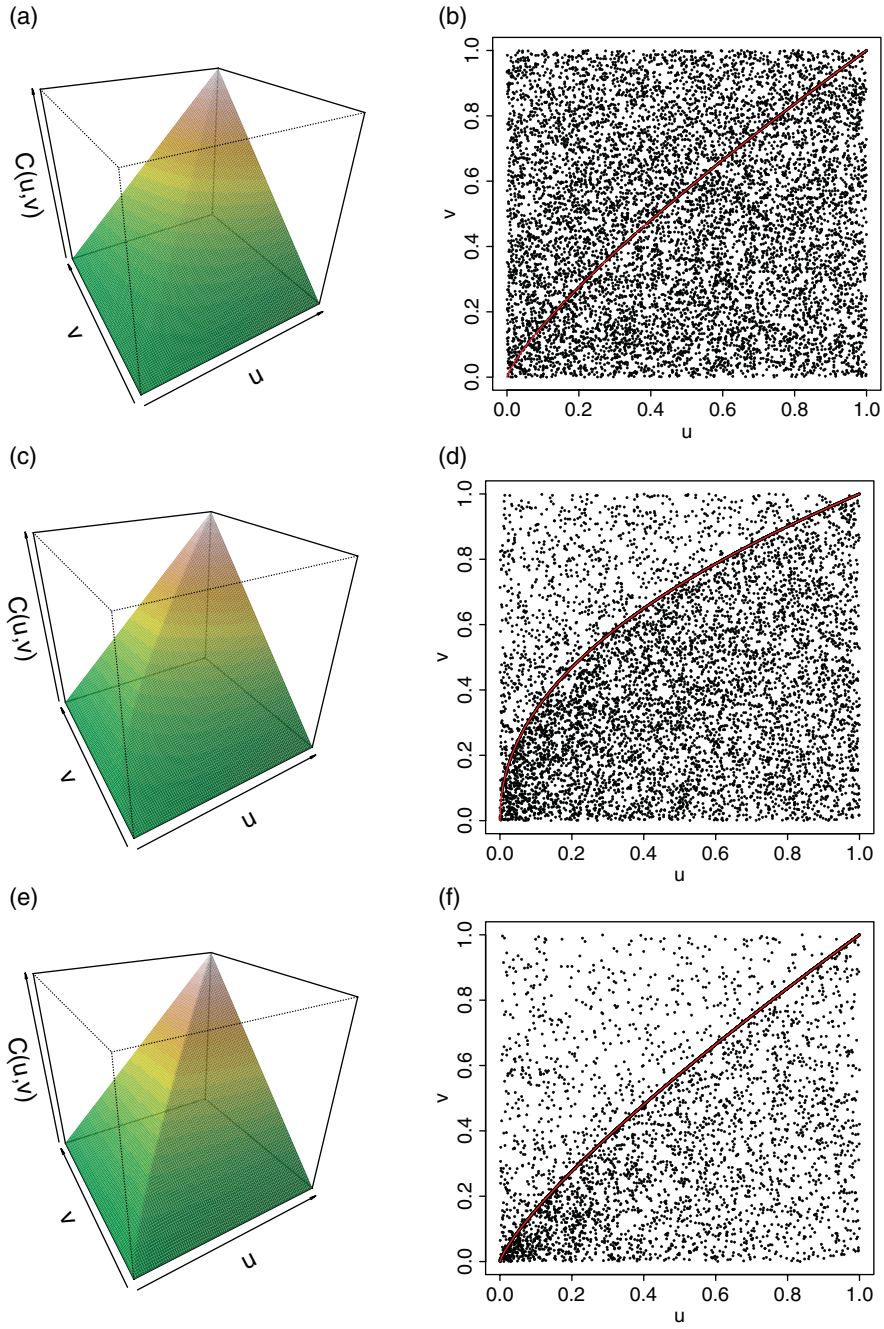


FIGURE 1: The M-O copula (left) and its MTD path (right) on a 10,000 point scatterplot for the specified M-O parameters (a, b) with $\kappa^* = 1.80$, 1.52, and 1.20 (top to bottom). (a) M-O for $(a, b) = (0.18, 0.225)$. (b) MTD for $(a, b) = (0.18, 0.225)$. (c) M-O for $(a, b) = (0.3529, 0.75)$. (d) MTD for $(a, b) = (0.3529, 0.75)$. (e) M-O for $(a, b) = (0.72, 0.9)$. (f) MTD for $(a, b) = (0.72, 0.9)$.

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2. Estimating the tail order of maximal dependence

This section is subdivided into three parts. In Section 2.1, we recall the definition of the tail order κ^* of *maximal* dependence, which we call TOMD for short. It is a counterpart of the earlier discussed tail order κ of *diagonal* dependence, TODD for short. In Section 2.2, we introduce the main idea of estimating κ^* without relying on explicit expressions of the underlying paths of MTD. In Section 2.3, we propose an empirical estimator of κ^*.

2.1. Preliminaries: the TOMD

Let $C : [0, 1]^2 \to [0, 1]$ be any copula. By definition (Furman *et al.*, 2015), a path of tail dependence is $(\varphi(u), u^2/\varphi(u))_{0\le u\le 1}$, where $\varphi : [0, 1] \to [0, 1]$ is a function that satisfies the following admissibility conditions:

(1) $\varphi(u) \in [u^2, 1]$ for every $u \in [0, 1]$;
(2) $\varphi(u) \to 0$ and $u^2/\varphi(u) \to 0$ when $u \downarrow 0$.

For example, $\varphi(u) = u$ is such a function, and it leads to the diagonal path $(u, u)_{0\le u\le 1}$ that we mentioned earlier in relation to the classical tail order κ.

Let $\mathcal{A}$ denote the set of all admissible functions. An MTD path, which is not necessarily unique, is $(\varphi^*(u), u^2/\varphi^*(u))_{0\le u\le 1}$ that arises from $\varphi^* \in \mathcal{A}$ given by

$$\varphi^*(u) = \arg\max_{\varphi\in\mathcal{A}} C\left(\varphi(u), u^2/\varphi(u)\right).$$

Denote

$$\Pi^*(u) = C(\varphi^*(u), u^2/\varphi^*(u)).$$

If there exists κ^* such that

$$\Pi^*(u) = \ell^*(u)u^{\kappa^*} \quad \text{when } u \downarrow 0 \tag{2.1}$$

for a slowly varying at 0 function ℓ^*, then κ^* is called (Furman *et al.*, 2015) the tail order of *maximal* dependence (TOMD). As illustrated in the aforementioned paper, there can be several MTD paths but there is only one TOMD. We also know from the paper that the TOMD may or may not coincide with the TODD, and the Gaussian copula is one of those rare cases when the two tail orders are the same (Furman *et al.*, 2016).

2.2. Estimating the TOMD: an idea

To introduce an idea for estimating the TOMD, we focus on the case

$$\Pi^*(u) = u^{\kappa^*}, \quad u \in [0, 1], \tag{2.2}$$

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where $\kappa^* \geq 1$ is a parameter. That is, for the sake of simplicity, but only until Section 4, we drop the slowly varying function ℓ^* from our main considerations, although we shall touch upon a practical aspect of this simplification sooner, at the end of the current section.

Property (2.2) is of course satisfied by the Marshall–Olkin (M-O) copula

$$C_{a,b}(u, v) = \min\left(u^{1-a}v, uv^{1-b}\right), \quad u, v \in [0, 1], \tag{2.3}$$

where $a, b \in [0, 1]$ are parameters, and we shall therefore conveniently use it in a simulation study, whose results will be reported in Section 3.2. Recall that the TOMD of this copula is (Furman *et al.*, 2015)

$$\kappa^* = 2 - \frac{2ab}{a+b}. \tag{2.4}$$

For theory and applications of the M-O copula with further references, we refer to Cherubini *et al.* (2013, 2015) and McNeil *et al.* (2015). For recent actuarial applications of the copula with generalizations and related references, we refer to Su and Furman (2017).

Next, under assumption (2.2), we have

$$\kappa^* = \frac{\log \Pi^*(u)}{\log u} \leq \frac{\log C\left(\varphi(u), u^2/\varphi(u)\right)}{\log u} \tag{2.5}$$

for all $u \in [0, 1]$ and every $\varphi \in \mathcal{A}$. With the notation $x = \varphi(u)$ and $y = u^2/\varphi(u)$, we therefore have

$$\kappa^* \leq \frac{\log C(x, y)}{\log \sqrt{xy}} = \frac{2 \log C(x, y)}{\log x + \log y}$$

for all $x, y \in [0, 1]$ such that $xy = u^2$. The latter bound suggests that given random pairs $(U_1, V_1), \ldots, (U_n, V_n) \sim C$, which we assume to be independent throughout this paper, the TOMD κ^* under model (2.2) can be estimated by

$$\widehat{\kappa}_n^* := \min_{i=1,\ldots,n} \frac{2 \log C_n(U_i, V_i)}{\log U_i + \log V_i}, \tag{2.6}$$

where

$$C_n(u, v) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{U_i \leq u, V_i \leq v\}$$

with $\mathbb{1}$ denoting the indicator function.

We shall find it more convenient, however, to work with a slight modification of the estimator $\widehat{\kappa}_n^*$. Namely, let

$$\widehat{\kappa}_n^*(\theta) := \min_{i=1,\ldots,n} \frac{2T_\theta \circ C_n(U_i, V_i)}{\log U_i + \log V_i}, \tag{2.7}$$

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where

$$T_\theta(t) = \begin{cases} (t^\theta - 1)/\theta & \text{when} \quad \theta > 0, \\ \log(t) & \text{when} \quad \theta = 0, \end{cases}$$

for all $t \in (0, 1]$. In the following paragraph, we shall explain the technical aspect that has lead us to the introduction of the function T_θ into considerations.

