DIFFRACTION BY A WIDE SLIT AND COMPLEMENTARY STRIP. I.

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Communicated by P. C. Clemmow

Received 26 February 1958

ABSTRACT. In recent years much attention has been directed towards the asymptotic solution of diffraction problems. In the present work, consideration is given to the relatively simple problem of the diffraction of an E-polarized plane wave by an infinite slit. The solution takes the form of a series in inverse powers of the ratio of slit-width to wavelength of the incident wave, and is based on the solution by successive substitutions of a pair of integral equations. The current densities induced on both halves of the screen are calculated, from which is deduced the electric field in the slit. The far-field is determined from the aperture distribution, and an asymptotic expression is found for the transmission coefficient as a function of the angle of incidence and the ratio of slit-width to wavelength.

While most previous work has been confined to near-normal incidence, the present theory is uniformly valid for all angles of incidence.

1. *Introduction.* The problem of the diffraction of a plane scalar or electromagnetic wave by a slit of infinite length in a plane screen (or, what is equivalent, by the complementary strip) has been the subject of numerous past investigations.

The first rigorous treatment (for normal incidence) was presented by Schwarzschild (19) who employed a method of successive approximations based on the Sommerfeld half-plane theory. Shortly thereafter, a solution was obtained by Sieger (20) by the method of separation of variables in the wave equation in elliptical coordinates; but many years passed before the Mathieu functions became sufficiently well tabulated to enable Strutt (22) to obtain some numerical data. Extensive numerical computations have since been performed by Morse and Rubenstein (15), to which later reference will be made.

Solutions as series of increasing powers of the ratio of slit-width to wavelength have been given by Lord Rayleigh ((17), (18)). These have been re-derived and extended in the past several years by Sommerfeld (21), Groschwitz and Hönl (7), Bouwkamp ((2), (3)), Müller and Westpfahl (16), and Hönl and Zimmer (8); see Bouwkamp (3).

The extent of application of such solutions is severely restricted by the slow convergence of the resultant series unless the width of the slit is quite small, and consequently it has been found necessary to proceed along somewhat different lines when the slit-width is greater than a few wavelengths.

Several attempts have been made to obtain solutions of an asymptotic nature which would prove useful even for moderately small values of the slit-width to wavelength ratio. In an approach to the problem proposed by Burger (4) and Timman (23), the steady-state, two-dimensional wave equation of elliptical character was transformed into a wave equation of hyperbolic type in three dimensions. It was demonstrated that the high-frequency solution could then be determined from the solution of the transformed problem for small values of the newly introduced variable.

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The solution obtained originally by Schwarzschild may be interpreted in terms of the diffraction by each half-plane of a sequence of excitations from the other. This same notion has been employed more recently by Karp and Russek ((10), (11)) and Clemmow (5) in connexion with asymptotic solutions to the problem for near-normal incidence.

An integral equation treatment has been given by Levine (13). Attention was confined to the case in which the normal derivative of the wave function vanished on the two halves of the screen, but consideration was given to grazing as well as to near-normal incidence.

It appears to have escaped notice that, despite the slowness of convergence, the series obtained by Schwarzschild may be employed directly in the determination of the asymptotic form of the solution as an expansion in inverse powers of the ratio of slit-width to wavelength, and this is the object of the present work.

More convenient than the method of Schwarzschild is an approach to the problem based on integral equations and suggested by Baker and Copson (1), although the two procedures are, in essence, equivalent. The case of normal incidence only was considered by Schwarzschild, but the formulation in terms of integral equations is made scarcely more complex by the generalization to arbitrary oblique (two-dimensional) incidence.

The problem to be solved is then the following. A plane electromagnetic wave falls onto a thin perfectly conducting plane containing a slit of infinite length. It is assumed that the problem is two-dimensional in nature. Attention is confined in the present paper to the consideration of E-polarized waves, while in Part II a similar study is made of the effect of an H-polarized incident field. In the case of E-polarization (studied in part by Baker and Copson, whose treatment is here briefly restated for completeness), the problem is reduced to the solution of a pair of simultaneous integral equations, while the case of H-polarization gives rise to a pair of differential-integral equations. The latter pair is readily integrated and the solutions of the integral equations for both polarizations are obtained by a method of successive substitutions.

In common with the solutions of Schwarzschild, the convergence of the resulting series is too slow for them to be of any great practical importance in their present form. Nevertheless, the manner for formation of the terms is such that the asymptotic expansions are readily obtained from the series by well-known methods.

In this way is deduced the asymptotic character of the distributions of tangential electric field in the slit. The asymptotic series for the far-fields at a quite general point behind the screen are also determined. In particular, the first few terms in the asymptotic expansions of the transmission coefficients for arbitrary angle of incidence are obtained. Discussion of the deductions for both polarizations, and certain comparisons with previous determinations are deferred to the conclusion of Part II.

2. *The solution for E-polarization.* 2·1. *Formulation of the problem in terms of a pair of integral equations.* Let the screen and slit occupy the plane $x = 0$ in a system (x, y, z) of rectangular Cartesian coordinates. The edges of the slit (the latter of width h) coincide with the lines $x = 0$, $y = 0$, and $x = 0$, $y = -h$. The configuration is illustrated in Fig. 1.

