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Livšic theorems and stable ergodicity for group extensions of hyperbolic systems with discontinuities

MATTHEW NICOL and ANDREW SCOTT
Department of Mathematics and Statistics, University of Surrey, Guildford, Surrey GU2 7XH, UK
(e-mail: m.nicol@surrey.ac.uk, a.scott@surrey.ac.uk)

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Abstract. We prove measurable Livšic theorems for Lie group valued cocycles over certain classes of uniformly hyperbolic maps with discontinuities, in particular the β-transformation, C^2 Markov maps and a simple class of toral linked twist maps. As an application, we establish stable ergodicity results for compact group extensions of certain classes of uniformly hyperbolic systems with discontinuities.

0. *Introduction*

Livšic theorems establish that solutions to dynamical cohomological equations which are assumed to have a certain degree of regularity have, in fact, a greater degree of regularity. For hyperbolic systems, the techniques in this area are still mainly based on the seminal ideas of Livšic [**13, 14**]. By measurable Livšic theorems we mean theorems that apply to dynamical cohomological equations in which solutions are *a priori* only measurable. Measurable Livšic theorems have applications to the study of ergodicity of compact group extensions of hyperbolic systems, since an obstruction to the ergodicity of such systems is given by the existence of group valued measurable solutions to a cohomological equation. Some good references on the state of the art for Livšic theorems are [**7, 15, 16, 17, 20, 23**].

Let G be a connected Lie group with Lie algebra $L(G)$ and a right invariant metric d_G. We are particularly interested in the case in which $G \subset U(d)$ is a compact connected Lie subgroup of the unitary group of $d \times d$ matrices with a bi-invariant metric induced by the norm $\|A\|^2 = \text{trace}(A\bar{A}^{\text{t}})$. We consider the regularity of G-valued measurable solutions h to the cocycle equation $g(x) = h(fx)h(x)^{-1}$ in two simple settings. First, where the base transformation $f: X \longrightarrow X$ is a piecewise C^2 expanding map of the interval and where g is a Lipschitz (or Hölder) G-valued cocycle. Second, where f is from a class of hyperbolic mappings of the two torus (a special case of which is a toral linked twist map of the two-dimensional torus X which is smooth except on finitely many lines of discontinuity) and g is a C^1 G-valued cocycle.

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In §2 we show, as applications of these results, that if G is a compact Lie group and $g\colon X \longrightarrow G$ is C^1, then in the C^1 topology a prevalent and residual set of group extensions of these base transformations are ergodic and if G is a semisimple Lie group, then an open, dense and prevalent set of group extensions of these base transformations are ergodic. If the base transformation is a Lasota–Yorke map, then our stable ergodicity results hold in the Hölder topology for Hölder cocycles.

Pollicott and Yuri [**24**, Theorem 1] have related results for the β-transformation. Suppose that $X := [0, 1) \pmod 1$, $f\colon X \longrightarrow X$ is a β-transformation, $g\colon X \longrightarrow \mathbb{R}$ is Hölder and μ is an equilibrium measure corresponding to the Hölder potential g. Pollicott and Yuri [**24**, Theorem 1] have shown that if $h\colon X \longrightarrow \mathbb{R}$ is an essentially bounded measurable solution to the cohomological equation $g(x) = h(fx) - h(x)$, then for any $\epsilon > 0$, there exists $h' = h$ μ-a.e. such that h' is of bounded variation on $[0, 1 - \epsilon]$. Jenkinson [**6**], using transfer operator techniques, has proved measurable Livšic theorems for smooth, compact Lie group valued cocycles over piecewise expanding Markov maps. Let $G = \mathbb{R}^n$, a compact Lie group or a connected Lie group satisfying a partial hyperbolicity and center-bunching condition (to be specified later). In Theorem 1 we show that if m is the unique smooth invariant probability measure for the β-transformation or a C^2 expanding Markov map, and $g\colon X \longrightarrow G$ is Hölder (respectively Lipschitz), then for any measurable solution h to $g(x) = h(fx)h(x)^{-1}$ there exists $h' = h$ m-a.e. and h' is Hölder (respectively Lipschitz) on X.

1. *Livšic Theorems*

Let G be a connected Lie group with Lie algebra denoted by $L(G)$ which we identify with the tangent space at the identity, T_eG. We let r_g denote right multiplication by $g \in G$. Given a norm $\|\cdot\|$ on T_eG, we define a norm on T_gG by $\|\mathbf{v}\|_g = \|r_{g^{-1}}\mathbf{v}\|_e$. This norm induces a right invariant metric d_G on G so that $d_G(gk, hk) = d_G(g, h)$ [**23**, §4]. Throughout this paper we will write $d(\cdot, \cdot)$ instead of $d_G(\cdot, \cdot)$ when it is clear from context that we mean the metric on G.

We define the adjoint map $\mathrm{Ad}\colon G \longrightarrow \mathrm{Aut}(L(G))$, for $g \in G$ and $X \in T_eG$, by

$$\mathrm{Ad}(g)\mathbf{v} = \frac{d}{dt}(g \exp(t\mathbf{v})g^{-1}).$$

A calculation [**23**, §4] shows that

$$d(gh, gk) \leqslant \|\mathrm{Ad}(g)\|\, d(h, k). \tag{1}$$

We make spectral assumptions on our dynamics and cocycle. In this paper we consider one-dimensional piecewise expanding maps and two-dimensional piecewise C^2 diffeomorphisms $f\colon X \longrightarrow X$, where X is either an interval, S^1 or $\mathbb{T}^2$. In both settings, f is ergodic with respect to an invariant measure m. We suppose that in the one-dimensional case $|f'(x)| \geq \lambda_u > 1$ for all but a finite number of points. We assume in the two-dimensional setting that for μ-a.e. point $x \in X$ there is a splitting of the tangent space $T_x = E^s_x \oplus E^u_x$ and two numbers $0 < \lambda_s < 1 < \lambda_u$ such that $|Df_x v| \leqslant \lambda_s |v|$ for all $v \in E^s_x$ and $|Df^{-1}_x v| \leqslant \lambda_u |v|$ for all $v \in E^u_x$.

We state our assumptions on partial hyperbolicity in the invertible case. In the one-dimensional case any assumptions involving the quantities μ_s, λ_s are vacuously satisfied.

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Let $g\colon X \longrightarrow G$ be Hölder of exponent $\gamma > 0$. We define

$$\mu_s := \lim_{n\to\infty} \Big(\sup_{x\in X} \|\mathrm{Ad}(g_n(x)^{-1})\| \Big)^{-1/n}$$

and

$$\mu_u := \lim_{n\to\infty} \Big(\sup_{x\in X} \|\mathrm{Ad}(g_n(x))\| \Big)^{1/n},$$

where $g_n = g(f^{n-1}x)\cdots g(fx)g(x)$.

We assume that the group extension is partially hyperbolic and satisfies the partial hyperbolicity condition

$$\lambda_s < \mu_s \leqslant 1 \leqslant \mu_u < \lambda_u. \tag{PH}$$

Define

$$\Theta_{PH} := \max\left\{ \frac{\log \mu_s}{\log \lambda_s}, \frac{\log \mu_u}{\log \lambda_u} \right\}.$$

The assumption (PH) implies that $\Theta_{PH} < 1$.

Note that if G is compact, nilpotent or abelian, then $\mu_s = \mu_u = 1$ and condition (PH) is satisfied.

1.1. *Lasota–Yorke maps.* We consider maps $f\colon [0,1] \longrightarrow [0,1]$ of the following form. Let $S = \{0 = s_0, s_1, \ldots, s_d = 1\}$ and suppose f restricted to each open interval (s_i, s_{i+1}) is C^2 (with C^2 extension to $[s_i, s_{i+1}]$ with left and right derivatives) and $f'(x) \geq \lambda_u > 1$ for all $x \in [0,1] \setminus S$. We call this class of maps Lasota–Yorke maps. It is known that such maps have a finite number of distinct ergodic absolutely continuous invariant measures and that if p is an f invariant density for such a measure (with respect to Lebesgue measure l), then the support of p is a finite union of open intervals Lebesgue a.e. [**1**, Theorems 8.2.1 and 8.2.2]. Furthermore, such an f-invariant density p is bounded below and above on its support so that there exists $M > 0$ such that

$$\frac{1}{M} \leqslant p(x) := \frac{dm}{dl} \leqslant M \quad \text{if } x \in \mathrm{supp}(p). \tag{2}$$

To simplify notation and to avoid trivialities, we assume that such a map is weak-mixing with a unique absolutely continuous invariant measure m with support $X \subset [0,1]$. This assumption implies that the map is exact.

We further note that Lasota–Yorke maps satisfy a bounded distortion condition [**1**, Lemma 6.1.1] in that there exists $R > 0$ such that for all n and all x, y, if $f^j(x)$ and $f^j(y)$ both belong to the same connected component of $X \setminus S$ for $j = 0, \ldots, n$, then

$$\frac{1}{R} \leqslant \frac{|(f^n)'(x)|}{|(f^n)'(y)|} \leqslant R. \tag{3}$$

The class of β-transformations are a subclass of Lasota–Yorke maps. Let $b < \beta < b+1$, where b is a positive integer. We let (x) denote $x \pmod 1$, although we write x instead of (x) when it is the argument of a function. Define the β-transformation $f\colon [0,1) \longrightarrow [0,1) \pmod 1$ by $f(x) = (\beta x)$. Hence $\lambda_u = \beta$. It is known that this system is exact with respect to a unique absolutely continuous invariant measure whose support is $[0,1) \pmod 1$.

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We also distinguish the subclass of Lasota–Yorke maps known as C^2 Markov maps, which have the additional property that for each i, $f(s_i, s_{i+1})$ is a union of intervals of form (s_j, s_{j+1}) [**1**, Ch. 9]. A C^2 Markov map is exact with respect to an absolutely continuous invariant measure with support $[0, 1]$.