The estimator $\widehat{\kappa}_n^*(\theta)$ is the ratio of a quantity based on T_θ for some small $\theta > 0$ in the numerator, and a quantity based on $T_0 = \log$ in the denominator. Having this combination of T_θ's does not affect the numerical performance of the estimator but considerably simplifies the proofs. To illustrate the technical challenges that we face, we can just look at the uniform empirical distribution function $C_n(u, 1)$, $0 < u < 1$. To assess how close $T_\theta \circ C_n(u, 1)$ and $T_\theta \circ C(u, 1)$ are, we employ the mean value theorem and look at the smallness of

$$\frac{|C_n(u, 1) - C(u, 1)|}{C(u, 1)^{1-\theta}(1 - C(u, 1))^{1-\theta}},$$

uniformly over $0 < u < 1$. When $n \to \infty$, we have uniform convergence to 0 as long as $\theta > 0$, but there cannot be such convergence when $\theta = 0$. These facts follow immediately from classical results of Lai (1974) on the weighted Glivenko–Cantelli Theorem. For a multivariate version of this theorem, we refer to Mason (1982). For further references and advances on the topic, we refer to Berghaus *et al.* (1987).

We are now ready to discuss practical implications of working with model (2.2) instead of (2.1). As we have seen, the simplification has helped us to elucidate the main idea of the estimator, but from the practical point of view, one would of course like to work with a great variety of copulas, and not just with the M-O copula, or its symmetrized version (e.g., Furman *et al.*, 2015). To see how to overcome this obstacle, we go back to general model (2.1) for which statement (2.5) turns into

$$\begin{aligned} \kappa^* &= \frac{\log \Pi^*(u) - \log \ell(u)}{\log u} \\ &\leq \frac{\log C\left(\varphi(u), u^2/\varphi(u)\right)}{\log u} - \frac{\log \ell(u)}{\log u} \approx \frac{\log C\left(\varphi(u), u^2/\varphi(u)\right)}{\log u} \end{aligned} \tag{2.8}$$

with the approximation holding for sufficiently small $u > 0$, due to (e.g., Bingham *et al.*, 1987, Proposition 1.3.6(i), p. 16)

$$\varrho(u) := \frac{\log \ell(u)}{\log u} \to 0 \quad \text{when } u \downarrow 0. \tag{2.9}$$

The right-hand side of (2.8) is the same as that of (2.5). Hence, the above introduced estimator of TOMD can work for general copulas as long as the ratio

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$\varrho(u)$ is small, for which $u > 0$ has to be small. This can be achieved by using only those random pairs $(U_1, V_1), \ldots, (U_n, V_n)$ that are close to the origin $(0, 0)$. We shall therefore employ a conditioning argument later in this paper.

We should point out right at the outset that the restriction of data to only those pairs that are close to the origin $(0, 0)$ is not an issue from the theoretical point of view but becomes a challenge from the practical point of view. Indeed, the closer we are to the origin, the fewer random pairs we have, and thus we possibly have a large statistical estimation error. If, however, we use all n observations, then the statistical error becomes minimal but the model bias due to possibly large ratio $\varrho(u)$ might be large, unless we of course deal with the M-O copula, for which $\varrho(u) = 0$ for all $u \in (0, 1)$. Hence, when dealing with real data, a very delicate balancing act needs to be accomplished, but the task is certainly doable as we shall see in Section 4.

2.3. In pursuit of efficiency: an improved estimator

Simulations have shown that the estimators $\widehat{\kappa}_n^*$ and $\widehat{\kappa}_n^*(\theta)$ do work, but they are rather inefficient. For this reason, we next introduce a modification of $\widehat{\kappa}_n^*(\theta)$.

Let the pairs $(U_1, V_1), \ldots, (U_n, V_n)$ be iid, and since we are working under model (2.2), we can utilize all the pairs. (Under model (2.1), we would need to restrict ourselves to only those pairs that are in a neighborhood of $(0, 0)$; the smaller the neighborhood, the smaller the bias.) Next we choose any fixed $m \in \{1, \ldots, n\}$ and randomly assign the pairs $(U_1, V_1), \ldots, (U_n, V_n)$ into $k = \lceil n/m \rceil$ groups, so that there can be at most one group (perhaps none) with less than m pairs, whereas all the other groups contain exactly m pairs. Denote these k groups by $G_1, \ldots, G_k$. We define *the average block-minima estimator* of the TOMD κ^* by

$$\widehat{\kappa}_n^*(m, \theta) = \frac{1}{k} \sum_{j=1}^{k} \min_{i \in G_j} \frac{2T_\theta \circ C_n(U_i, V_i)}{\log U_i + \log V_i}. \tag{2.10}$$

Note that the earlier introduced estimator $\widehat{\kappa}_n^*(\theta)$ is $\widehat{\kappa}_n^*(m, \theta)$ when $m = n$, and thus $k = 1$. In view of the aforementioned deficiencies of $\widehat{\kappa}_n^*(\theta)$, however, in what follows (unless explicitly noted otherwise) we work with fixed m, which can in principle be allowed to depend on n, but this would require more complex arguments, which we wish to avoid at this stage of our research.

In a nutshell, the estimator $\widehat{\kappa}_n^*(m, \theta)$ works because:

(1) for every integer $m \geq 1$ and real $\theta > 0$, and when $n \to \infty$, the estimator $\widehat{\kappa}_n^*(m, \theta)$ converges to the limit

$$\kappa^*(m, \theta) = \mathbb{E}\left[\min_{i=1,\ldots,m} \frac{2T_\theta \circ C(U_i, V_i)}{\log U_i + \log V_i}\right]; \tag{2.11}$$

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(2) the TOMD κ^* can be written as

$$\kappa^* = \operatorname{ess\,inf} \frac{2\log C(U,V)}{\log U + \log V} \approx \operatorname{ess\,inf} \frac{2T_\theta \circ C(U,V)}{\log U + \log V}, \tag{2.12}$$

with the approximation holding for small $\theta > 0$.

Thus, κ^* can be approximated by $\kappa^*(m,\theta)$ for sufficiently large $m \geq 1$ and small $\theta > 0$.

Rigorous formulations and assumptions needed for the validity of the above two facts make up the contents of two theorems in the next section. Namely, Theorem 3.1 deals with consistency, whereas Theorem 3.2 deals with asymptotic bias. Their proofs, which are rather involved, are in the Appendix, where we also place graphs that visualize the asymptotic distribution (i.e., normality) of the estimator.

3. Estimator's statistical properties and performance

We split this section into two parts: Section 3.1 deals with theoretical aspects of the estimator $\widehat{\kappa}_n^*(m,\theta)$, and Section 3.2 illustrates the estimator's performance on simulated data.

3.1. Theoretical results

Throughout, we use the function

$$g(u,v) = \frac{\mathbb{1}\{0 < C(u,v) < 1\}}{C(u,v)(1 - C(u,v))},$$

which is defined for all $u, v \in [0,1]$.

Theorem 3.1. *Let C be positively quadrant dependent (PQD), that is, $C(u,v) \geq uv$ for every $u, v \in [0,1]$ and let $\theta \in (0,1)$ be any constant. If $\mathbb{E}[g(U,V)^{1-\theta}] < \infty$, then for every integer $m \geq 1$, we have*

$$\widehat{\kappa}_n^*(m,\theta) \xrightarrow{\mathrm{p}} \kappa^*(m,\theta) \tag{3.1}$$

when $n \to \infty$, where $\xrightarrow{\mathrm{p}}$ denotes convergence in probability.

We next discuss how to verify the finiteness of $\mathbb{E}[g(U,V)^{1-\theta}]$, for which we employ Kendall's cumulative distribution function (cdf) K_C (Nelsen *et al.*, 2003). It is the cdf of the random variable $C(U,V)$ when the pair (U,V) follows the joint cdf C. For example, when C is the M-O copula, the Kendall's cdf is (details in Section A.1)

$$K_{a,b}(t) = t - \left(2 - \frac{2}{\kappa^*}\right) t \log(t) \tag{3.2}$$

for all $t \in (0,1]$, where κ^* is the TOMD given equation (2.4).

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Note 3.1. The appearance of the TOMD κ^* and not of the classical TODD $\kappa = \min\{a, b\}$ in formula (3.2) for $K_{a,b}$ lends additional support for the naturalness of the TOMD as an index of tail dependence.

Lemma 3.1. *Let K_C denote the Kendall's cdf and let $\tau \in (0, 1)$ be any constant. If the integral $\int_{(0,1]} t^{-\tau}\, \mathrm{d}K_C(t)$ is finite, then the moment $\mathbb{E}[g(U, V)^\tau]$ is finite, where the random pair (U, V) follows the joint cdf C.*

To illustrate, when C is the comonotonic copula, we have $\Pi^*(u) = u$ and $K_C(t) = t$, and when C is the independence copula, we have $\Pi^*(u) = u^2$ and $K_C(t) = t - t \log t$ (Nelsen *et al.*, 2001). In both cases, Lemma 3.1 and thus Theorem 3.1 apply. The following corollary to Lemma 3.1 is convenient.

Corollary 3.1. *If the derivative K'_C of the Kendall's cdf exists in a neighborhood of 0 and is regularly varying of order $\alpha \in (-1, 0]$, that is, $K'_C(t) = t^\alpha \ell_C(t)$, where ℓ_C is a slowly varying at 0 function, then the moment $\mathbb{E}[g(U, V)^\tau]$ is finite for all $\tau \in (0, 1 + \alpha)$.*

Theorem 3.1 does not imply that $\widehat{\kappa}_n^*(m, \theta)$ is a consistent estimator of κ^* because $\kappa^*(m, \theta)$ is not generally equal to κ^*. Nevertheless, the following theorem shows that $\kappa^*(m, \theta)$ approaches κ^* when m gets sufficiently large and $\theta > 0$ sufficiently small. This may suggest to use the maximally large m, that is, $m = n$, but this would lead to $k = \lceil n/m \rceil = 1$ and thus turn the estimator $\widehat{\kappa}_n^*(m, \theta)$ into just one summand, thus effectively eliminating the important averaging effect that reduces the estimator's statistical error.

Theorem 3.2. *Let C be PQD, and let the functions $\varphi^*(u)$ and $\psi^*(u) := u^2/\varphi^*(u)$ be strictly increasing. Then $\kappa^*(m, \theta)$ can be made as close to κ^* as desired by taking sufficiently large $m \geq 1$ and sufficiently small $\theta > 0$.*

We next illustrate these theoretical results on simulated data.