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For the present it will be assumed that the incident electric vector, $\mathbf{E}^i$, of unit intensity, is polarized parallel to the axis of z, and the normal to the wave front makes an angle α with the positive x-axis:

$$\mathbf{E}^i = (0, 0, 1)\, e^{ik(x\cos\alpha + y\sin\alpha)}, \tag{1}$$

where $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$. (The rationalized MKS system of units is employed throughout. It is assumed that steady-state conditions prevail, and the time-dependence factor $e^{i\omega t}$ is suppressed.)

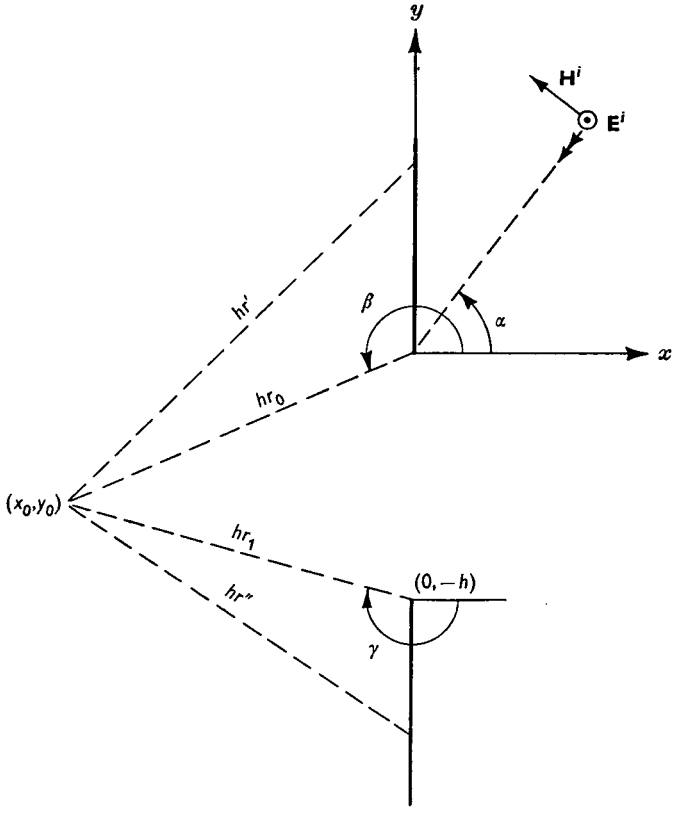


Fig. 1. An E-polarized wave incident on a slit of width h with associated symbols.

In the case under consideration, H_y is the only non-vanishing component of magnetic field tangential to the screen. It follows that an electric current, of density $I_z(y;\alpha)$, flows on the screen in the direction of the z-axis. The total electric field (which has a z-component only) at a point (x_0, y_0) is then given by the following expression:

$$E_z(x_0, y_0) = e^{ik(x_0\cos\alpha + y_0\sin\alpha)} - \frac{kZ_0}{4}\left\{\int_{-\infty}^{-h} + \int_0^{\infty}\right\} I_z(y;\alpha)\, H_0^{(2)}(khr)\, dy. \tag{2}$$

$H_0^{(2)}(khr)$ is the Hankel function of the second kind of order zero, and hr is the distance from (x_0, y_0) to the variable point of integration on the screen

$$(hr)^2 = x_0^2 + (y - y_0)^2. \tag{3}$$

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Z_0 is the impedance of free space and is defined in terms of the permeability μ_0 and permittivity ϵ_0 by

$$Z_0 = (\mu_0/\epsilon_0)^{\frac{1}{2}}.$$

Application of the boundary condition that $E_z(x_0, y_0) = 0$ for all points on the screen yields a pair of simultaneous integral equations for the unknown current density on each half of the screen (see Baker and Copson (1))

$$\left.\begin{aligned} \int_0^\infty I^+(y;\alpha)\, H_0^{(2)}(k\,|y-y_0|)\,dy + \int_0^\infty I^-(y;\alpha)\, H_0^{(2)}[k(y+y_0+h)]\,dy &= k^{-1}\, e^{iky_0 \sin\alpha}, \\ \int_0^\infty I^-(y;\alpha)\, H_0^{(2)}(k\,|y-y_0|)\,dy + \int_0^\infty I^+(y;\alpha)\, H_0^{(2)}[k(y+y_0+h)]\,dy &= k^{-1}\, e^{-ik(y_0+h)\sin\alpha}, \end{aligned}\right\} \tag{4}$$

where $y_0 > 0$ and

$$I^+(y;\alpha) = \tfrac{1}{4} Z_0 I_z(y;\alpha),$$

$$I^-(y;\alpha) = \tfrac{1}{4} Z_0 I_z(-y-h;\alpha)$$

for $y > 0$.