In Theorems 1 and 2, X denotes the support of the corresponding exact absolutely continuous invariant measure for the maps, i.e. $[0, 1)$ (mod 1) for the β-transformation, $[0, 1]$ for a C^2 Markov map and an interval for a weak-mixing Lasota–Yorke map. We prove the following regularity results.

THEOREM 1. *Let $f: X \to X$ be a β-transformation or a C^2 Markov map, G a connected Lie group and $g: X \longrightarrow G$ a cocycle which is Lipschitz or Hölder of exponent γ. Assume condition (PH) and if g is γ-Hölder, assume $\gamma > \Theta_{PH}$. Then, if $g: X \to G$ is Lipschitz (respectively Hölder) and $h: X \to G$ is a measurable solution to the equation*

$$g(x) = h(fx)h(x)^{-1},$$

then there exists h' which is Lipschitz (respectively Hölder) on X and $h' = h$ m-a.e.

THEOREM 2. *Let $f: X \to X$ be a weak-mixing Lasota–Yorke map with respect to an absolutely continuous invariant measure m with support X. Assume the same (PH) condition on $g: X \longrightarrow G$ as in Theorem 1. Then if $g: X \to G$ is Lipschitz (respectively Hölder) and $h: X \to G$ is a measurable solution to the equation*

$$g(x) = h(fx)h(x)^{-1},$$

then there exists h' such that $h' = h$ m-a.e. and for m-a.e. $x \in X$ there exists an interval $U(x)$ with $h'|U(x)$ Lipschitz (respectively Hölder). The Lipschitz (respectively Hölder) constant and exponent is uniform over all such intervals $U(x)$. Furthermore, the intervals $U(x)$ are independent of h.

Remark. It is easy to construct non-trivial examples of Lipschitz $\mathbb{R}^d$ or group-valued coboundaries where the base transformation is the β-transformation and the cobounding function is also Lipschitz. For example, let $h(x)$ be an $\mathbb{R}^d$ valued Lipschitz function on X with the additional property that $h(\beta) = h(0) = h(1)$ and define $g(x) = h(fx) - h(x)$. Since h is Lipschitz on X and f has a discontinuity only at the point $0 = 1$, the composition $h \cdot f$ is Lipschitz except possibly at the point $0 = 1$. By checking left and right limits, it follows that $h \cdot f$ is Lipschitz on X and hence $g(x)$ is Lipschitz on X. Hölder and Lie group valued examples may be constructed in the same manner.

The proof of Theorem 2 proceeds in three steps. The proof of Theorem 1 requires an additional step. The proof for Lipschitz group extensions is almost the same as that for Hölder group extensions, so we prove only the Hölder case.

Step 1: the natural extension. Suppose that $f: X \longrightarrow X$ is a Lasota–Yorke map. We suppose that $g: X \longrightarrow G$ is Hölder of exponent γ and satisfies the (PH) conditions with $\gamma > \Theta_{PH}$. Since Lasota–Yorke maps are non-invertible, we form the natural extension. We recall its construction. Let

$$\Omega: = \{(x_0, x_1, x_2, \dots) : x_n = f(x_{n+1}), x_n \in X, n = 0, 1, 2, \dots\}$$

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with $\sigma : \Omega \longrightarrow \Omega$ defined by $\sigma((x_0, x_1, x_2, \dots)) = (f(x_0), x_0, x_1, \dots)$. The map σ is invertible on Ω. Given the invariant measure m, we define the invariant measure for the natural extension $\tilde{m}$ on Ω by defining it first on cylinder sets $C(A_0, A_1, \dots, A_k)$ by

$$\tilde{m}(C(A_0, A_1, \dots, A_k)) = m(f^{-k}(A_0) \cap f^{-k+1}(A_1) \cap \cdots \cap f^{-k+i}(A_i) \cap \cdots \cap A_k)$$

and then extending it to Ω using Kolmogorov's extension theorem. We think of $(x_0, x_1, x_2, \dots)$ as being an inverse branch of $x_0 \in X$ under the mapping $f : X \longrightarrow X$. The map $\sigma : \Omega \longrightarrow \Omega$ is ergodic with respect to $\tilde{m}$ [**26**]. The natural extension is in fact Bernoulli [**1**, §8.3], but in this section we only use the fact that σ^{-1} and σ are ergodic with respect to $\tilde{m}$.

Step 2: a Borel–Cantelli argument. Let B denote the image of S under f; strictly speaking we take the images of the points s_i under the appropriate C^2 extension of $f|_{(s_i, s_{i+1})}$. In other words,

$$B := \left\{ \lim_{x \searrow s_i} f(x), \lim_{x \nearrow s_i} f(x) : i = 0, 1, \dots, k \right\}$$

Note that B may consist of twice as many points as S. Let $x = (x_0, x_1, \dots)$ be a point in Ω such that $x_i \notin B$ for all $i \geqslant 0$ and $x_0 \notin S$. A full measure set of points satisfies this condition.

Take a point $x \in \Omega$ and an open interval I disjoint from S which contains x_0. Consider the preimage $f^{-1}I$, consisting of a finite number of intervals. Divide this partition further by deleting points in S. Thus we have

$$f^{-1}(I) \setminus S = I_{(1)} \dot{\cup} I_{(2)} \dot{\cup} \cdots \dot{\cup} I_{(l_0)},$$

where all $I_{(i)}$ are open intervals which may or may not map onto I. Note that if $I \cap B = \emptyset$, then $f(I_{(i)}) = I$ for all i.

Now take one of these intervals $I_{(i_1)}$ and carry out the same procedure as before giving us a partition into open intervals

$$f^{-1}(I_{(i_1)}) \setminus S = I_{(i_1,1)} \dot{\cup} I_{(i_1,2)} \dot{\cup} \cdots \dot{\cup} I_{(i_1,l_{(i_1)})},$$

where each of the intervals will map onto $I_{(i)}$ if $I_{(i)} \cap B = \emptyset$.

We can repeat this process indefinitely giving us partitions, in general, of the form

$$I_{(i_1,i_2,\dots,i_{n-1})} \setminus S = I_{(i_1,i_2,\dots,i_{n-1},1)} \dot{\cup} I_{(i_1,i_2,\dots,i_{n-1},2)} \dot{\cup} \cdots \dot{\cup} I_{(i_1,i_2,\dots,i_{n-1},l_{(i_1,i_2,\dots,i_{n-1})})}.$$

Again, if $I_{(i_1,i_2,\dots,i_{n-1})} \cap B = \emptyset$, then all of the intervals $I_{(i_1,i_2,\dots,i_{n-1},j)}$ will map onto I.

This process enables us to define modified inverse branches of the interval I. For ease of notation we write I_n for such a branch (that is, we define $I_n := I_{(i_1,i_2,\dots,i_n)}$ with $I_{i_0} = I$). In particular, $x_n \in I_n$ for all $n \geqslant 0$.

It may happen that for each $i \geqslant 0$ we have that $f(I_{i+1})$ is strictly contained in I_i. The following lemma shows that for $\tilde{m}$-almost every $x \in \Omega$ there exists $N(x)$ such that for all $i > N(x)$, $f(I_{i+1}) = I_i$.

LEMMA 1. *For $\tilde{m}$-a.e. $x \in \Omega$ there exists $N = N(x) \in \mathbb{N}$ such that for all $n \geqslant N$, $f(I_{n+1}) = I_n$.*

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Proof. We define a sequence of cylinder sets E_n by

$$E_n = C(\cdot, \dots, N(B, \lambda_u^{-n}))$$

where $N(B, \lambda_u^{-n}) = \{x \in X : d(x, B) < \lambda_u^{-n}\}$ is in the nth position. Note that $\tilde{m}(E_n) \leqslant M\lambda_u^{-n}$ for $M > 0$ so that

$$\sum_{n=0}^{\infty} \tilde{m}(E_n) < \infty.$$

Thus we can use the Borel–Cantelli lemma to conclude that $\tilde{m}$-almost every $x = (x_0, x_1, \dots)$ belongs to only a finite number of E_n. Thus there exists $N = N(x) \in \mathbb{N}$ such that for all $n \geqslant N$, $I_n \cap B = \emptyset$ which implies $I_n = f(I_{n+1})$, proving the lemma. □

If we take the open interval neighbourhood I still smaller, then we have an open interval J_0 and inverse interval branch satisfying $x_n \in J_n$ and $f^n(J_n) = J_0$ for all $n \geqslant 0$. Note also that by construction $J_n \cap S = \emptyset$ for all $n \geq 0$.

Step 3: h has a Hölder version on J_0. Take such an $x_0 \in J_0$. In what follows, if we refer to a point $y_0 \in J_0$ we naturally associate with it the inverse branch $(y_0, y_1, \dots)$ such that $y_n \in J_n$ for all $n \geqslant 0$ as constructed in Step 2.

Now we take points $y = (y_0, y_1, \dots, y_n, \dots), z = (z_0, z_1, \dots, z_n, \dots) \in \Omega$ such that $y_k, z_k \in J_k$ for all $k = 0, 1, \dots$. We remark that the set of points, $y = (y_0, y_1, \dots y_n) \in \Omega$ such that for all $i \in \mathbb{N}$, $g(y_{i-1}) = h(y_i)h(y_{i-1})$, is of full measure, and without loss of generality we will only consider such points.

On J_0 we define a function $\Phi^u : J_0 \longrightarrow G$ by

$$\Phi^u(y_0) = \lim_{n\to\infty} g_n(y_0)g_n(x_0)^{-1},$$

where $g_n(x_0) = g(x_1) \cdots g(x_n)$. This is well defined. In fact

$$\begin{aligned}
&d(g_{n+1}(y_0)g_{n+1}(x_0)^{-1}, g_n(y_0)g_n(x_0)^{-1}) \\
&\quad = d(g_n(y_0)g(y_{n+1})g(x_{n+1})^{-1}g_n(x_0)^{-1}, g_n(y_0)g_n(x_0)^{-1}) \\
&\quad = d(g_n(y_0)g(y_{n+1})g(x_{n+1})^{-1}g_n(x_0)^{-1}, g_n(y_0)g(x_{n+1})g(x_{n+1})^{-1}g_n(x_0)^{-1}) \\
&\quad = d(g_n(y_0)g(y_{n+1}), g_n(y_0)g(x_{n+1})) \quad \text{(right invariance)} \\
&\quad \leqslant \|\mathrm{Ad}(g_n(y_0))\| d(g(y_{n+1}), g(x_{n+1})) \quad \text{(by (1))} \\
&\quad \leqslant C((\mu_u)\lambda_u^{-\gamma})^n \\
&\quad \leqslant C\kappa^n,
\end{aligned}$$

where $|\kappa| < 1$ (since $\gamma > \Theta_{\mathrm{PH}}$). Thus the sequence $g_n(y_0)g_n(x_0)^{-1}$ is Cauchy and so converges.