3.2. An illustrative simulation study

To illustrate the performance of $\widehat{\kappa}_n^*(m, \theta)$, we simulate data from the M-O copula (equation (2.3)) whose TOMD is given by equation (2.4). Hence, $\Pi^*_{a,b}(u) = u^{\kappa^*}$. Since

$$K'_{a,b}(t) = \left(\frac{2}{\kappa^*} - 1\right) - \left(2 - \frac{2}{\kappa^*}\right) \log(t) \tag{3.3}$$

is slowly varying at 0, Theorem 3.1 applies. Furthermore, since the M-O copula is PQD, Theorem 3.2 also applies. Based on these results, and always setting the parameter θ to

$$\theta_0 := 10^{-6},$$

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TABLE 1

THE PARAMETER (a, b) CHOICES WITH κ^* VALUES AND MTD STRENGTH.

(a, b)	(0.18, 0.225)	(0.3529, 0.75)	(0.72, 0.9)
κ^*	1.80	1.52	1.20
MTD Strength	Weak	Medium	Strong

TABLE 2

SUMMARY STATISTICS FOR $\widehat{\kappa}_n^*(m, \theta_0)$ BASED ON IID DATA OF SIZE $n = 10{,}000$ WITH THE p-VALUES OF THE A-D AND C-VM TESTS FOR NORMALITY SHADED WHEN THEY FALL BELOW 0.05.

m	$k = \lceil n/m \rceil$	κ^*	Mean	StDev	A-D	C-vM
		1.80	1.8267	0.0186	0.7448	0.6912
3	3334	1.52	1.5466	0.0172	0.7114	0.5860
		1.20	1.1946	0.0151	0.6905	0.7290
		1.80	1.8073	0.0191	0.6053	0.5469
5	2000	1.52	1.5182	0.0181	0.3325	0.2770
		1.20	1.1819	0.0165	0.1222	0.1910
		1.80	1.7969	0.0195	0.4578	0.4273
7	1429	1.52	1.5065	0.0190	0.1679	0.1476
		1.20	1.1761	0.0179	0.0221	0.0469
		1.80	1.7872	0.0200	0.2899	0.2859
10	1000	1.52	1.4972	0.0203	0.0497	0.0502
		1.20	1.1700	0.0200	0.0026	0.0075
		1.80	1.7700	0.0219	0.0790	0.0959
20	500	1.52	1.4817	0.0244	0.0018	0.0040
		1.20	1.1563	0.0258	0.0000	0.0002

we expect the simulated values of $\widehat{\kappa}_n^*(m, \theta_0)$ to be close to the true theoretical ones. We verify this conjecture using a simulated experiment.

We set the M-O parameter (a, b) to the values specified in Table 1, with the corresponding values of κ^* calculated according to equation (2.4). Our adopted classification of the MTD strength according to the values of $\kappa^* \in [1, 2]$ is of course subjective: we say that MTD is weak when $\kappa^* \in [5/3, 2]$, medium when $\kappa^* \in [4/3, 5/3]$, and strong when $\kappa^* \in [1, 4/3]$. To study the effect of m on the estimator $\widehat{\kappa}_n^*(m, \theta_0)$, we set $m = 3, 5, 7, 10$, and 20. We measure the estimator's performance using the mean (Mean) and the standard deviation (StDev) based on 1,000 simulated values of $\widehat{\kappa}_n^*(m, \theta_0)$. We also calculate the p-values based on the Anderson–Darling (A-D) and Cramér–von Mises (C-vM) goodness-of-fit tests when the simulated values of $\widehat{\kappa}_n^*(m, \theta_0)$ are fitted to the normal distribution. The results for the sample sizes $n = 10{,}000$ and $n = 20{,}000$ are reported in Tables 2 and 3. The corresponding histograms and fitted normal densities are depicted in Figures A.3 and A.4. Based on these numerical results and graphs, we make the following observations.

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TABLE 3

SUMMARY STATISTICS FOR $\widehat{\kappa}_n^*(m,\theta_0)$ BASED ON IID DATA OF SIZE $n=20,000$ WITH THE p-VALUES OF THE A-D AND C-VM TESTS FOR NORMALITY SHADED WHEN THEY FALL BELOW 0.05.

m	$k=\lceil n/m\rceil$	κ^*	Mean	StDev	A-D	C-vM
3	6667	1.80	1.8296	0.0132	0.7623	0.6902
		1.52	1.5493	0.0124	0.9995	0.9984
		1.20	1.1983	0.0109	0.4018	0.3724
5	4000	1.80	1.8117	0.0136	0.8813	0.8560
		1.52	1.5226	0.0132	0.9594	0.9294
		1.20	1.1878	0.0120	0.0515	0.0608
7	2857	1.80	1.8025	0.0139	0.8974	0.8980
		1.52	1.5125	0.0138	0.5907	0.6090
		1.20	1.1836	0.0130	0.0077	0.0128
10	2000	1.80	1.7944	0.0144	0.9443	0.9683
		1.52	1.5052	0.0148	0.1778	0.2272
		1.20	1.1793	0.0145	0.0006	0.0017
20	1000	1.80	1.7813	0.0158	0.8076	0.8567
		1.52	1.4938	0.0178	0.0032	0.0088
		1.20	1.1697	0.0187	0.0000	0.0000

First, the simulated means are quite close to the corresponding values of κ^*. Note also that the differences between them get larger when the values of κ^* get smaller, which may suggest that the stronger the tail dependence is, the more observations are needed to accurately estimate κ^*. Tables 2 and 3 suggest that $m=3$ is not sufficiently large for the cases $\kappa^*=1.80$ and $\kappa^*=1.52$, because the differences between the simulated means and the theoretical values do not seem to diminish when the sample size increases. The choice $m=5$ seems appropriate as it leads to fairly accurate estimates of κ^* in all the cases considered. Setting m larger than 5 does not seem to significantly improve the estimates. Furthermore, we see from Figures A.3 and A.4 that the simulated distributions of $\widehat{\kappa}_n^*(m,\theta_0)$ exhibit uni-modality, even normality, which suggests that the simulated means get reasonably close to the theoretical values of κ^*.

4. ESTIMATING THE TOMD WHEN ℓ IS PRESENT

In this section, we adjust the methodology discussed in the previous sections in order to estimate TOMD when, unlike in the case of the M-O copula, a slowly varying function might be present. At the core of this adjustment is a conditioning argument. Namely, we restrict ourselves to only those observations that are near the origin $(0,0)$, which effectively shifts our attention from the original copula to the one conditioned on observations below certain prespecified thresholds. Since the conditional copula is the ratio of probabilities,

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this leads us – via equation (2.1) – to a ratio of the type $\ell(qu)/\ell(q)$, where $q > 0$ is a small parameter whose value could, for example, be a confidence level set by convention (e.g., 0.05), or by regulation (e.g., Basel Committee on Banking Supervision, 2016, 2019). The ratio $\ell^*(qu)/\ell^*(q)$ is close to 1 when $q > 0$ is small, due to the very definition of slowly varying at 0 function (e.g., Bingham *et al.*, 1987, Definition, p. 6).

4.1. Estimating the TOMD

In view of the equation $\Pi^*(u) = \ell^*(u)u^{\kappa^*}$ when $u \downarrow 0$, the conditional maximal tail probability is

$$\frac{C(\varphi^*(qu), q^2u^2/\varphi^*(qu))}{C(\varphi^*(q), q^2/\varphi^*(q))} = \frac{\ell^*(qu)(qu)^{\kappa^*}}{\ell^*(q)q^{\kappa^*}} \approx u^{\kappa^*}$$

when $0 < q \approx 0$. Hence, for any φ, we have the approximate bound

$$\frac{C(\varphi(qu), q^2u^2/\varphi(qu))}{C(\varphi^*(q), q^2/\varphi^*(q))} \lessapprox u^{\kappa^*}.$$

By letting $\tilde{u} = \varphi(qu)/\varphi^*(q)$ and $\tilde{v} = u^2\varphi^*(q)/\varphi(qu)$, where

$$\varphi^*(q) = \arg\max_x C(x, q^2/x),$$

the above approximate bound turns into

$$\frac{C(\varphi^*(q)\tilde{u}, q^2\tilde{v}/\varphi^*(q))}{C(\varphi^*(q), q^2/\varphi^*(q))} \lessapprox u^{\kappa^*}.$$

Note the equation $u^{\kappa^*} = (\tilde{u}\tilde{v})^{\kappa^*/2}$. With the notation

$$C_q^*(\tilde{u}, \tilde{v}) := \frac{C(\varphi^*(q)\tilde{u}, q^2\tilde{v}/\varphi^*(q))}{C(\varphi^*(q), q^2/\varphi^*(q))},$$

we arrive at the bound

$$\kappa^* \lessapprox \frac{2\log C_q^*(\tilde{u}, \tilde{v})}{\log\tilde{u} + \log\tilde{v}}.$$

Hence, given random pairs $(U_1, V_1), \ldots, (U_n, V_n)$ and a small risk level $q > 0$, the TOMD κ^* can be estimated via the following procedure:

(i) using all the pairs available in the entire unit square $[0, 1]^2$, compute an estimate of $\varphi^*(q)$ by maximizing $C_n(x, q^2/x)$ with respect to $x \in [q^2, 1]$, where C_n is the empirical copula constructed from all the n pairs that are available in the entire unit square $[0, 1]^2$;

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TABLE 4

SUMMARY OF SIMULATION RESULTS WHEN $m = 5$ AND $q = 0.05$.

(γ_0, γ_1)	κ^*	Mean	StDev	A-D	C-vM
(0.1, 0.8)	1.8	1.7914	0.0366	0.9565	0.9507
(0.4, 0.8)	1.5	1.5071	0.0219	0.9596	0.9183
(0.4, 0.2)	1.2	1.2086	0.0127	0.8601	0.8792

(ii) extract the set $\mathcal{P}_q$ of all those pairs (U_i, V_i) that are in the rectangle $[0, \varphi^*(q)] \times [0, q^2/\varphi^*(q)]$, and let $m_q := \#(\mathcal{P}_q)$ denote the number of pairs in $\mathcal{P}_q$;

(iii) randomly assign the pairs that are in $\mathcal{P}_q$ into $\lceil m_q/m \rceil$ disjoint groups, whose index sets we denote by $G_1, \ldots, G_{\lceil m_q/m \rceil} \subseteq \{1, 2, \ldots, n\}$, and they are such that there is at most one group with less than m elements, with all the other groups having exactly m elements;

(iv) compute the average block-minima estimator of the TOMD κ^* by the formula

$$\widehat{\kappa}^*_{m_q}(m, \theta, q) = \frac{1}{\lceil m_q/m \rceil} \sum_{j=1}^{\lceil m_q/m \rceil} \min_{i \in G_j} \frac{2T_\theta \circ C^*_{q,m_q}(\tilde{U}_i, \tilde{V}_i)}{\log \tilde{U}_i + \log \tilde{V}_i},$$

where $\tilde{U}_i = U_i/\varphi^*(q)$, $\tilde{V}_i = V_i\varphi^*(q)/q^2$, and

$$C^*_{q,m_q}(u, v) = \frac{1}{m_q} \sum_{(U_k, V_k) \in \mathcal{P}_q} \mathbb{1}\{\tilde{U}_k \le u, \tilde{V}_k \le v\}.$$

4.2. An illustrative simulation study

To demonstrate how the estimator $\widehat{\kappa}^*_{m_q}(m, \theta, q)$ works in practice, we simulate $n = 500{,}000$ independent pairs $(U_i, V_i) \sim C_{\gamma_0,\gamma_1}$ from the generalized Clayton copula

$$C_{\gamma_0,\gamma_1}(u, v) = u^{\gamma_1/\gamma^*}\left(u^{-1/\gamma^*} + v^{-1/\gamma_0} - 1\right)^{-\gamma_0}, \quad u, v \in [0, 1],$$

where $\gamma_0 > 0$ and $\gamma_1 \ge 0$ are the model parameters, with their values specified in Table 4, and $\gamma^* := \gamma_0 + \gamma_1$. As calculated by Furman *et al.* (2015), the TOMD for this copula is

$$\kappa^* = 1 + \frac{\gamma_1}{\gamma_1 + 2\gamma_0}.$$

We repeat the procedure 1000 times and thus obtain 1000 values of $\widehat{\kappa}^*_{m_q}(m, \theta_0, q)$ with the parameter values set to $m = 5$ and $q = 0.05$. Summary statistics are reported in Table 4, from which we see that the A-D and

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TABLE 5

SUMMARY OF SIMULATION RESULTS WHEN $m = 5$ AND $q = 0.1$.

(γ_0, γ_1)	κ^*	Mean	StDev	A-D	C-vM
(0.1, 0.8)	1.8	1.8072	0.0200	0.9436	0.8645
(0.4, 0.8)	1.5	1.5142	0.0136	0.7527	0.7505
(0.4, 0.2)	1.2	1.2111	0.0081	0.7281	0.7883

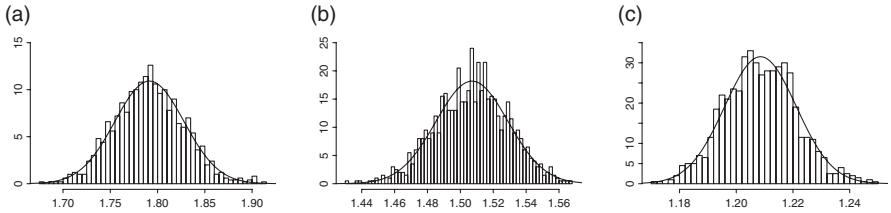


FIGURE 2: Fits of simulated $\widehat{\kappa}^*_{m_q}(m, \theta_0, q)$ values to the normal distribution when $m = 5$ and $q = 0.05$. (a) $m = 5$, $\kappa^* = 1.8$. (b) $m = 5$, $\kappa^* = 1.5$. (c) $m = 5$, $\kappa^* = 1.2$

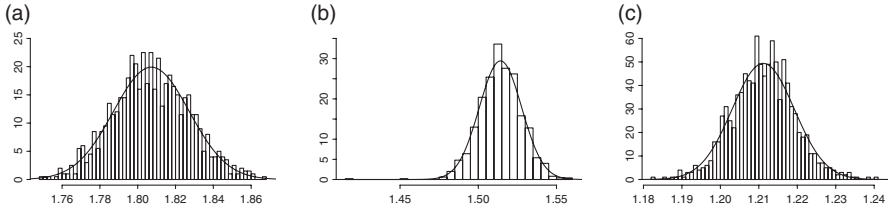


FIGURE 3: Fits of simulated $\widehat{\kappa}^*_{m_q}(m, \theta_0, q)$ values to the normal distribution when $m = 5$ and $q = 0.1$. (a) $m = 5$, $\kappa^* = 1.8$. (b) $m = 5$, $\kappa^* = 1.5$. (c) $m = 5$, $\kappa^* = 1.2$.

C-vM goodness-of-fit p-values retain the null hypothesis of normality. Fits of the estimator values to the normal distribution are depicted in Figure 2.

For comparison, using the same parameter (γ_0, γ_1) values as before, and the same $m = 5$ but now with the increased threshold $q = 0.1$ and thus a larger m_q, we obtain 1000 values of the TOMD estimator $\widehat{\kappa}^*_{m_q}(m, \theta_0, q)$ whose summary statistics are reported in Table 5. The A-D and C-vM goodness-of-fit p-values retain the null hypothesis of normality. Fits of the estimator values to the normal distribution are depicted in Figure 3.

By increasing the risk level q from 0.05 to 0.1, we have increased the number m_q of pairs and thus potentially reduced the estimator's statistical error, but this could have increased the deterministic bias tackled in Theorem 3.2. As we have noted earlier, it is a delicate task to strike the right balance between the competing forces of having small $q > 0$ and large m_q.

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5. CONCLUSION

We have proposed an empirical estimator of the tail order of maximal dependence. The estimator successfully avoids the complexity of deriving explicit formulas of the underlying paths of MTD. To test the estimator's practical performance, we have conducted a simulation study and discussed our findings. There are still a number of interesting problems for future research, including these:

(1) Explore estimator's statistical properties and practical performance under various dynamical DGP's (i.e., data generating processes) such as those arising from short- and long-range memory time series.
(2) Relax the currently imposed fixedness of m by describing those sequences $m = m_n \in \{1, \ldots, n\}$ which grow to infinity together with $n \to \infty$ and maintain asymptotic unbiasedness of the proposed estimator.
(3) When searching for efficient and reliable estimators, explore different block designs and their combinations using, for example, generalized means, as illustrated by Vovk and Wang (2020) in a statistically different but philosophically closely related context.
(4) Explore the roles of the functions T_θ and $T_0 = \log$ when defining estimators.
(5) Derive asymptotic normality/distribution of the estimator and explore ways for practical assessment of its standard error.
(6) In addition to κ^*, Furman *et al.* (2015) also discuss other indices related to maximal tail dependence, such as λ_L^* and χ_L^*. We have indications, but no proofs yet, that techniques of the present paper could be adapted to estimate those indices as well.

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REFERENCES

BASEL COMMITTEE ON BANKING SUPERVISION (2016) *Minimum Capital Requirements for Market Risk (January, 2016)*. Bank for International Settlements, Basel. https://www.bis.org/bcbs/publ/d352.htm.

BASEL COMMITTEE ON BANKING SUPERVISION (2019) *Minimum Capital Requirements for Market Risk (February, 2019)*. Bank for International Settlements, Basel. https://www.bis.org/bcbs/publ/d457.htm.

https://doi.org/10.1017/asb.2020.21 Published online by Cambridge University Press

BERGHAUS, B., BÜCHER, A. and VOLGUSHEV, S. (2017) Weak convergence of the empirical copula process with respect to weighted metrics. *Bernoulli*, **23**, 743–772.
BINGHAM, N.H., GOLDIE, C.M. and TEUGELS, J.L. (1987) *Regular Variation.* Cambridge: Cambridge University Press.
CHERUBINI, U., LUCIANO, E. and VECCHIATO, W. (2013) *Copula Methods in Finance.* Chichester: Wiley.
CHERUBINI, U., DURANTE, F. and MULINACCI, S. (2015) *Marshall Olkin Distributions – Advances in Theory and Applications.* New York: Springer.
COLES, S., HEFFERNAN, J. and TAWN, J. (1999) Dependence measures for extreme value analyses. *Extremes*, **2**, 339–365.
DRAISMA, G., DREES, H., FERREIRA, A. and DE HAAN, L. (2004) Bivariate tail estimation: dependence in asymptotic independence. *Bernoulli*, **10**, 251–280.
DURANTE, F. and SEMPI, C. (2015) *Principles of Copula Theory.* Boca Raton: Chapman and Hall/CRC.
FRAHM, G., JUNKER, M. and SCHMIDT, R. (2005) Estimating the tail-dependence coefficient: Properties and pitfalls. *Insurance: Mathematics and Economics*, **37**, 80–100.
FURMAN, E., KUZNETSOV, A., SU, J. and ZITIKIS, R. (2016) Tail dependence of the Gaussian copula revisited. *Insurance: Mathematics and Economics*, **69**, 97–103.
FURMAN, E., SU, J. and ZITIKIS, R. (2015) Paths and indices of maximal tail dependence. *ASTIN Bulletin: The Journal of the International Actuarial Association*, **45**, 661–678.
HUA, L. and JOE, H. (2011) Tail order and intermediate tail dependence of multivariate copulas. *Journal of Multivariate Analysis*, **102**, 1454–1471.
JOE, H. (1993) Parametric families of multivariate distributions with given margins. *Journal of Multivariate Analysis*, 46, 262–282.
JOE, H. (2014) *Dependence Modeling with Copulas.* Boca Raton: Chapman and Hall/CRC.
KRUPSKII, P. and JOE, H. (2019) Nonparametric estimation of multivariate tail probabilities and tail dependence coefficients. *Journal of Multivariate Analysis*, **172**, 147–161.
LAI, T.L. (1974) Convergence rates in the strong law of large numbers for random variables taking values in Banach spaces. *Bulletin of the Institute of Mathematics Academia Sinica*, **2**, 67–85.
MASON, D.M. (1982) Some characterizations of almost sure bounds for weighted multidimensional empirical distributions and a Glivenko-Cantelli theorem for sample quantiles. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, **59**, 505–513.
MCNEIL, A.J., FREY, R. and EMBRECHTS, P. (2015) *Quantitative Risk Management: Concepts, Techniques and Tools* (Revised Edition). Princeton: Princeton University Press.
NELSEN, R.B., QUESADA-MOLINA, J.J., RODRÍGUEZ-LALLENA, J.A. and ÚBEDA-FLORES, M. (2001) Distribution functions of copulas: A class of bivariate probability integral transforms. *Statistics and Probability Letters*, **54**, 277–282.
NELSEN, R.B., QUESADA-MOLINA, J.J., RODRÍGUEZ-LALLENA, J.A. and ÚBEDA-FLORES, M. (2003) Kendall distribution functions. *Statistics and Probability Letters*, **65**, 263–268.
NELSEN, R.B. (2006) *An Introduction to Copulas*, 2nd edition. New York: Springer.
SU, J. and FURMAN, E. (2017) Multiple risk factor dependence structures: Copulas and related properties. *Insurance: Mathematics and Economics*, **74**, 109–121.
VOVK, V. and WANG, R. (2020) Combining *p*-values via averaging. *Biometrika* (to appear).
WANG, R. and ZITIKIS, R. (2020) An axiomatic foundation for the Expected Shortfall. *Management Science* (to appear).