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Next we show that Φ_u is Hölder. Let $y_0, z_0 \in J_0$, then

$$\begin{aligned} d(g_n(y_0)g_n(x_0)^{-1}, g_n(z_0)g_n(x_0)^{-1}) &= d(g_n(y_0), g_n(z_0)) \\ &\leqslant \sum_{i=0}^{n-1} d(g_i(y_0)g(y_{i+1})g_{n-i-1}(z_{i+1}), g_i(y_0)g(z_{i+1})g_{n-i-1}(z_{i+1})) \\ &\leqslant \sum_{i=0}^{n-1} \|\mathrm{Ad}(g_i(y_0))\|\, d(g(y_{i+1}), g(z_{i+1})) \\ &\leqslant \sum_{i=0}^{n-1} C(\mu_u)^{i+1}\lambda_u^{-(i+1)\gamma}\, d(y_0, z_0)^{\gamma}. \end{aligned}$$

Letting $n \to \infty$ gives $d(\Phi_u(y_0), \Phi_u(z_0)) \leqslant K d(y_0, z_0)^{\gamma}$, K constant. It is clear that if g is Lipschitz, then Φ_u is also Lipschitz.

Since h is measurable, by Lusin's theorem there is a positive measure set H such that h restricted to H is uniformly continuous. We claim the following.

LEMMA 2. *Given $\eta > 0$ for infinitely many $n > 0$,*

$$m\{y_0 \in J_0 : y_n \in J_n \cap H\} > (1 - \eta)m(J_0).$$

Let l denote Lebesgue measure. We use the Lebesgue density theorem to find positive numbers $\rho_k \longrightarrow 0$ such that if $H_k \subset H$ is defined as the set

$$H_k := \left\{ p \in H : \frac{l((p-\rho, p+\rho) \cap H)}{l(p-\rho, p+\rho)} > 1 - \frac{1}{2^k} \quad \text{for all} \quad 0 < \rho \leqslant \rho_k \right\}$$

then $l(H_k) > (1 - 1/2^k)l(H)$. The sets H_k contain points of 'uniform density'. Note that by the ergodicity of $\sigma^{-1}\colon \Omega \longrightarrow \Omega$, a full measure set of $x = (x_0, x_1, \dots)$ have the property that for all k, $x_n \in H_k$ for infinitely many n and without loss of generality we assume this property for $x \in \Omega$.

Fix k large enough so that $M^2R^2/2^{k-2} < \eta$ and choose n large enough so that $\lambda_u^{-n} < \rho_k$ and also $x_n \in H_k$. This guarantees that $J_n \subset (x_n - \rho_k, x_n + \rho_k)$ and consequently

$$l(J_n \cap H^c) \leqslant l(J_n \cap H_k^c) < \frac{1}{2^{k-1}} l(J_n).$$

Thus by bounded distortion

$$l(f^n(J_n \cap H^c)) \leqslant \frac{R^2}{2^{k-1}} l(J_0),$$

where l is Lebesgue measure. Since $1/M \leqslant dm/dl \leq M$,

$$m(f^n(J_n \cap H^c)) \leq \frac{M^2R^2}{2^{k-1}} m(J_0).$$

Since we have chosen k so that $M^2R^2/2^{k-2} < \eta$, this proves the lemma.

Given $0 < \delta < 1$, we may choose a subsequence n_i such that

$$m(f^{n_i}(J_{n_i} \cap H^c)) < \frac{\delta}{2^i} m(J_0)$$

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and hence

$$m\{y_0 \in J_0 : y_{n_i} \in J_{n_i} \cap H\} > \left(1 - \frac{\delta}{2^i}\right) m(J_0).$$

Since $\sum_{i=1}^{\infty} \delta/2^i < 1$, we may choose a point $t_0 \in J_0$ such that $t_{n_i} \in J_{n_i} \cap H$ for all n_i.

Define $\Psi^{u,n}(y_0) = g_n(y_0)g_n(t_0)^{-1}$. Then $\lim_{n\to\infty} \Psi^{u,n}(y_0) = \Phi^u(y_0)\Phi^u(t_0)^{-1}$. Furthermore,

$$h(y_0) = \Psi^{u,n}(y_0)\Psi^{u,n}(t_0)^{-1}h(t_0)h(t_n)^{-1}h(y_n).$$

This may seem at first a cumbersome way of expressing $h(y_0)$, but it simplifies the following calculations. Using the triangle inequality we estimate

$$\begin{aligned} d(h(y_0), h(z_0)) \le\ & d(\Psi^{u,n}(y_0)\Psi^{u,n}(t_0)^{-1}h(t_0)h(t_n)^{-1}h(y_n), \Psi^{u,n}(z_0)\Psi^{u,n}(t_0)^{-1}h(t_0)) \\ & + d(\Psi^{u,n}(y_0)\Psi^{u,n}(t_0)^{-1}h(t_0)h(t_n)^{-1}h(z_n), \Psi^{u,n}(y_0)\Psi^{u,n}(t_0)^{-1}h(t_0)) \\ & + d(\Psi^{u,n}(y_0)\Psi^{u,n}(t_0)^{-1}h(t_0)h(t_n)^{-1}h(z_n), \\ & \quad \Psi^{u,n}(z_0)\Psi^{u,n}(t_0)^{-1}h(t_0)h(t_n)^{-1}h(z_n)). \end{aligned}$$

By the right-invariance of the metric, the last term may be written

$$\begin{aligned} & d(\Psi^{u,n}(y_0)\Psi^{u,n}(t_0)^{-1}h(t_0)h(t_n)^{-1}h(z_n), \Psi^{u,n}(z_0)\Psi^{u,n}(t_0)^{-1}h(t_0)h(t_n)^{-1}h(z_n)) \\ & \quad = d(\Psi^{u,n}(y_0)\Psi^{u,n}(t_0)^{-1}h(t_0), \Psi^{u,n}(z_0)\Psi^{u,n}(t_0)^{-1}h(t_0)). \end{aligned}$$

The group valued function of y_0, $\Psi^{u,n}(y_0)\Psi^{u,n}(t_0)^{-1}h(t_0)$, is uniformly bounded and converges to the Hölder function of y_0, $\Phi^u(y_0)\Phi^u(t_0)^{-1}h(t_0)$, as $n \to \infty$. Furthermore, $\Psi^{u,n}(z_0)\Psi^{u,n}(t_0)^{-1}h(t_0)$ converges to $\Phi^u(z_0)\Phi^u(t_0)^{-1}h(t_0)$. We now consider the first two terms. Given $\eta > 0$, there exists N such that if $n_i \ge N$ and $z_{n_i}, y_{n_i} \in H$, then

$$d(\Phi^{u,n}(y_0)\Phi^{u,n}(t_0)^{-1}h(t_0)h(t_n)^{-1}h(y_n), \Phi^{u,n}(y_0)\Phi^{u,n}(t_0)^{-1}h(t_0)) < \frac{\eta}{2}$$

and

$$d(\Phi^{u,n}(y_0)\Phi^{u,n}(t_0)^{-1}h(t_0), \Phi^{u,n}(y_0)\Phi^{u,n}(t_0)^{-1}h(t_0)h(t_n)^{-1}h(z_n)) < \frac{\eta}{2}.$$

Recall that

$$m\{y_0 \in J_0 : y_{n_i} \in J_{n_i} \cap H\} > \left(1 - \frac{\delta}{2^i}\right) m(J_0).$$

Given $\eta > 0$, by taking n_i sufficiently large,

$$\begin{aligned} (m \times m)\{(y_0, z_0) \in J_0 \times J_0 : d(h(y_0), h(z_0)) \le K' d(y_0, z_0)^\gamma + \eta\} & \\ \ge (1 - 2\eta)(m \times m)(J_0 \times J_0). & \end{aligned}$$

Since η was arbitrary, this implies that

$$(m \times m)\{(y_0, z_0) \in J_0 \times J_0 : d(h(y_0), h(z_0)) \le K' d(y_0, z_0)^\gamma\} = (m \times m)(J_0 \times J_0),$$

and hence $h|J_0$ has a Hölder version. Note that the interval J_0 was defined independently of the function h†. Thus for m-a.e. x_0 there exists an interval $J_0 := U(x_0)$ on which the restriction of any coboundary h has a Hölder version. This proves Theorem 2.

† Suppose h' is another measurable solution and I_0 is an interval on which h' is Lipschitz (respectively Hölder). There exists a $J_n \subset I_0$ since $x = (x_0, x_1, \dots, x_n, \dots)$ has a dense orbit. h' has a Lipschitz (respectively Hölder) version on J_n which may be extended to J_0 as $f^j(J_n) \cap S = \emptyset$ for $j = 0, \dots, n$. Hence h' has a Lipschitz (respectively Hölder) version on J_0.

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The β-transformation and Markov maps: extending h as a Hölder function from J_0 to X. We now show how in the case that the base transformation is a Markov map or a β-transformation, the version h may be extended to be Hölder on X. We consider the β-transformation first. Suppose that there exists an open interval $J_0 = (\gamma_1, \gamma_2) \subset (0, 1)$ on which h has a Hölder version. The key idea is to extend h to a larger domain by using the equation $h(f^i p) = g(f^{i-1}p) \cdots g(fp)g(p)h(p)$. There exists a first $i = j$, say, such that $0 \in (\beta^j\gamma_1, \beta^j\gamma_2)$, in which case h has a uniformly Hölder version on $(0, \beta^j\gamma_2)$. We now extend h as a Hölder function from $(0, \beta^j\gamma_2)$ to $(0, 1)$. Since there exists a first integer r such that $(0, 1) \subset f^r(0, \beta^j\gamma_2)$, h has a Hölder version on $(0, 1)$. Since h is Hölder on a neighbourhood of the point $1/\beta$ using the equation $h(fx) = g(x)h(x)$, we may extend h as a Hölder function on X.