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APPENDIX A. TECHNICAL APPENDICES

In Section A.1, we present a number of technical results that will later be used for proving Theorems 3.1 and 3.2 in Section A.2.

A.1. Auxiliary results

We start with three calculus-type propositions, which are included here to make the proofs in subsequent appendices more self-contained and thus easier checked.

Proposition A.1. *For all real $x_i, y_i \in \mathbb{R}$ and integer $m \geq 1$, we have*

$$\Big| \min_{i=1,\dots,m} x_i - \min_{i=1,\dots,m} y_i \Big| \leq \max_{i=1,\dots,m} |x_i - y_i|.$$

Proof. Let j and k be those that satisfy the equations $\min_{i=1,\dots,m} x_i = x_j$ and $\min_{i=1,\dots,m} y_i = y_k$. If $x_j \geq y_k$, then $|x_j - y_k| = x_j - y_k \leq x_k - y_k$, but if $x_j < y_k$, then $|x_j - y_k| = y_k - x_j \leq y_j - x_j$. This finishes the proof. □

Proposition A.2. *Let $\theta \in (0,1)$. The bound $x^{1-\theta}|y^\theta - x^\theta| \leq |y - x|$ holds for all $x, y \in (0,\infty)$.*

Proof. Denote $\Delta := x^{1-\theta}|y^\theta - x^\theta|$. If $y > x$, then $\Delta = x^{1-\theta}y^\theta - x \leq y - x = |y - x|$, but if $x > y$, then $\Delta = x - x^{1-\theta}y^\theta \leq x - y = |y - x|$. This finishes the proof. □

Proposition A.3. *For all $x \leq 0$, we have $|e^x - 1 - x| \leq 0.5x^2$.*

Proof. The functions $f(x) = e^x - 1 - x$ and $g(x) = 0.5x^2$ are nonnegative and satisfy $f(0) = 0 = g(0)$. Furthermore, their difference $g(x) - f(x)$ is nonincreasing. Hence,

$$g(x) - |f(x)| = g(x) - f(x) \geq g(0) - f(0) = 0,$$

which completes the proof. □

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For any copula C, its Kendall's cdf is, by definition, the distribution of the random variable $C(U, V)$, where (U, V) follows the joint cdf C.

Proposition A.4. *The Kendall's cdf $K_{a,b}$ of the M-O copula $C_{a,b}$ is given by formula (3.2).*

Proof. Let (U, V) denote the random pair whose joint cdf is the M-O copula $C_{a,b}$. Then (e.g., Cherubini *et al.*, 2013, p. 128)

$$\mathbb{P}(U \le u \mid V = v) = u^{1-a}\mathbb{1}\{u \in [v^{b/a}, 1]\} + (1-b)uv^{-b}\mathbb{1}\{u \in [0, v^{b/a})\}$$

for all $u, v \in [0, 1]$. Using this equation, we next derive formula (3.2) for the Kendall's cdf $K_{a,b}$ by establishing the corresponding formula for the Kendall's survival function $\overline{K}_{a,b} = 1 - K_{a,b}$. Namely, for $t \in (0, 1)$, we have

$$\begin{aligned}
\overline{K}_{a,b}(t) &= \mathbb{P}\left(C_{a,b}(U, V) > t\right) \\
&= \mathbb{P}\left(U^{1-a}V > t,\, UV^{1-b} > t\right) \\
&= \int_t^1 \mathbb{P}\left(U > \max\left((t/v)^{1/(1-a)}, tv^{b-1}\right) \mid V = v\right) \mathrm{d}v \\
&= \int_t^1 \Big(1 - \max\left(t/v, t^{1-a}v^{(b-1)(1-a)}\right)\mathbb{1}\{\max\left((t/v)^{1/(1-a)}, tv^{b-1}\right) \in [v^{b/a}, 1]\} \\
&\qquad - (1-b)\max\left((t/v)^{1/(1-a)}, tv^{b-1}\right)v^{-b}\mathbb{1}\{\max\left((t/v)^{1/(1-a)}, tv^{b-1}\right) \in [0, v^{b/a})\}\Big)\,\mathrm{d}v.
\end{aligned}$$

Since $(t/v)^{1/(1-a)} \ge v^{b/a}$ is equivalent to the bound $t^{1/(b/a-b+1)} \ge v$, and $tv^{b-1} \ge v^{b/a}$ is also equivalent to the same bound $t^{1/(b/a-b+1)} \ge v$, we have the following equivalence relationships:

$$\max\left((t/v)^{1/(1-a)}, tv^{b-1}\right) \in [v^{b/a}, 1] \iff t^{1/(b/a-b+1)} \ge v \iff t/v \ge t^{1-a}v^{(b-1)(1-a)}.$$

Hence,

$$\begin{aligned}
\overline{K}_{a,b}(t) &= 1 - t - t\int_t^{t^{1/(b/a-b+1)}} v^{-1}\,\mathrm{d}v - (1-b)t\int_{t^{1/(b/a-b+1)}}^1 v^{-1}\,\mathrm{d}v \\
&= 1 - t - t\int_t^1 v^{-1}\,\mathrm{d}v + bt\int_{t^{1/(b/a-b+1)}}^1 v^{-1}\,\mathrm{d}v \\
&= 1 - t + t\log(t) - \frac{ab}{a+b-ab}t\log(t) \\
&= 1 - t + \left(2 - \frac{a+b}{a+b-ab}\right)t\log(t) \\
&= 1 - t + \left(2 - \frac{2}{\kappa^*}\right)t\log(t).
\end{aligned}$$

This completes the proof of formula (3.2) and establishes Proposition A.4. □

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Proof of Lemma 3.1. Note that $K_C(t) \geq t$ for all $t \in (0,1)$. Hence,

$$\begin{aligned}\mathbb{E}[g(U,V)^\tau \mathbb{1}\{C(U,V) > 1/2\}] &= \int_{(1/2,1]} \frac{1}{t^\tau (1-t)^\tau} \, \mathrm{d}K_C(t) \\ &\leq 2^\tau \int_{(1/2,1]} \frac{1}{(1-K_C(t))^\tau} \, \mathrm{d}K_C(t) < \infty\end{aligned}$$

for every $\tau \in (0,1)$. The expectation $\mathbb{E}[g(U,V)^\tau \mathbb{1}\{C(U,V) < 1/2\}]$ is finite whenever $\int_{(0,1]} t^{-\tau} \, \mathrm{d}K_C(t) < \infty$, which establishes Lemma 3.1. □

Proposition A.5. *Let (U,V) follow the joint distribution C. If the moment $\mathbb{E}[g(U,V)^\tau]$ is finite for some $\tau \in (0,1)$, then when $n \to \infty$,*

$$\Delta_n(\tau) := \sup_{u,v \in [0,1]} g(u,v)^\tau |C_n(u,v) - C(u,v)| \xrightarrow{\mathrm{p}} 0.$$

Proof. We begin by noting that for every sufficiently large constant c (e.g., $c > 4^\tau$), there exists $\gamma_c > 0$ such that

$$c = (1/2 + \gamma_c)^{-\tau} (1/2 - \gamma_c)^{-\tau},$$

and $\gamma_c \to 1/2$ when $c \to \infty$. Hence, $g(u,v)^\tau > c$ if and only if

$$C(u,v) \in (0, 1/2 - \gamma_c) \cup (1/2 + \gamma_c, 1).$$

Therefore,

$$\Delta_n(\tau) \leq c I_{1n}(c) + I_{2n}(c) + I_{3n}(c), \tag{A.1}$$

where

$$\begin{aligned}I_{1n}(c) &= \sup_{u,v \in [0,1]} |C_n(u,v) - C(u,v)|, \\ I_{2n}(c) &= \sup_{u,v \in [0,1]: C(u,v) < 1/2 - \gamma_c} g(u,v)^\tau |C_n(u,v) - C(u,v)|, \\ I_{3n}(c) &= \sup_{u,v \in [0,1]: C(u,v) > 1/2 + \gamma_c} g(u,v)^\tau |C_n(u,v) - C(u,v)|.\end{aligned}$$

For every fixed $c > 0$, the two-dimensional version of the Glivenko–Cantelli theorem implies

$$I_{1n}(c) \xrightarrow{\mathrm{p}} 0 \tag{A.2}$$

when $n \to \infty$. As to $I_{2n}(c)$, we estimate it from above by $J_n(c) + J(c)$, where

$$\begin{aligned}J_n(c) &= \sup_{u,v \in [0,1]: C(u,v) < 1/2 - \gamma_c} g(u,v)^\tau \, C_n(u,v), \\ J(c) &= \sup_{u,v \in [0,1]: C(u,v) < 1/2 - \gamma_c} g(u,v)^\tau \, C(u,v).\end{aligned}$$

To tackle $J_n(c)$, we first note that when $C(u,v) < 1/2 - \gamma_c$, the function $g(u,v)^\tau$ is nonincreasing in each argument. Hence,

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$$J_n(c) \le \frac{1}{n}\sum_{i=1}^{n} \sup_{u,v\in[0,1]:C(u,v)<1/2-\gamma_c} g(u,v)^\tau \mathbb{1}\{U_i \le u, V_i \le v\}$$
$$\le \frac{1}{n}\sum_{i=1}^{n} g(U_i,V_i)^\tau \sup_{u,v\in[0,1]:C(u,v)<1/2-\gamma_c} \mathbb{1}\{U_i \le u, V_i \le v\}$$
$$\le \frac{1}{n}\sum_{i=1}^{n} g(U_i,V_i)^\tau \sup_{u,v\in[0,1]:C(u,v)<1/2-\gamma_c} \mathbb{1}\{C(U_i,V_i) \le C(u,v)\}$$
$$\le \frac{1}{n}\sum_{i=1}^{n} g(U_i,V_i)^\tau \mathbb{1}\{C(U_i,V_i) < 1/2-\gamma_c\}.$$

By the law of large numbers, for every $\epsilon > 0$, we can find a sufficiently large $c < \infty$ such that

$$\mathbb{P}\big(J_n(c) \ge \epsilon\big) \to 0$$

when $n \to \infty$, because $\gamma_c \to 1/2$ when $c \to \infty$ and thus $\mathbb{E}[g(U,V)^\tau \mathbb{1}\{C(U_1,V_1) < 1/2-\gamma_c\}]$ can be made as small as desired due to $\mathbb{E}[g(U,V)^\tau] < \infty$. Furthermore,

$$J(c) \le \sup_{u,v\in[0,1]:C(u,v)<1/2-\gamma_c} \frac{C(u,v)^{1-\tau}}{(1-C(u,v))^\tau} \le \frac{(1/2-\gamma_c)^{1-\tau}}{(1/2+\gamma_c)^\tau} \to 0$$

when $c \to \infty$. Hence, for every $\epsilon > 0$, we can find a sufficiently large $c < \infty$ such that, when $n \to \infty$,

$$\mathbb{P}\big(I_{2n}(c) \ge \epsilon\big) \to 0. \tag{A.3}$$

Finally, to tackle $I_{3n}(c)$, we start with the bound $I_{3n}(c) \le K_n(c) + K(c)$, where

$$K_n(c) = \sup_{u,v\in[0,1]:C(u,v)>1/2+\gamma_c} g(u,v)^\tau (1 - C_n(u,v)),$$
$$K(c) = \sup_{u,v\in[0,1]:C(u,v)>1/2+\gamma_c} g(u,v)^\tau (1 - C(u,v)).$$

We have

$$K_n(c) \le \frac{1}{n}\sum_{i=1}^{n} \sup_{u,v\in[0,1]:C(u,v)>1/2+\gamma_c} g(u,v)^\tau \big(\mathbb{1}\{U_i > u\} + \mathbb{1}\{V_i > v\}\big).$$

When $C(u,v) > 1/2+\gamma_c$, the function $g(u,v)^\tau$ is nondecreasing in each argument and thus

$$K_n(c) \le \frac{1}{n}\sum_{i=1}^{n} g(U_i,1)^\tau \sup_{u\in[0,1]:u>1/2+\gamma_c} \mathbb{1}\{U_i > u\} + \frac{1}{n}\sum_{i=1}^{n} g(1,V_i)^\tau \sup_{v\in[0,1]:v>1/2+\gamma_c} \mathbb{1}\{V_i > v\}$$
$$\le \frac{1}{n}\sum_{i=1}^{n} g(U_i,1)^\tau \mathbb{1}\{U_i > 1/2+\gamma_c\} + \frac{1}{n}\sum_{i=1}^{n} g(1,V_i)^\tau \mathbb{1}\{V_i > 1/2+\gamma_c\}.$$

By the law of large numbers, for every $\epsilon > 0$, we can find a sufficiently large $c < \infty$ such that

$$\mathbb{P}\big(K_n(c) \ge \epsilon\big) \to 0$$

when $n \to \infty$, because $\gamma_c \to 1/2$ when $c \to \infty$ and thus the expectations $\mathbb{E}[g(U,1)^\tau \mathbb{1}\{U > 1/2+\gamma_c\}]$ and $\mathbb{E}[g(1,V)^\tau \mathbb{1}\{V > 1/2+\gamma_c\}]$ can be made as small as desired, due to

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$$\mathbb{E}[g(U,1)^\tau] = \mathbb{E}[g(1,V)^\tau] = \text{Beta}(1-\tau, 1-\tau) < \infty.$$

Furthermore,

$$K(c) \le \sup_{u,v\in[0,1]:C(u,v)>1/2+\gamma_c} \frac{(1-C(u,v))^{1-\tau}}{C(u,v)^\tau} \le \frac{(1/2-\gamma_c)^{1-\tau}}{(1/2+\gamma_c)^\tau} \to 0$$

when $c \to \infty$. Consequently, for every $\epsilon > 0$, we can find a sufficiently large $c < \infty$ such that

$$\mathbb{P}\big(I_{3n}(c) \ge \epsilon\big) \to 0 \tag{A.4}$$

when $n \to \infty$. Bound (A.1) and statements (A.2)–(A.4) establish Proposition A.5. □

A.2. Proofs of Theorems 3.1 and 3.2

Proof of Theorem 3.1. Since there is at most one group G_j whose cardinality is less than m, without loss of generality we can, and thus do, assume $n = mk$. Denote

$$\widehat{\xi}_{i,n,\theta} = \frac{2T_\theta \circ C_n(U_i, V_i)}{\log U_i + \log V_i} \quad \text{and} \quad \xi_{i,\theta} = \frac{2T_\theta \circ C(U_i, V_i)}{\log U_i + \log V_i}$$

for all $i = 1, \dots, n$. By Proposition A.1, for every $j = 1, \dots, k$, we have

$$\begin{aligned}\Big|\min_{i\in G_j} \widehat{\xi}_{i,n,\theta} - \min_{i\in G_j} \xi_{i,\theta}\Big| &\le \max_{i\in G_j} \big|\widehat{\xi}_{i,n,\theta} - \xi_{i,\theta}\big| \\ &= \frac{2}{\theta} \max_{i\in G_j} \left|\frac{C_n(U_i,V_i)^\theta - C(U_i,V_i)^\theta}{\log U_i + \log V_i}\right| \\ &\le \frac{2}{\theta} \max_{i\in G_j} \frac{|C_n(U_i,V_i) - C(U_i,V_i)|}{C(U_i,V_i)^{1-\theta}|\log U_i + \log V_i|},\end{aligned} \tag{A.5}$$

where the last inequality is due to Proposition A.2. Since C is PQD, we have $uv \le C(u,v)$ and thus $|\log(uv)| \ge |\log C(u,v)|$ for all $u, v \in [0,1]$. Bound (A.5) implies

$$\Big|\min_{i\in G_j} \widehat{\xi}_{i,n,\theta} - \min_{i\in G_j} \xi_{i,\theta}\Big| \le \frac{2}{\theta} \max_{i\in G_j} \frac{|C_n(U_i,V_i) - C(U_i,V_i)|}{C(U_i,V_i)^{1-\theta}|\log C(U_i,V_i)|}. \tag{A.6}$$

Furthermore, since $|\log C(u,v)| \ge 1 - C(u,v)$ for every $u, v \in [0,1]$, bound (A.6) implies

$$\begin{aligned}\Big|\min_{i\in G_j} \widehat{\xi}_{i,n,\theta} - \min_{i\in G_j} \xi_{i,\theta}\Big| &\le \frac{2}{\theta} \max_{i\in G_j} \frac{|C_n(U_i,V_i) - C(U_i,V_i)|}{C(U_i,V_i)^{1-\theta}[1 - C(U_i,V_i)]} \\ &\le \frac{2}{\theta} \Delta_n(1-\theta) \max_{i\in G_j} [1 - C(U_i,V_i)]^{-\theta},\end{aligned} \tag{A.7}$$

where $\Delta_n(\cdot)$ is defined in Proposition A.5. Hence,

$$\begin{aligned}\left|\frac{1}{k}\sum_{j=1}^k \min_{i\in G_j} \widehat{\xi}_{i,n,\theta} - \frac{1}{k}\sum_{j=1}^k \min_{i\in G_j} \xi_{i,\theta}\right| &\le \frac{1}{k}\sum_{j=1}^k \Big|\min_{i\in G_j} \widehat{\xi}_{i,n,\theta} - \min_{i\in G_j} \xi_{i,\theta}\Big| \\ &\le \Delta_n(1-\theta)\frac{2}{k\theta}\sum_{j=1}^k \max_{i\in G_j} [1 - C(U_i,V_i)]^{-\theta}\end{aligned}$$

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$$\leq \Delta_n(1-\theta)\frac{2}{k\theta}\sum_{j=1}^{k}\sum_{i\in G_j}[1-C(U_i,V_i)]^{-\theta}$$

$$\leq \Delta_n(1-\theta)\frac{2m}{n\theta}\sum_{i=1}^{n}[1-C(U_i,V_i)]^{-\theta}. \tag{A.8}$$

Since the moment $\mathbb{E}[g(U,V)^{1-\theta}]$ is finite by assumption, the quantity $\Delta_n(1-\theta)$ converges to 0 in probability. Hence, the entire right-hand side of bound (A.8) converges to 0 in probability if the average $n^{-1}\sum_{i=1}^{n}[1-C(U_i,V_i)]^{-\theta}$ converges to a finite number. This follows from the classical law of large numbers because

$$\mathbb{E}[(1-C(U,V))^{-\theta}] = \int_{(0,1)}\frac{1}{(1-t)^{\theta}}\,\mathrm{d}K_C(t)$$

$$\leq \int_{(0,1)}\frac{1}{(1-K_C(t))^{\theta}}\,\mathrm{d}K_C(t) < \infty.$$

Therefore, the left-hand side of bound (A.8) converges to 0 in probability, which means

$$\widehat{\kappa}_n^*(m,\theta) - \frac{1}{k}\sum_{j=1}^{k}\min_{i\in G_j}\xi_{i,\theta} \xrightarrow{\mathrm{p}} 0$$

when $k\to\infty$ ($n\to\infty$), thus concluding the proof of Theorem 3.1. □

Proof of Theorem 3.2. First we show that for every $m\geq 1$, we have

$$\kappa^*(m,\theta)\to\kappa^*(m,0) \tag{A.9}$$

when $\theta\downarrow 0$. For this, we first employ Proposition A.1 and have

$$|\kappa^*(m,\theta)-\kappa^*(m,0)| \leq \mathbb{E}\left(\max_{i=1,\ldots,m}\left|\frac{2T_\theta\circ C(U_i,V_i)}{\log U_i+\log V_i}-\frac{2T_0\circ C(U_i,V_i)}{\log U_i+\log V_i}\right|\right). \tag{A.10}$$