Now suppose that (f, X, m) is a C^2 Markov map. First note that we may assume that $|f'(x)| > 2$ for all x (where we take right or left derivatives at endpoints of (s_i, s_{i+1})). If this is not the case, then we may take a power f^n of f for which $|f'(x)| > 2$ and a refinement $\{(\tilde{s}_j, \tilde{s}_{j+1})\}$ of the partition $\{(s_i, s_{i+1})\}$ which is Markov for f^n. If $g(x) = h(fx)h(x)^{-1}$, then $h(f^n x)h(x)^{-1} = g(f^{n-1}) \cdots g(x)$, where $g(f^{n-1}) \cdots g(x)$ is Hölder on the interior of each partition element $(\tilde{s}_j, \tilde{s}_{j+1})$. Hence, if we obtain a Hölder version of h on X solving $h(f^n x)h(x)^{-1} = g(f^{n-1}) \cdots g(x)$ (recall that h also solves $g(x) = h(fx)h(x)^{-1}$), then we have the Hölder version we are seeking. Suppose that $|f'(x)| \geq 2 + \alpha$, where $\alpha > 0$. Take an interval J_0 such that $f^n(J_n) = J_0$ and on which h is Hölder. We may extend h as a Hölder solution to $f(J_0)$ unless $f(J_0) \cap S \neq \emptyset$. If $f(J_0) \cap S = \emptyset$, then the length of J_0 has been increased by a factor of at least two. However, if $f(J_0) \cap S \neq \emptyset$ then either: (i) $(s_j, s_{j+1}) \subset f(J_0)$ for some j; or (ii) we may divide $f(J_0)$ into two intervals one of which, say $J(1)$, has length strictly greater than that of J_0 by a factor of $1 + \alpha/2$. Continuing this process, we produce an interval $J(i)$ such that h is Hölder on $J(i)$ and $(s_j, s_{j+1}) \subset f(J(i))$. Thus we may extend h as a Hölder function to (s_j, s_{j+1}) and then the Markov property allows us to extend h as a Hölder version on X. The argument in the case of a Lipschitz extension is identical. This proves Theorem 1.

Remark. The extension of h as a Lipschitz or Hölder version from an interval J_0 to X, the support of the absolutely continuous invariant measure, depends upon the form of the Lasota–Yorke map and must be investigated on a case-by-case basis. In the case of the β-transformation and Markov maps this is straightforward, but we were not able to prove the existence of a Hölder version on X for a general Lasota–Yorke map.

1.2. *Two-dimensional systems.* In this section we consider two-dimensional examples of uniformly hyperbolic systems with discontinuities, concentrating on a simple class of toral link twist maps. First we describe a setting slightly more general than the toral linked twist maps we consider in which our results hold.

1.2.1. *Uniformly hyperbolic toral maps with discontinuities.* Suppose f is a piecewise C^2 Lebesgue measure-preserving mapping of the two-torus $\mathbb{T}^2$.

We suppose the following.

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(I) f is piecewise C^2 in the sense that there exists a finite set of C^2 one-dimensional submanifolds $\{S_1^+, \ldots, S_r^+\}$ with $S_i^+ \cap S_j^+$, $i \neq j$, either empty or consisting of a finite set of points and f is C^2 on $\mathbb{T}^2/\big(\bigcup_{j=1}^r S_j^+\big)$ with C^2 extension to the boundary of open components of $\mathbb{T}^2/\big(\bigcup_{j=1}^r S_j^+\big)$. Similarly there exists a finite set of C^2 one-dimensional submanifolds $\{S_1^-, \ldots, S_t^-\}$ with $S_i^- \cap S_j^-$, $i \neq j$, either empty or consisting of a finite set of points and f^{-1} is C^2 on $\mathbb{T}^2/\big(\bigcup_{j=1}^t S_j^-\big)$ with C^2 extension to the boundary of open components of $\mathbb{T}^2/\big(\bigcup_{j=1}^t S_j^-\big)$.

(II) There exist piecewise continuous Df invariant cone fields $C(x)$, $D(x)$ in the tangent space to $\mathbb{T}^2$ such that $DfC(x) \subset C(fx)$ (strictly) and $Df^{-1}D(x) \subset D(f^{-1}x)$ (strictly), and that vectors inside the cone $C(x)$ (respectively $D(x)$) are strictly expanded in length by Df (respectively Df^{-1}). Furthermore, suppose that the angle between $C(x)$ and $D(x)$ is strictly bounded away from zero.

(III) The submanifolds S_j^- have no tangents contained in the cones $D(x)$ and the submanifolds S_j^+ have no tangents contained in the cones $C(x)$.

By results of Pesin [**22**] and Katok and Strelcyn [**8**], through almost every point $p \in \mathbb{T}^2$ there are local stable and unstable manifolds of dimension one and the foliations are absolutely continuous. We note here that such maps satisfy conditions A–F of [**12**]. We call this class of maps *uniformly hyperbolic toral maps with discontinuities*.

To fix ideas, we first analyse in detail a special class of such maps, discontinuous toral linked twist maps of the two torus. The structure of the toral linked twist maps enables us, as in the case of the β-transformation, to show that measurable solutions to cohomological equations are Lipschitz on $\mathbb{T}^2$. For hyperbolic toral maps with discontinuities, we are only able to show that almost every point $x \in \mathbb{T}^2$ has a neighbourhood $U(x)$ and a version $h' = h$ m-a.e. for which h' is Lipschitz on each $U(x)$.

The toral linked twist mapping (TLTM) we consider has been analyzed in detail by Liverani and Wojtkowski [**12**] as a particularly simple setting in which to study the Hopf method.

Let $a > 0$ and let $f_i\colon \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $f_1(x, y) = (x + ay, y)$ and $f_2(x, y) = (x, y + ax)$. Let $f\colon \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ be the toral map $f := f_2 \cdot f_1$. This map has linearization

$$\begin{pmatrix} 1 & a \\ a & 1 + a^2 \end{pmatrix}.$$

Note that if a is a non-negative integer, then f is a hyperbolic toral automorphism. Here, though, we are interested in the case in which a is not an integer. The map f has a finite set of discontinuity lines, preserves Lebesgue measure m and is Bernoulli [**12**]. Our analysis and results hold equally well for a TLTM in which $f_2(x, y) = (x, y + bx)$ for $a \neq b > 0$, but in the tradition of Liverani and Wojtowski we will focus on the simplest case notationally.

Let R^+ and R^- denote the (finite) set of discontinuity lines for f and f^{-1}, respectively (see Figure 1).

If $h\colon \mathbb{T}^2 \to G$ $(\mathbb{R}^d)$ is measurable we say that h has a Lipschitz version if there exists $h'\colon \mathbb{T}^2 \longrightarrow G$ $(\mathbb{R}^d)$ such that $h' = h$ m-a.e. and h' is Lipschitz.

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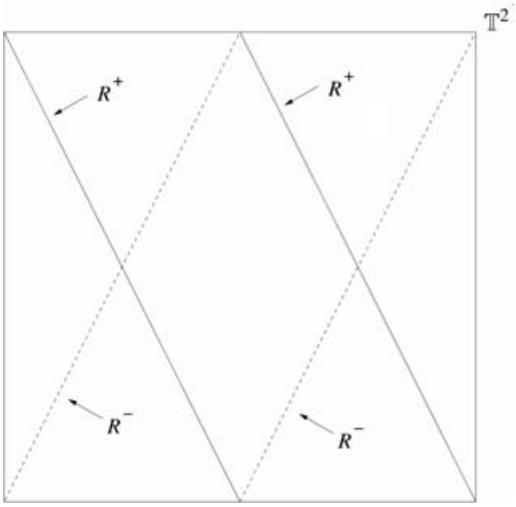


FIGURE 1. Discontinuity lines for the case $a = \frac{1}{2}$.

THEOREM 3. *Let $f: \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ be a toral linked twist map. Let G be a connected Lie group and $g: \mathbb{T}^2 \longrightarrow G$ satisfy (PH). If $g: \mathbb{T}^2 \longrightarrow G$ is Lipschitz, then any measurable solution $h: \mathbb{T}^2 \longrightarrow G$ of*

$$g(x) = h(fx)h(x)^{-1} \tag{4}$$

has a Lipschitz version $h' = h$ m-a.e.

THEOREM 4. *Let $f: \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ be a uniformly hyperbolic toral map with discontinuities. Let G be a connected Lie group and $g: \mathbb{T}^2 \longrightarrow G$ satisfy (PH). If $g: \mathbb{T}^2 \longrightarrow G$ is Lipschitz, then any measurable solution $h: \mathbb{T}^2 \longrightarrow G$ of*

$$g(x) = h(fx)h(x)^{-1} \tag{5}$$

has a version $h' = h$ m-a.e. such that m-a.e. $x \in \mathbb{T}^2$ has a neighbourhood $U(x)$ on which h' is Lipschitz with the same Lipschitz constant. Furthermore, the neighbourhoods $U(x)$ are independent of h.

Proof of Theorem 3. Our proof of Theorem 3 follows, with little modification, the exposition of [**12**, §1]. Our first step is to construct, for m-almost every $x \in \mathbb{T}^2$ stable and unstable leaves $W^s(x)$ and $W^u(x)$. We may construct unstable leaves for m-a.e. x such that

$$x \notin \bigcup_{i=0}^{\infty} f^i R^-$$

(we say that x is smooth in the past). Similarly we may construct stable leaves for m-a.e. x such that

$$x \notin \bigcup_{i=0}^{\infty} f^{-i} R^+$$

(we say that x is smooth in the future).