Next we apply Proposition A.3 with $x=\theta\log(C(U_i,V_i))\leq 0$ and have

$$|\kappa^*(m,\theta)-\kappa^*(m,0)| \leq \theta\mathbb{E}\left(\max_{i=1,\ldots,m}\left|\frac{(\log C(U_i,V_i))^2}{\log U_i+\log V_i}\right|\right)$$

$$\leq \theta\mathbb{E}\left(\max_{i=1,\ldots,m}\left|\log(U_iV_i)\right|\right),$$

where the last inequality is due to C being PQD. Hence,

$$\left|\kappa^*(m,\theta)-\kappa^*(m,0)\right| \leq \theta\mathbb{E}\left(\max_{i=1,\ldots,m}(-\log U_i)\right)+\theta\mathbb{E}\left(\max_{i=1,\ldots,m}(-\log V_i)\right)$$

$$= 2m\theta\int_0^{\infty}x(1-e^{-x})^{m-1}e^{-x}\,\mathrm{d}x.$$

For every m, the right-hand side of the latter bound can be made as small as desired by choosing a sufficiently small $\theta>0$. This concludes the proof of statement (A.9).

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We next prove that $\kappa^*(m,0)$ can be made as close to κ^* as desired by choosing a sufficiently large m. We start with the equation

$$\kappa^*(m,0) = \int_0^2 \mathbb{P}(\xi > x)^m \, \mathrm{d}x, \tag{A.11}$$

where

$$\xi := \frac{2\log C(U,V)}{\log U + \log V} \in [0,2] \tag{A.12}$$

with $(U,V) \sim C$. Since $\Pi^*(u) = u^{\kappa^*}$ for all $u \in [0,1]$ by assumption, we have

$$\kappa^* \le \frac{2\log C(u,v)}{\log u + \log v}$$

for all $u, v \in [0,1]$, and thus $\kappa^* \le \operatorname{ess\,inf} \xi$. Hence, equation (A.11) becomes

$$\kappa^*(m,0) = \kappa^* + \int_{\kappa^*}^2 \mathbb{P}(\xi > x)^m \, \mathrm{d}x, \tag{A.13}$$

from which we see that we only need to show that the integral on the right-hand side converges to 0 when $m \to \infty$. In doing so, we can assume without loss of generality that $\kappa^* < 2$, which effectively eliminates the independence copula, that is, from now on we can, and thus do, assume that C is not the independence copula.

By the Lebesgue dominated convergence theorem, the integral converges to 0 provided that $\mathbb{P}(\xi > x) \in (0,1)$ for all $x \in (\kappa^*, 2)$. In other words, we need to show that

$$\mathbb{P}(\xi \le \kappa^* + h) > 0 \quad \text{for every} \quad h \in (0, 2-\kappa^*). \tag{A.14}$$

We start with the bound

$$\begin{aligned} \mathbb{P}(\xi \le \kappa^* + h) &= \mathbb{P}\left(\frac{2\log C(U,V)}{\log U + \log V} \le \frac{2\log w_0^{\kappa^*+h}}{\log w_0^2} \right) \\ &\ge \mathbb{P}\left((U,V) \in B_h \right), \end{aligned} \tag{A.15}$$

where the area B_h (see Figure A.1) is defined by the equation

$$B_h = \left\{ (u,v) \in [0,1]^2 : uv \le w_0^2, \ C(u,v) > w_0^{\kappa^*+h} \right\}.$$

Since C is PQD but not the independence copula, we can find $u_0, v_0 \in (0,1)$ such that

$$u_0 v_0 < C(u_0, v_0) = (u_0 v_0)^{\kappa^*/2}.$$

With the notation $w_0 = \sqrt{u_0 v_0}$, we have $u_0 = \varphi^*(w_0)$, $v_0 = w_0^2/\varphi^*(w_0)$, and

$$C\left(\varphi^*(w_0), w_0^2/\varphi^*(w_0) \right) = w_0^{\kappa^*}.$$

Since $\kappa^* < 2$, $\varphi^*(w_0)$ is neither w_0^2 nor 1. Therefore, there is $\eta \in (0,2)$ such that $\varphi^*(w_0) = w_0^\eta$ due to $w_0^2 < \varphi^*(w_0) < 1$. Define the function $\Phi_{w_0}^* : [0,2] \to [0,1]$ by the formula

$$\Phi_{w_0}^*(x) = C\left(w_0^x, w_0^{2-x} \right).$$

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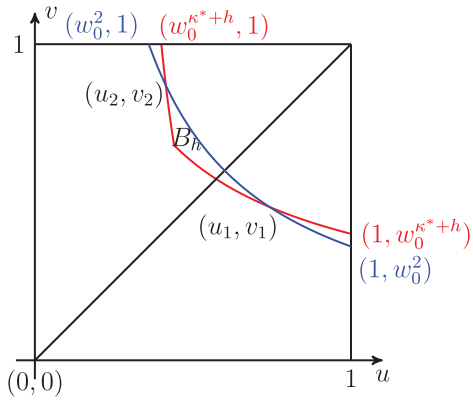


FIGURE A.1: The area B_h with its associated curves and points.

Note the values:

$$\Phi^*_{w_0}(0) = w_0^2, \quad \Phi^*_{w_0}(\eta) = w_0^{\kappa^*} \quad \text{and} \quad \Phi^*_{w_0}(2) = w_0^2.$$

Moreover, for $0 \le x_1 \le x_2 \le 2$, we have

$$\begin{aligned}\left|\Phi^*_{w_0}(x_2) - \Phi^*_{w_0}(x_1)\right| &\le |w^{x_2} - w^{x_1}| + |w^{2-x_2} - w^{2-x_1}| \\ &= (w_0^{x_1} + w_0^{2-x_2})(1 - w_0^{x_2-x_1}) \\ &\le 2(1 - w_0^{x_2-x_1}),\end{aligned}$$

which shows that $\Phi^*_{w_0}$ is continuous and thus uniformly continuous on the compact interval $[0, 2]$. Therefore, we can find $x_1 \in (0, \eta)$ and $x_2 \in (\eta, 2)$ such that

$$C(u_1, v_1) = C(u_2, v_2) = w_0^{\kappa^*+h},$$

where $u_i = w_0^{x_i}$ and $v_i = w_0^{2-x_i}$ for $i = 1$ and 2. This implies that the curve $uv = w_0^2$ and the contour $C(u, v) = w_0^{\kappa^*+h}$ intersect at the points (u_1, v_1) and (u_2, v_2).

We next show that into the area B_h we can squeeze a rectangle E_h of positive Lebesgue area such that $\mathbb{P}\left((U, V) \in E_h\right) > 0$. In view of bound (A.15), this will imply statement (A.14) needed to complete the proof of Theorem 3.2. Denote

$$u_h = \varphi^*(w_0^{(\kappa^*+h)/\kappa^*}) \quad \text{and} \quad v_h = w_0^{2(\kappa^*+h)/\kappa^*} / \varphi^*(w_0^{(\kappa^*+h)/\kappa^*}). \tag{A.16}$$

We have $C(u_h, v_h) = \Pi^*(w_0^{(\kappa^*+h)/\kappa^*}) = w_0^{\kappa^*+h}$. Define the rectangle by the equation

$$E_h := (u_h, u_0] \times (v_h, v_0] \subset B_h$$

(see Figure A.2). The functions $\varphi^*(u)$ and $\psi^*(u) := u^2/\varphi^*(u)$ are strictly increasing, and so $u_h < u_0$, $v_h < v_0$. Moreover, since

$$w_0^{2(\kappa^*+h)/\kappa^*} \le \varphi^*(w_0^{(\kappa^*+h)/\kappa^*}) \le 1,$$

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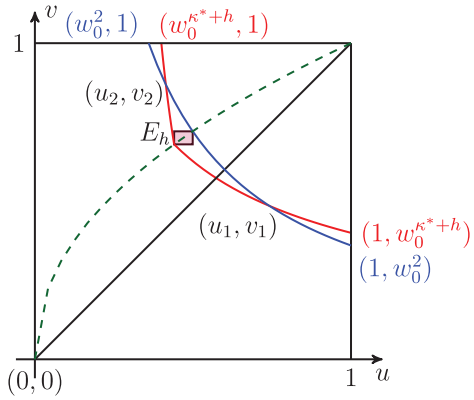


FIGURE A.2: The rectangle E_h with its associated curves and points.

we have

$$(u_h, v_h) = (w_0^{\eta_h}, w_0^{2(\kappa^*+h)/\kappa^* - \eta_h})$$

for some $\eta_h \in [0, 2(\kappa^* + h)/\kappa^*]$. Since $u_h < u_0$ and $v_h < v_0$, we have the bounds $\eta_h > \eta$ and

$$\frac{2(\kappa^* + h)}{\kappa^*} - \eta_h > 2 - \eta,$$

which imply

$$\eta < \eta_h < \eta + \frac{2h}{\kappa^*}. \tag{A.17}$$

Hence,

$$\begin{aligned}
\mathbb{P}\big((U, V) \in E_h\big) &= C(u_0, v_0) - C(u_0, v_h) - C(u_h, v_0) + C(u_h, v_h) \\
&= w_0^{\kappa^*} - C(u_0, v_h) - C(u_h, v_0) + w_0^{\kappa^*+h} \\
&\geq w_0^{\kappa^*} - (w_0^{\eta} w_0^{2(\kappa^*+h)/\kappa^* - \eta_h})^{\kappa^*/2} - (w_0^{\eta_h} w_0^{2-\eta})^{\kappa^*/2} + w_0^{\kappa^*+h} \\
&= w_0^{\kappa^*} - w_0^{\kappa^*+h+\kappa^*(\eta-\eta_h)/2} - w_0^{\kappa^*-\kappa^*(\eta-\eta_h)/2} + w_0^{\kappa^*+h} \\
&= w_0^{\kappa^*}(1 - w_0^{-\kappa^*(\eta-\eta_h)/2}) + w_0^{\kappa^*+h}(1 - w_0^{\kappa^*(\eta-\eta_h)/2}) \\
&= (w_0^{\kappa^*+\kappa^*(\eta_h-\eta)/2} - w_0^{\kappa^*+h})(w_0^{\kappa^*(\eta-\eta_h)/2} - 1),
\end{aligned}$$

which is positive, because bounds (A.17) imply that the two factors on the right-hand side of the bound are strictly positive. This completes the proof of Theorem 3.2. □

A.3. Histograms and normal fits

In this appendix, we depict histograms and fitted normal densities of the values of $\widehat{\kappa}_n^*(m, \theta_0)$ (given by equation (2.10)) that arise from simulated iid data, as discussed in Section 3.

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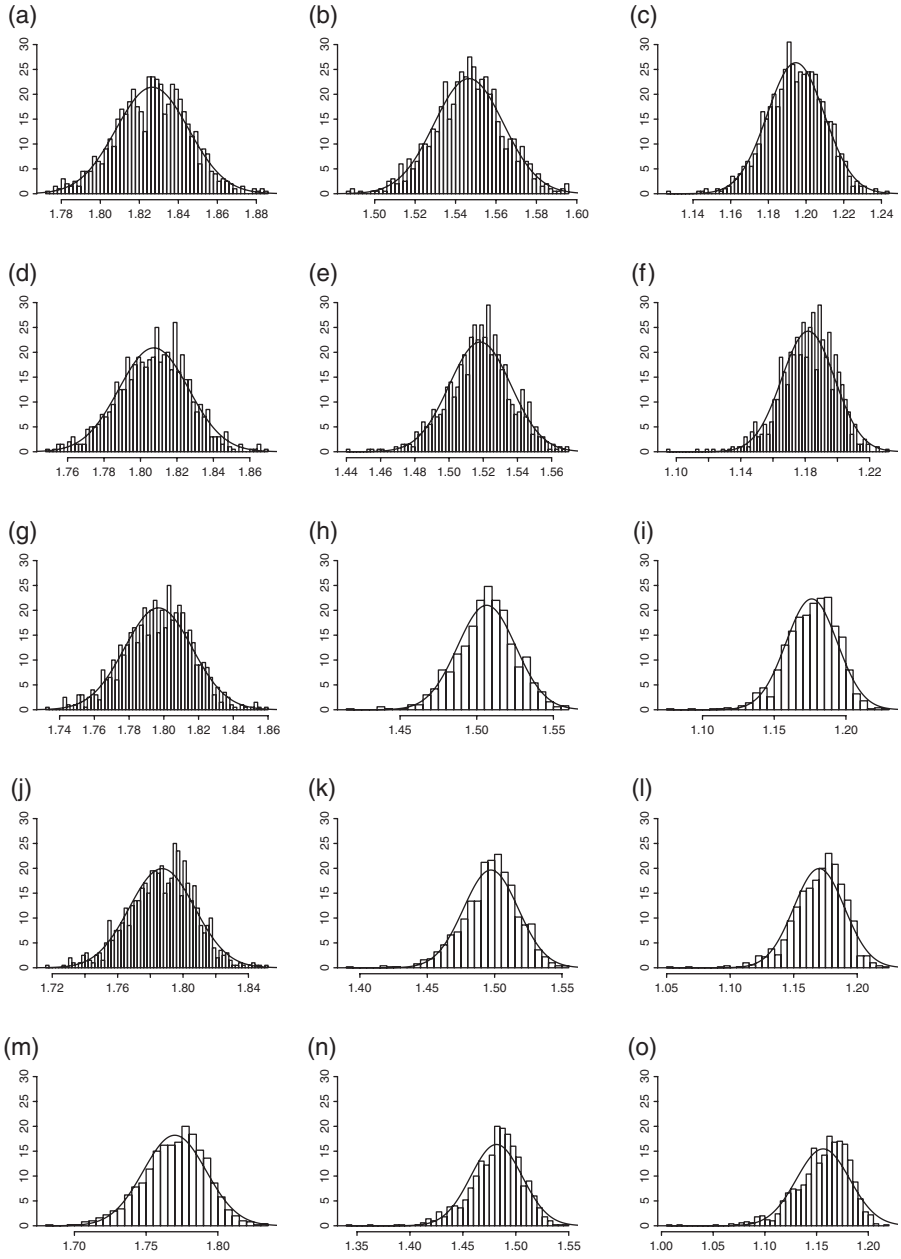


FIGURE A.3: Fits of simulated $\widehat{\kappa}_n^*(m, \theta_0)$ to the normal distribution for iid data. (a) $m = 3$, $\kappa^* = 1.80$. (b) $m = 3$, $\kappa^* = 1.52$. (c) $m = 3$, $\kappa^* = 1.20$. (d) $m = 5$, $\kappa^* = 1.80$. (e) $m = 5$, $\kappa^* = 1.52$. (f) $m = 5$, $\kappa^* = 1.20$. (g) $m = 7$, $\kappa^* = 1.80$. (h) $m = 7$, $\kappa^* = 1.52$. (i) $m = 7$, $\kappa^* = 1.20$. (j) $m = 10$, $\kappa^* = 1.80$. (k) $m = 10$, $\kappa^* = 1.52$. (l) $m = 10$, $\kappa^* = 1.20$. (m) $m = 20$, $\kappa^* = 1.80$. (n) $m = 20$, $\kappa^* = 1.52$. (o) $m = 20$, $\kappa^* = 1.20$.

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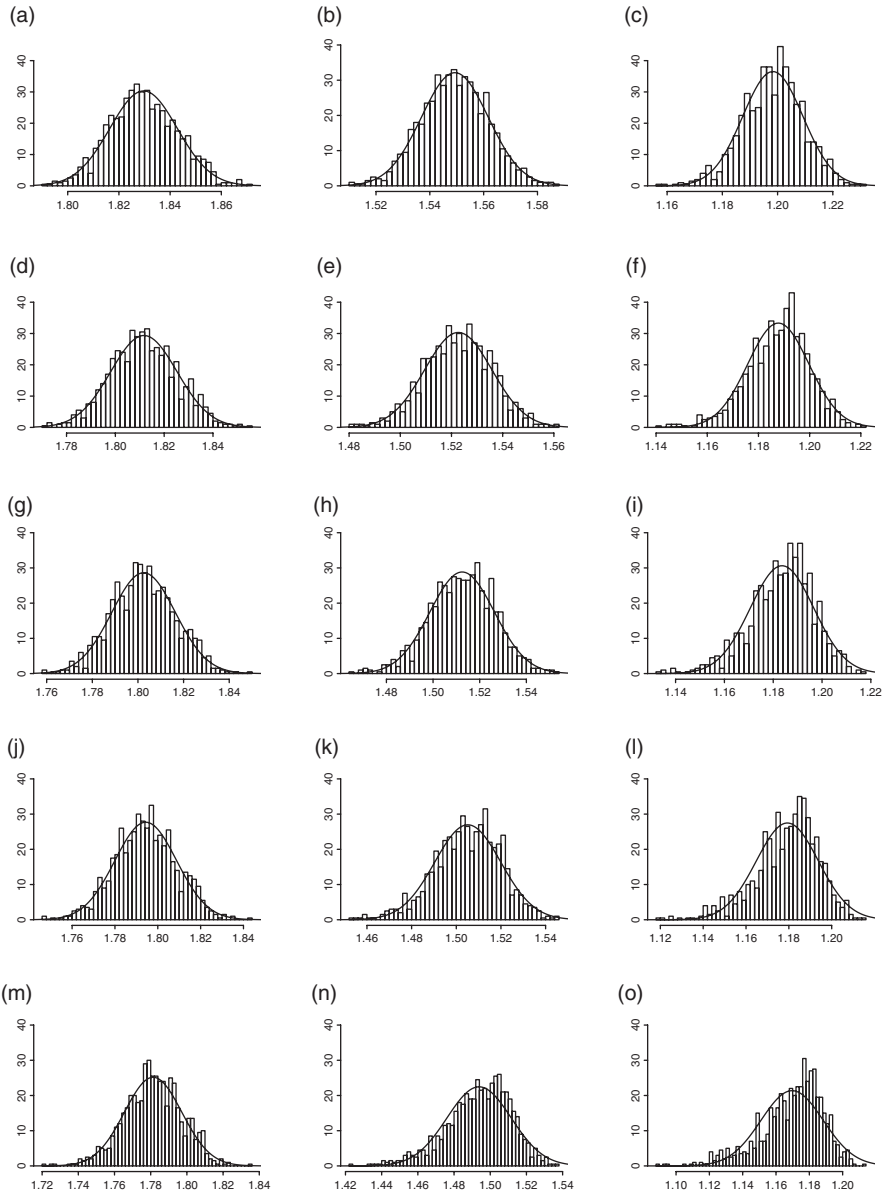


FIGURE A.4: Fits of simulated $\widehat{\kappa}_n^*(m, \theta_0)$ to the normal distribution for iid data. (a) $m = 3$, $\kappa^* = 1.80$. (b) $m = 3$, $\kappa^* = 1.52$. (c) $m = 3$, $\kappa^* = 1.20$. (d) $m = 5$, $\kappa^* = 1.80$. (e) $m = 5$, $\kappa^* = 1.52$. (f) $m = 5$, $\kappa^* = 1.20$. (g) $m = 7$, $\kappa^* = 1.80$. (h) $m = 7$, $\kappa^* = 1.52$. (i) $m = 7$, $\kappa^* = 1.20$. (j) $m = 10$, $\kappa^* = 1.80$. (k) $m = 10$, $\kappa^* = 1.52$. (l) $m = 10$, $\kappa^* = 1.20$. (m) $m = 20$, $\kappa^* = 1.80$. (n) $m = 20$, $\kappa^* = 1.52$. (o) $m = 20$, $\kappa^* = 1.20$.

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