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The stable and unstable leaves are line segments along which we have exponential contraction under f (respectively f^{-1}) for $W^s(x)$ (respectively $W^u(x)$). In addition, these leaves may be constructed so that all forward iterates of $W^s(x)$ and all backward iterates of $W^u(x)$ avoid the singular set.

The argument giving regularity on local unstable manifolds is the same as that for local stable manifolds, so we only give the argument for local stable manifolds. Fix $x \in X$ and on $W^s(x)$ define a function $\Phi_x^s\colon W^s(x) \longrightarrow G$ by

$$\Phi_x^s(y) = \lim_{n\to\infty} g_n(y)^{-1} g_n(x),$$

where $g_n(x) = g(f^{n-1}x)\cdots g(x)$. Note that

$$\begin{aligned}
&d(g_{n+1}(y)^{-1}g_{n+1}(x), g_n(y)^{-1}g_n(x))\\
&\quad = d(g_n(y)^{-1}g(f^n y)^{-1}g(f^n x)g_n(x), g_n(y)^{-1}g_n(x))\\
&\quad = d(g_n(y)^{-1}g(f^n y)^{-1}g(f^n x)g_n(x), g_n(y)^{-1}g(f^n x)^{-1}g(f^n x)g_n(x))\\
&\quad = d(g_n(y)^{-1}g(f^n y)^{-1}, g_n(y)^{-1}g(f^n x)^{-1}) \quad \text{(right invariance)}\\
&\quad \le \|\mathrm{Ad}(g_n(y)^{-1})\| d(g(f^n y)^{-1}, g(f^n x)^{-1}) \quad \text{(by (1))}\\
&\quad \le C((\mu_s)^{-1}\lambda_s)^n\\
&\quad \le C\kappa^n,
\end{aligned}$$

where $|\kappa| < 1$.

Thus the sequence $g_n(y)^{-1}g_n(x)$ is Cauchy and so converges. Next we show that Φ_x^s is Lipschitz. Let $y, z \in W^s(x)$, then for large n.

$$\begin{aligned}
&d(g_n(y)^{-1}g_n(x), g_n(z)^{-1}g_n(x)) = d(g_n(y)^{-1}, g_n(z)^{-1})\\
&\quad \leqslant \sum_{i=0}^{n-1} d(g_i(y)^{-1}g(f^i y)^{-1}g_{n-i-1}(f^{i+1}z)^{-1}, g_i(y)^{-1}g(f^i z)^{-1}g_{n-i-1}(f^{i+1}z)^{-1})\\
&\quad \leqslant \sum_{i=0}^{n-1} \|\mathrm{Ad}(g_i(y))^{-1}\| d(g(f^i y)^{-1}, g(f^i z)^{-1})\\
&\quad \leqslant \sum_{i=0}^{n-1} C\mu_s^i\lambda_s^i\, d(y,z).
\end{aligned}$$

Since $\mu_s\lambda_s < 1$, letting $n \to \infty$ gives $d(\Phi_x^s(y), \Phi_x^s(z)) \leqslant Kd(y,z)$, K constant.

Let $Y' := \{x \in X \mid g(x) = h(fx)h(x)^{-1}\}$ and define $Y = \bigcap_{n\in\mathbb{Z}} f^{-n}Y'$. Then $m(Y) = 1$, the cocycle equation holds on Y and Y is f-invariant. For $x, y \in Y$ we may iterate the cocycle equation to give

$$g_n(y)^{-1}g_n(x) = h(y)h(f^n y)^{-1}h(f^n x)h(x)^{-1}.$$

By Lusin's theorem, we can find a set H, $m(H) > \frac{1}{2}$ such that h and h^{-1} are uniformly continuous when restricted to H. For almost all stable leaves $W^s(x)$ and any full measure set A, the intersection of $W^s(x)$ with A has full m_x measure, where m_x is the conditional

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induced measure on $W^s(x)$ (i.e. scaled one-dimensional Lebesgue measure). In particular, if we let $\mathcal{A}$ be the set of points $x \in \mathbb{T}^2$ such that

$$m(H) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_H(f^i x)$$
$$= \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_H(f^{-i} x)$$

(by the ergodicity of f a set of full measure) then we can conclude that for m almost every $W^s(x)$, $m_x(\mathcal{A} \cap W^s(x)) = m_x(W^s(x))$. Since $m(H) > \frac{1}{2}$ for each x, $y \in W^s(x) \cap \mathcal{A}$ there exists a sequence $n_i \to \infty$ such that $f^{n_i} x$, $f^{n_i} y \in H$. Thus if we let $n \to \infty$ through the sequence $\{n_i\}$ we obtain

$$h(y)h(x)^{-1} = \Phi_x^s(y).$$

Hence for m-almost all stable leaves $W^s(x)$ and m_x-almost all $y \in W^s(x)$, $h(y) = \Phi_x^s(y)h(x)$. Thus there is version of h whose restriction to m-a.e. stable leaf $W^s(x)$ is Lipschitz. A similar argument shows that there is a version of h whose restriction to m-a.e. stable leaf $W^s(x)$ and m-a.e. unstable leaf $W^u(x)$ is Lipschitz.

The Sinai method. Our next aim is to prove that almost every x has a neighbourhood $U(x)$ on which h has a Lipschitz version. For this we recall the Sinai method used in [**12**] to prove the local ergodicity of f.

In the proof of local ergodicity two sectors, $\mathcal{K}_1$ and $\mathcal{K}_2$, containing the stable and unstable directions respectively, are chosen. These sectors are chosen sufficiently narrow that for any square intersected by a line with direction in $\mathcal{K}_i$, $i = 1, 2$, the areas shown in Figure 2 are no more than $\frac{1}{8}$ of the area of the square. The constants M and N are then chosen to be the smallest positive integers so that all line segments $\{f^{-i}R^+\}_{i=M}^{\infty}$ or $\{f^i R^-\}_{i=N}^{\infty}$ fall inside $\mathcal{K}_1$ or $\mathcal{K}_2$, respectively. We may choose such M, N as R^+ is not contained in the unstable direction and R^- is not contained in the stable direction.

The neighbourhood of $x, \mathcal{U}(x)$, is defined as the largest neighbourhood not intersecting the collection of lines

$$L := \bigcup_{i=0}^{N-1} f^{-i} R^+ \cup \bigcup_{i=0}^{M-1} f^i R^-.$$

Let $x \in \mathbb{T}^2/L$ and $\mathcal{U}(x)$ be such a neighbourhood of x. To simplify the exposition we assume $\mathcal{U}(x)$ has the shape of a square. We construct a sequence of open coverings of $\mathcal{U}(x)$ by dividing it into n^2 squares of equal size and then expanding the squares by a factor of two, keeping centres fixed to obtain an open cover Γ_n. We make the following definitions.

Definition. Let H be a square in Γ_n, and y be a point in H which has a stable leaf and an unstable leaf. If $W^s(y)$ (respectively $W^u(y)$) intersects both horizontal (vertical) edges of H then we say that $W^s(y)$ (respectively $W^u(y)$) is long or connecting in H. Otherwise, $W^s(y)$ (respectively $W^u(y)$)) is called short in H.

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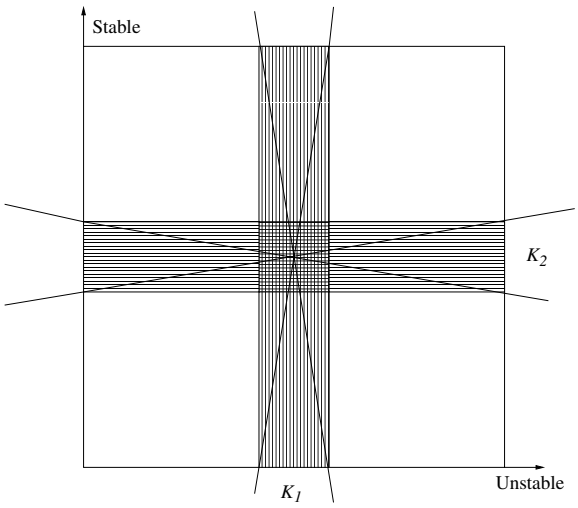


FIGURE 2. Areas shaded are each less than $\frac{1}{8}$ of the area of the square.

A square $H \in \Gamma_n$ is called connecting if:

$$m(\{y \in H : W^s(y) \text{ and } W^u(y) \text{ is long in H}\}) \geq \tfrac{3}{4}m(H).$$

Otherwise H is called non-connecting.

A stable (unstable) leaf is said to completely intersect a row (column) of squares if it is connecting in one of the squares in the row (column).

Note that if H_1, H_2 are two adjacent connecting squares then

$$m\{p \in H_1 \cap H_2 : W^u(p) \text{ and } W^s(p) \text{ are long in } H_1 \text{ and } H_2\} \geq \tfrac{1}{4}m(H_1).$$

This implies that it is possible to construct a $W^{u,s}$ path between two adjacent connecting squares.

A standard argument, a nice account of which is given in [**12**] to demonstrate the Sinai–Chernov method, shows the following.

PROPOSITION 5. [**12**, Sinai Theorem] *Let $x \in \mathbb{T}^2$ be smooth in the past and future. Then*

$$\lim_{n\to\infty} n\, m(\cup H \subset \Gamma_n \mid H \text{ is non-connecting}) = 0.$$

Let $\mathcal{U}(x)$ be a square neighbourhood of a point x constructed as above and on which the Sinai theorem holds. Suppose y and z are points in $\mathcal{U}(x)$ which are smooth in both the past and future and such that h is uniformly Lipschitz on their stable and unstable leaves (we will call such points *typical*). Also, assume that $W^u(y)$ and $W^s(z)$ are such that m_y, m_z-a.e. point on them is typical and contained in $\mathcal{A}$. The set of such points is of full measure in $\mathcal{U}(x)$.

As a consequence of Proposition 5 it is straightforward to show the following.

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LEMMA 3. *There exists $n \in \mathbb{N}$ such that at least one row of connecting squares in Γ_n is completely intersected by $W^s(z)$ and one column of connecting squares by $W^u(y)$.*

Proof. Suppose that for any n, all rows which are completely intersected by $W^s(z)$ contain at least one bad square. Since the number of such rows grows linearly with n and the measure of one square is $4m(\mathcal{U}(x))/n^2$, this implies that $\lim_{n\to\infty} n\, m(\cup H \subset \Gamma_n \mid \text{H is bad}) = O(1)$, contradicting the Sinai theorem. The statement regarding $W^u(y)$ is proven similarly. □

We may choose a row and column of connecting squares which are intersected completely by $W^s(z)$ and $W^u(y)$, respectively. We are thus able to construct a $W^{u,s}$ path of stable and unstable leaves which links $W^s(z)$ and $W^u(y)$. By the absolute continuity of the foliation, the intersections of the leaves of the path may be chosen to be typical points $\{x_i\}$, $i = 1, \ldots, N$ say, which also lie in $\mathcal{A}$. More explicitly, $x_1 \in W^s(z)$, $x_N \in W^u(y)$ and $x_i \in (W^s(x_{i-1}) \cup W^u(x_{i-1})) \cap (W^s(x_{i+1}) \cup W^u(x_{i+1}))$, $i = 2, \ldots, N-1$. Note that h is uniformly Lipschitz (with same Lipschitz constant K) along each leaf $W^r(x_i)$ where $r \in \{s, u\}$.

Furthermore, since $x_i \in \mathcal{A}$, $d(h(x_i), h(x_{i+1}) \le K d(x_i, x_{i+1})$ for each $i = 0, \ldots, N+1$ (where for convenience we take $x_0 = z$ and $x_{N+1} = y$). Since $\sum_{i=0}^{N+1} d(x_i, x_{i+1}) \le \sqrt{2}d(z, y)$, it follows that h has a Lipschitz version on $\mathcal{U}(x)$ and hence h has a version which is Lipschitz on each connected open component of $\mathbb{T}^2/L$.

We now show that h is uniformly Lipschitz on $\mathbb{T}^2$. Let C_1 and C_2 be two such components separated by a line segment $l \in f^{M-1}R^-$. Take a point z on l, not lying on any other line in L and a small neighbourhood A of z.

The point fz is contained in the interior of one of the components of $\mathbb{T}^2/L$. For A sufficiently small, $f(A)$ will lie wholly inside one of the components. This is because the only invariant lines have direction vectors equal to the eigenvectors of the linearization of f. As g is C^1 on A we can thus conclude, in view of the equation $g(x) = h(fx)h(x)^{-1}$, that h is Lipschitz on A, allowing us to extend the Lipschitz version of h to the union of C_1 and C_2. We then argue inductively to take care of all the lines

$$l \in \bigcup_{i=0}^{M-1} f^i R^-.$$

In a similar way we can also inductively extend h as a Lipschitz version across the boundaries of the components given by the lines

$$l \in \bigcup_{i=0}^{N-1} f^{-i} R^+$$

using the identity $h(x) = g(f^{-1}x)h(f^{-1}x)$. Thus h has a Lipschitz extension to $\mathbb{T}^2$. This concludes the proof of Theorem 3. □

Proof of Theorem 4. In our set-up, conditions (I)–(III) imply conditions A–F of [**12**]. The same argument as in the case of the toral linked twist map shows that there is a full measure set $A \subset \mathbb{T}^2$ and a version of h which has the same Lipschitz constant C when restricted to $W^u_\epsilon(x)$ and $W^s_\epsilon(x)$ for $x \in A$ (note that ϵ does not indicate the induced length

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of local stable or unstable manifolds, over which there is no uniformity, merely that they are not global). Take a neighbourhood $U(x)$ of x and two points $y, z \in A \cap U(x)$ for which there exists a $W^{u,s}$ path which links $W^s_\epsilon(y)$ and $W^u_\epsilon(z)$. The absolute continuity of the foliations into local stable and unstable leaves allows us to assume that if x_i, $i = 1, \ldots, N$ are the intersections of the leaves of the $W^{u,s}$ path, then $x_i \in A$. Conditions (I) and (II) imply that $\sum_{i=1}^N d(x_i, x_{i+1}) \leq K d(y, z)$ and hence

$$\begin{aligned} d(h(y), h(z)) &\leq \sum_{i=1}^N d(h(x_i), h(x_{i+1})) \\ &\leq \sum_{i=1}^N C d(x_i, x_{i+1}) \\ &\leq KC d(y, z). \end{aligned}$$

Thus h has a version h' which is Lipschitz on each neighbourhood $U(x)$ with the same Lipschitz constant. □

2. *Ergodicity of compact Lie group extensions of hyperbolic systems with discontinuities* Suppose that $f: X \to X$ is an ergodic dynamical system with respect to an invariant measure m. Let G be a compact Lie group and let dh denote Haar measure on G. Suppose that $g: X \longrightarrow G$ is measurable.

PROPOSITION 6. (Keynes and Newton [**9**]; see also [**18**])

(a) *The compact group extension $T(x, g) = (f(x), g(x)g)$ is ergodic with respect to $m \times dh$ if and only if the only solutions to*

$$h(fx) = R(g(x))h(x), \tag{6}$$

where R is an irreducible (unitary) representation of degree d and $h: X \longrightarrow \mathbb{C}^d$ is measurable, are h constant and R the trivial representation.

(b) *Suppose that $f: X \to X$ is weak-mixing and $T: X \times G \to X \times G$ is ergodic. Then $T(x, g) = (f(x), g(x)g)$ is weak-mixing with respect to $m \times dh$ if and only if for any $e^{i\alpha} \neq 1$ and any non-trivial one-dimensional representation χ of G, the equation*

$$h(fx) = e^{i\alpha}\chi(g(x))h(x) \quad m\text{-a.e.} \tag{7}$$

has no non-trivial measurable solution $h: X \to \mathbb{C}$.

We now give two results which pertain to the class of systems we have considered. For a discussion of the notion of prevalence see [**5**]. We now consider C^1 rather than Lipschitz extensions because it is simpler to state our results in the C^1 topology, although in the case of Lasota–Yorke maps analogous statements may be made in the Hölder topology.

THEOREM 7. *Suppose that $f: X \to X$ is a Lasota–Yorke map which is weak mixing with respect to an absolutely continuous invariant measure m with support X. Let F denote the class of C^1 cocycles $g: X \longrightarrow G$.*

(a) *If G is semisimple, then there exists a C^1 open and dense subset F' of F such that $T(x, g) = (fx, g(x)g)$ is Bernoulli.*

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(b) *If G is a compact connected Lie group, then there exists a C^1 prevalent and residual subset F' of F such that if $g \in F'$, then $T(x, g) = (fx, g(x)g)$ is Bernoulli.*

THEOREM 8. *Suppose that $f\colon \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ is a uniformly hyperbolic toral map with discontinuities. Let F denote the class of C^1 cocycles $g\colon X \to G$.*

(a) *If G is semisimple, then there exists a C^1 open and dense subset F' of F such that $T(x, g) = (fx, g(x)g)$ is Bernoulli.*

(b) *If G is a compact connected Lie group, then in the C^1 topology there exists a prevalent and residual subset F' of F such that if $g \in F'$, then $T(x, g) = (fx, g(x)g)$ is Bernoulli.*

In the following we establish Theorems 7 and 8. We consider first abelian and semisimple extensions and then, following Field and Parry [**4**, §5.1], (see also [**17, 19**]) use the fact that a compact connected Lie group is covered by the direct product of a compact group with a semisimple Lie group to obtain our results for general compact connected Lie groups.

2.1. *Proof of Theorems 7 and 8.* We may modify the proof of the Livšic regularity theorems above to conclude that if $f\colon X \to X$ is a map satisfying the conditions of Theorem 7 or Theorem 8 and there exists a measurable solution h to equation (6), then there exists a solution which is Lipschitz on an open set. The modification is straightforward and we sketch the main points only in the case of Lasota–Yorke maps. For m-a.e. $x_0 \in X$ there exists a sequence of intervals $\{J_n\}$ and an inverse branch $x_n \in J_n$, $f^n J_n = J_0$. Let $z_0, y_0 \in J_0$ and $f^n(z_n) = z_0$ and $f^n(y_n) = y_0$ define the inverse branches of z_0 and y_0, respectively, with the property $z_n, y_n \in J_n$. We calculate

$$\begin{aligned}\|h(z_0) - h(y_0)\| &= \|R(g(z_1))\cdots R(g(z_{n-1}))h(z_n) - R(g(y_1))\cdots R(g(y_{n-1}))h(y_n)\| \\ &\leqslant \Bigg[\sum_{j=0}^{n-2} \|R(g(y_1))\cdots R(g(y_j))(R(g(z_{j+1})) \\ &\quad - R(g(y_{j+1})))R(g(z_{j+2}))\cdots R(g(z_{n-1}))h(z_n)\|\Bigg] \\ &\quad + \|R(g(y_1))\cdots R(g(y_{n-1}))(h(z_n) - h(y_n))\|.\end{aligned}$$

The sum

$$\sum_{j=0}^{n_k-2} \|R(g(y_1))\cdots R(g(y_j))(R(g(z_{j+1})) - R(g(y_{j+1})))R(g(z_{j+2}))\cdots R(g(z_{n_{k-1}-1}))h(z_{n_k})\|$$

is bounded above by $Kd(z_0, y_0)$ for some $K > 0$, since $R(g(z_{j+2}))\cdots R(g(z_{n_{k-1}-1}))$ $h(z_{n_k})$ is uniformly bounded in k and $d(z_j, y_j) < \lambda_u^{-j} d(z, y)$. Since h is measurable there is a set $H, m(H) > \frac{1}{2}$, such that h restricted to H is uniformly continuous (hence bounded). The same argument as that of Step 3 in §1.1 using the Lebesgue density theorem and the

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fact that f has bounded distortion shows that given $\eta > 0$,

$$\begin{aligned} m \times m\{(y_0, z_0) : \|R(g(y_1)) \cdots R(g(y_{n-1}))(h(z_n) - h(y_n))\| \leqslant \eta\} \\ \geq (1 - 2\eta)(m \times m)(J_0 \times J_0), \end{aligned}$$

and hence for m-a.e. $x_0 \in X$ there is an interval $x_0 \in J_0$ such that h restricted to J_0 is Lipschitz.

Thus in equation (6) we may assume that m-a.e. point has a neighbourhood $U(x)$ on which h is Lipschitz. The same analysis shows that in equation (7) we may also assume that m-a.e. point x has a neighbourhood $U(x)$ on which h is Lipschitz.

2.2. *Abelian and semisimple extensions of Lasota–Yorke maps.* In the case that $G = \mathbb{T}^r$ is a toral group we may, following Field and Parry [**4**, Theorem 2.4.2, Remark 2.4.3] recast equation (6) in multiplicative form as the statement that the group extension is ergodic iff there are no non-trivial solutions $h\colon X \longrightarrow \mathbb{T}^r$ to the equation $h(fx)h(x)^{-1} = g(x)^k$ for any integer $k = (k_1, \ldots, k_r)$ and representation $g^k = (g_1^{k_1}, \ldots, g_r^{k_r})$. By Theorem 2 we may assume h has the property that m-a.e. point x has a neighbourhood on which h is uniformly Lipschitz. If p is a periodic point of period n that avoids the singular set for f, then $h(fx)h(x)^{-1} = g(x)^k$ implies that $Id = (g(f^{n-1}p) \cdots g(p))^k$. However, $(g(f^{n-1}p) \cdots g(p))^k = Id$ only if $g(f^{n-1}p) \cdots g(p)$ is not a generator of $\mathbb{T}^r$.

Lasota–Yorke maps have a dense set of periodic points. We may choose a periodic orbit (i.e. a point x such that $f^{n(x)}x = x$) that avoids the singular set $\{0\}$ and whose iterates are contained in an open set on which h is Lipschitz. For a prevalent and residual set of extensions $g\colon X \longrightarrow G$, $g(f^{n-1}x) \cdots g(x)$ will generate the toral group G [**4**]. Thus for a prevalent and residual set of extensions g, the toral extension is ergodic (for a discussion of the notion of prevalence see [**5**]).

To establish the prevalence and residualness of weak-mixing, first note that we may as well assume that T is ergodic as a prevalent and residual set of extensions are ergodic. It is easy to see that we may choose two periodic points x, y with distinct orbits of periods m, n say, both of whose iterates are contained in neighbourhoods on which h is Lipschitz. If T is not weak-mixing then neither $e^{im\alpha}g(f^{m-1}x) \cdots g(x)$ nor $e^{in\alpha}g(f^{n-1}y) \cdots g(y)$ generate G, which implies that $g(f^{m-1}x) \cdots g(x)$ is rationally related to $g(f^{n-1}y) \cdots g(y)$. For a prevalent and residual set of extensions $g(f^{m-1}x) \cdots g(x)$ will not be rationally related to $g(f^{n-1}y) \cdots g(y)$ and hence a prevalent and residual set of extensions are weak-mixing. A result of Rudolph [**25**, Theorem 1] states that a weak-mixing compact group extension of a Bernoulli dynamical system is Bernoulli, and hence we have shown a prevalent and residual set of extensions are Bernoulli.

Assume now that G is semisimple. Kuranishi [**11**] has shown that an open and dense set of pairs (g_1, g_2) of elements of G generate G (Field has shown the set of such pairs is Zariski open [**3**]). We may choose a periodic point for f, x_0, and a point $y = (y_0, y_1, \ldots) \in \Omega$ homoclinic to an inverse branch of x_0, $x = (x_0, x_1, \ldots, x_0, \ldots)$. Presumably it is well known that such homoclinic points exist for piecewise C^2 expanding maps, but we were unable to find a reference in the literature so we have included a proof here.

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To see that we may assume the existence of such homoclinic points and inverse branches note that by taking a suitable power of f we may assume $|f'(x)| > 2$.

As we saw before, assuming $|f'(x)| > 2$, for each i and every subinterval $I \subset (s_i, s_{i+1})$, there exists $J(0) \subset I$ such that $f^{k_1}(J(0)) \supset (s_j, s_{j+1})$ for some j, k_1 and $f^i(J(0)) \cap S = \emptyset$ for $i = 0, \ldots, k_1$. Hence we may extend h to be Lipschitz on (s_j, s_{j+1}). Furthermore, for any periodic point z_0 and for any sufficiently small interval $I(z_0)$, $z_0 \in I$, there exists an inverse branch $\{I_n(z_0)\}$ of $f^{-n}(I(z_0))$ which contains an iterate of the point z_0 at each stage, i.e. $f^j(z_0) \in I_n(z_0)$ for some j, and which limits on z_0 in the sense that $\lim_{n\to\infty} l(I_n(z_0)) = 0$.

So we may take a subinterval $J(0) \subset I(z_0) \subset (s_{j_0}, s_{j_0+1})$, $z_0 \notin J(0)$ and an integer k_0 such that $f^{k_0}(J(0)) \supset (s_{j_1}, s_{j_1+1})$ for some integers k_0, j_0. If we may take $j_1 = j_0$, then there exists a point $y_0 \in J(0)$ distinct from z_0 for which $f^{k_0}(y_0) = z_0$ and which has an inverse branch limiting on z_0 so we have found a point homoclinic to z_0. It may be the case that $j_1 \neq j_0$. In this case we choose a periodic orbit $z_1 \in (s_{j_1}, s_{j_1+1})$ with corresponding interval $I(z_1)$ and subinterval $J(1) \subset I(z_1)$ such that $f^{k_1} J(1) \supset (s_{j_2}, s_{j_2+1})$. Continuing this process, since there are only finitely many intervals (s_i, s_{i+1}), we find a cycle of (s_j, s_{j+1}) interval indices $(j_0 j_1 \cdots j_t \cdots j_t)$. By refining the interval $I(t) \subset J(z_t) \subset (s_t, s_{t+1})$, we find $I(t)$, $z_t \notin I(t)$, such that $f^{k_t} I(t) \supset (s_t, s_{t+1})$ and $f^i(I(t)) \cap S = \emptyset$ for $i = 0, \ldots, k_t$.

Hence we may find a periodic point $x_0 \in (s_i, s_{i+1})$ for some i on which h has a Lipschitz version and an interval $I \subset (s_i, s_{i+1})$ such that $f^k(I) \supset (s_i, s_{i+1})$ and an inverse branch of I limits on x_0.

We assume without loss of generality that x_0 is a fixed point (otherwise take an appropriate power of the mapping) and that $\lim_{n\to\infty} f^n y_0 = x_0$ and $\lim_{n\to\infty} y_n = x_0$. We may also assume by construction that forward iterates of y_0 and x_0 avoid $\{0\}$ and $x_k, y_k \notin B$ for any $k > 0$.

Consider the cohomology equation

$$R(g(z))h(z) = h(fz),$$

where R is an irreducible unitary representation and h is Lipschitz on an open dense set of intervals (by Livšic's theorem) and of constant modulus $|h| = 1$ (since R is unitary). To simplify notation we write $R(g)$ as g. Iterating the cohomology equation, we obtain $h(y_0) = g(f^{n-1}y_n) \cdots g(y_n)h(y_n)$ and $g(f^{n-1}y_0) \cdots g(fy_0)h(y_0) = h(f^n y_0)$ for each positive integer n. Since the representation is linear (hence smooth), the cocycles g,h are Lipschitz and the products $g(f^{n-1}y_n) \cdots g(y_n)$ and $g(f^{n-1}y_0) \cdots g(fy_0)$ converge. Let $\gamma^- = \lim_{n\to\infty} g(f^{n-1}y_n) \cdots g(y_n)$ and $\gamma^- = \lim_{n\to\infty} g(f^{n-1}y_0) \cdots g(y_0)$. Since

$$\lim_{n\to\infty} h(f^n y_0) = \lim_{n\to\infty} h(y_n) = h(x_0),$$

we have the equations $h(x_0) = \gamma^+ h(y_0)$, $h(y_0) = \gamma^- h(x_0)$. Furthermore, since x_0 is fixed, $g(x_0)h(x_0) = h(x_0)$. Thus $h(x_0) = \gamma^+ h(y_0) = \gamma^+\gamma^- h(x_0)$ and $h(x_0) = g(x_0)h(x_0)$.

Since generating pairs are open and dense for semisimple Lie groups, it is easy to show that for a C^1 open and dense set of extensions the pair $(g(x_0), \gamma^+\gamma^-)$ generate G and hence the entire group fixes $h(x_0)$. This is a contradiction as $|h(x_0)| = 1$ and the representation was assumed irreducible.

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Since the only characters of semisimple Lie groups are trivial, ergodicity implies weak-mixing, so that an open and dense set of extensions are weak-mixing (hence Bernoulli [**25**]).

2.3. *Abelian and semisimple extensions of uniformly hyperbolic toral maps with discontinuities.* We may, using the same arguments as in the case of Lasota–Yorke maps, assume that m-a.e. $x \in T^2$ has a neighbourhood $U(x)$ on which solutions to equations (6) and (7) are Lipschitz.

The proofs to establish prevalence and residualness of the Bernoullicity of compact connected Lie group extensions and the openness and density of the Bernoullicity of semisimple group extensions proceed in much the same way as in the setting of the β-transformation, with the proviso that our arguments using periodic and homoclinic points rely on the work of Krüger and Troubetskoy [**10**], who constructed Markov partitions for a large class of hyperbolic systems with discontinuities. We have included the details in Appendix A.

2.4. *General compact lie group extensions.* Our proof follows that of Field and Parry [**4**, §5.1], who use a structure theorem for general compact Lie groups to deduce the result from the corresponding result for semisimple and abelian Lie groups. We only sketch the details. If G is a compact connected Lie group, then there is a covering homeomorphism $\pi : \tilde{G} \longrightarrow G$, where $\tilde{G}$ is the direct product of a torus $\mathbb{T}^r$ and a compact semisimple Lie group Γ.

Define $A = \pi(\mathbb{T}^r \times I_\Gamma)$ and $B = \pi(I_{\mathbb{T}^r} \times \Gamma)$. Then A and B are normal subgroups of G. Furthermore A is abelian (in fact $A \simeq \mathbb{T}^r$) and B is semisimple. Let $\pi_A : G \longrightarrow G/A$ and $\pi_B : G \longrightarrow G/B$ be the quotient homeomorphisms. If we denote $G/B = \tilde{A}$ and $G/A = \tilde{B}$, then $\tilde{A} \simeq \mathbb{T}^r$ and $\tilde{B}$ is semisimple. We consider only the case of Lasota–Yorke maps as the case of uniformly hyperbolic toral maps is identical. If $g : X \longrightarrow G$ is a C^1 cocycle, we define $g_A = \pi_A g$ and $g_B = \pi_B g$. Note that g_A (respectively g_B): $X \longrightarrow \tilde{B}$ (respectively g_A) is a C^1 cocycle and hence defines a $\tilde{B}$ (respectively $\tilde{A}$) extension. For a prevalent set P of cocycles g, both extensions are Bernoulli. Let Σ be the isotropy subgroup of an ergodic component of the g-extension $T(x, g) = (fx, g(x)g)$ and assume that $g \in P$. Then $\pi_A(\Sigma) = \tilde{A}$ and $\pi_B(\Sigma) = \tilde{B}$, which implies $\Sigma = G$. Hence $T(x, g) = (fx, g(x)g)$ is ergodic. Recall that if T is ergodic then T is weak-mixing if and only if for any $e^{i\alpha} \neq 1$ and any non-trivial one-dimensional representation χ of G, the equation

$$h(fx) = e^{i\alpha}\chi(g(x))h(x) \quad m\text{-a.e.}$$

has no non-trivial measurable solution $h : X \to \mathbb{C}$. It follows from the measurable Livšic theorem we have proved that we may assume that h is Lipschitz. Since B is semisimple, $\chi|B$ is trivial and hence χ determines a character $\chi_A : \tilde{A} \longrightarrow S^1$ by $\chi_A(gB) = \chi(g)$, $g \in A$. Since the g_A extension is weak-mixing, χ_A is trivial and $\alpha = 1$, and hence T is weak-mixing and by Rudolph's theorem [**25**, Theorem 1] also Bernoulli. The argument in the case of uniformly hyperbolic toral maps with discontinuities is identical.

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3. *Discussion*

The method of proof of Theorems 1, 3, 4, 7 and 8 is simple and robust as it is based upon the Hopf method. It allows the generalization to a wider class of base transformations, but the statements of analogous results in a general setting are complicated and certain details of the proofs, particularly those relating to the geometry of the mappings, may need to be checked on a case-by-case basis.

In the context of one-dimensional mappings it would be interesting to know if measurable solutions to cohomological equations for Lasota–Yorke maps have regular solutions on X, the support of the absolutely continuous invariant measure, as in the case of Markov maps and β-transformations. We were unable to prove this. A natural question to ask is what can be said about the regularity of measurable solutions to cohomological equations involving non-uniformly expanding maps or more general invariant measures than smooth measures.

Our results in two dimensions are limited to discontinuous uniformly hyperbolic systems with Lipschitz foliations preserving volume measures and it would be interesting to see whether similar regularity theorems hold for non-uniformly hyperbolic systems with discontinuities. Two crucial features of our proof would break down in these settings. The first is that if contraction (expansion) is not uniform, then we would lose control over the uniformity of the Lipschitz constant for h restricted to stable (unstable) manifolds. Second, we use a strong form of local accessibility, i.e. m-a.e. point p has a neighbourhood $U(p)$ such that a.e. pair of points $x, y \in U(p)$ may be joined by a $W^{u,s}$ path (we assume no uniformity over the number n of legs needed), but we do assume that we may choose the path so that $\sum_{i=1}^{n} d(w_i, w_{i+1}) < Kd(x, y)$ where K is constant. A weaker assumption such as $\sum_{i=1}^{N-1} d(x_i, x_{i+1}) \leq Kd(y, z)^{\alpha}$ for $\alpha > 0$ would yield Hölder regularity, but it is not clear to us for which discontinuous systems this assumption would hold. Another natural question in this context would be to consider whether *a priori* measurable solutions have smooth versions. The results and techniques of de la Llave, Marco and Moriyón [**2**], and Nitica and Török [**17**] are relevant to this question.

A different approach would be to try to establish measurable Livšic theorems for systems for which a Markov tower construction holds, thereby reducing to a base transformation on which we have uniform estimates.

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A. *Appendix*

Krüger and Troubetskoy have shown that for a class of non-uniformly hyperbolic systems with singularities (described axiomatically in [**10**, §1]), periodic orbits are dense [**10**, Lemma 5] and there exists a countable Markov partition for the system. By assumption, the two-dimensional systems we consider fall within this class and hence automatically periodic points are dense. In this appendix, we show that the existence of a Markov partition with the properties Krüger and Troubetskoy establish is enough to prove our stable ergodicity results.

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More precisely, Krüger and Troubetskoy have shown that for m-a.e. $x \in M := \mathbb{T}^2 \setminus \left(\bigcup_{i \geq 0} f^{-i} R^+ \cup f^- R^- \right)$ there exists a unique corresponding symbol sequence from at most a countable alphabet given by the Markov partition. Furthermore, (M, f) is conjugate to a topological Markov chain (Σ, σ) in the sense that there is a subset $B \subset M$ of measure one such that the conjugating map $\phi\colon (B, f) \longrightarrow (\phi B, \sigma)$ satisfies that ϕ^{-1} is continuous. We will denote ϕB by $\tilde{B}$.

Choose a periodic point $\omega \in \Sigma$ (not necessarily in $\tilde{B}$) which lies in a neighbourhood $U \subset \Sigma$ such that ϕ^{-1} restricted to $\tilde{B} \cap U$ is continuous. Since $\tilde{B}$ is dense in Σ we may define $p = \phi^{-1}(\omega)$ by continuity if $\omega \notin \tilde{B}$. Suppose for the moment that $\omega \in \tilde{B}$ and hence $\phi^{-1}(\omega) \in M$. Without loss of generality, we suppose that ω is a fixed point for σ. Since m-a.e. point x in $\phi^{-1}U$ satisfies $\phi f^j(x) = \sigma^j \phi(x)$ for all $j \in \mathbb{Z}$ by using the continuity of ϕ^{-1} and the symbolic dynamics of (B, σ), we may find a sequence of points $\omega_j \in \tilde{B}$ satisfying $\phi^{-1}\sigma^i w_j = f^i \phi^{-1} w_j$ for all $i \in \mathbb{Z}$, $j \geq 0$ and which have the properties that $\lim_{j \longrightarrow \infty} f\phi^{-1}(\omega_j) = f\phi^{-1}(\omega)$ and $\lim_{j \to \infty} \phi^{-1}(\omega_j) = \phi^{-1}(\omega)$. Furthermore, by the continuity of ϕ^{-1} and f it follows that $\phi^{-1}(\omega) = \lim_{j \to \infty} \phi^{-1}(\omega_j) = \lim_{j \to \infty} f\phi^{-1}(\omega_j) = f\phi^{-1}(\omega)$. Thus if $\phi^{-1}(\omega) \in M$, then $p = \phi^{-1}(\omega)$ is a periodic orbit under f. Hence by the Livšic theorem, a coboundary must satisfy the equation $g(p)h(p) = h(p)$. If $\phi^{-1}(\omega) \notin M$, then we may define $\phi^{-1}(\omega_j) = p_j$ and $f\phi^{-1}(\omega_j) = fp_j$ and note that $h(fp_j) = g(p_j)h(p_j)$. Taking limits as $j \to \infty$, we obtain $h(p) = g(p)h(p)$ as before, since h and g are Lipschitz. The argument to establish that for any non-trivial irreducible representation h must satisfy the equation $R(g(p))h(p) = h(p)$ is identical. Hence, arguing as in §2.2, we establish the prevalence and residualness of Bernoulli abelian extensions.

To establish our genericity results for the Bernoulliness of semisimple extensions, we choose a point $\tilde{\omega} \in U$ homoclinic to ω under σ. It may not be true that $\phi^{-1}\sigma^j\tilde{\omega} = f^j\phi^{-1}\tilde{\omega}$ for all j, so it is not possible to state that $\phi^{-1}(\tilde{\omega})$ is homoclinic to $p = \phi^{-1}(\omega)$ under f. However since $\phi f^j(x) = \sigma^j\phi(x)$ for all $j \in \mathbb{Z}$ holds for m-a.e. $x \in \phi^{-1}U$, we may find a sequence of points $\{\tilde{\omega}_r\}$ which have the property that for each $j \in \mathbb{Z}$, $\phi^{-1}\sigma^j\tilde{\omega}_r = f^i\phi^{-1}\tilde{\omega}_r$ and defining $y_i^r = f^i\phi^{-1}\tilde{\omega}_r$:

(a) for each i, $\{y_i^r\}$ is Cauchy in r;

(b) $\lim_{r \to \infty} y_{\pm r}^r \to p$.

The Lipschitz function h satisfies for each r the equations $h(y_r^r) = g(y_{r-1}^r) \cdots g(y_0^r)h(y_0^r)$ and $h(y_0^r) = g(y_{-1}^r) \cdots g(y_{-r}^r)h(y_{-r}^r)$. We may choose the rate of convergence in (a) so that $\lim_{r \to \infty} g(y_{-1}^r) \cdots g(y_{-r}^r) := \gamma^-$ and $\lim_{r \to \infty} g(y_1^r) \cdots g(y_r^r) := \gamma^+$ both exist. Thus, by continuity $h(p) = \gamma^+\gamma^- h(p)$, since $\lim_{r \to \infty} y_{\pm r}^r \to p$. As we noted before, a C^1 open and dense set of cocycles have the property that $\gamma^+\gamma^-$ and $g(p)$ generate G and hence a C^1 open dense set of semisimple extensions are Bernoulli.

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