2·2. *Solution of the integral equations.* The integral equations (4) may be transformed into two of known form by an application of a formula of Fox (6)

$$H_0^{(2)}[k(y+y_0+h)] = \frac{1}{\pi}\int_0^\infty \frac{e^{-ik(y+z+h)}}{y+z+h}\left(\frac{y+h}{z}\right)^{\frac{1}{2}} H_0^{(2)}(k\,|z-y_0|)\,dz. \tag{5}$$

Then

$$\left.\begin{aligned} \int_0^\infty [I^+(y;\alpha) + i^-(y;\alpha)]\, H_0^{(2)}(k\,|y-y_0|)\,dy &= k^{-1}\, e^{iky_0\sin\alpha}, \\ \int_0^\infty [I^-(y;\alpha) + i^+(y;\alpha)]\, H_0^{(2)}(k\,|y-y_0|)\,dy &= k^{-1}\, e^{-ik(y_0+h)\sin\alpha} \end{aligned}\right\} \tag{6}$$

when $y_0 > 0$, where

$$i^+(y;\alpha) = \frac{1}{\pi}\int_0^\infty I^+(z;\alpha)\frac{e^{-ik(y+z+h)}}{y+z+h}\left(\frac{z+h}{y}\right)^{\frac{1}{2}} dz,$$

$$i^-(y;\alpha) = \frac{1}{\pi}\int_0^\infty I^-(z;\alpha)\frac{e^{-ik(y+z+h)}}{y+z+h}\left(\frac{z+h}{y}\right)^{\frac{1}{2}} dz$$

for $y > 0$.

The equations (6) are of the Wiener–Hopf type; they arise in the problem of the diffraction of a plane wave by a half-plane, and have solutions

$$I^+(y;\alpha) + i^-(y;\alpha) = \psi(y;\alpha),$$

$$I^-(y;\alpha) + i^+(y;\alpha) = e^{-ikh\sin\alpha}\,\psi(y;-\alpha),$$

where $\psi(y;\alpha) = \frac{1}{2}\cos\alpha\, e^{iky\sin\alpha} - \pi^{-\frac{1}{2}} e^{\frac{1}{4}\pi i}\cos\alpha\, e^{iky\sin\alpha} F[(2ky)^{\frac{1}{2}}\sin(\frac{1}{4}\pi + \frac{1}{2}\alpha)]$

$$+ (2\pi ky)^{-\frac{1}{2}} e^{-\frac{1}{4}\pi i} e^{-iky}\sin(\tfrac{1}{4}\pi - \tfrac{1}{2}\alpha) \tag{7}$$

and the Fresnel integral is defined by

$$F(x) = \int_x^\infty e^{-i\lambda^2}\,d\lambda.$$

It is apparent that $4\psi(y;\alpha)/Z_0$ is equal to the density of the current induced on the surface of the (isolated) half-plane $x = 0$, $y \geqslant 0$ by the incident plane wave.

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It remains, then, to consider the equations

$$\left.\begin{aligned} I^{+}(y;\alpha) &= \psi(y;\alpha) - \frac{1}{\pi}\int_0^\infty I^{-}(z;\alpha)\,\frac{e^{-ik(y+z+h)}}{y+z+h}\left(\frac{z+h}{y}\right)^{\frac{1}{2}} dz, \\ I^{-}(y;\alpha) &= e^{-ikh\sin\alpha}\psi(y;-\alpha) - \frac{1}{\pi}\int_0^\infty I^{+}(z;\alpha)\,\frac{e^{-ik(y+z+h)}}{y+z+h}\left(\frac{z+h}{y}\right)^{\frac{1}{2}} dz \end{aligned}\right\} \tag{8}$$

when $y > 0$.

Equations (8) may be solved by a process of successive substitutions, as outlined by Baker and Copson (1). Such a procedure leads to series solutions of the form

$$I^{\pm}(y;\alpha) = \sum_{n=0}^{\infty} I_n^{\pm}(y;\alpha), \tag{9}$$

where

$$I_n^{\pm}(y;\alpha) = -\frac{1}{\pi}\int_0^\infty I_{n-1}^{\mp}(z;\alpha)\,\frac{e^{-ik(y+z+h)}}{y+z+h}\left(\frac{z+h}{y}\right)^{\frac{1}{2}} dz \tag{10}$$

for $n \geqslant 1$ and

$$I_0^{+}(y;\alpha) = \psi(y;\alpha),$$
$$I_0^{-}(y;\alpha) = e^{-ikh\sin\alpha}\psi(y;-\alpha).$$

These series may be shown to converge uniformly and absolutely for $h > 0$, and to satisfy the integral equations (8). In the following section, the asymptotic expansions of the solution in inverse power of kh will be obtained from the series (9).

Note added in proof. A set of equations similar to (8) but involving the current induced on the shaded side of the screen has been derived recently by G. A. Grinberg (*J. Tech. Phys.* (*U.S.S.R.*), 27 (1957), 2595–2605) in connexion with the same problem. The addition and subtraction of these equations then leads to two independent integral equations for the sum and difference of the unknowns, for which an approximate solution may be found. Little numerical work is performed in this paper.

2·3. *The asymptotic expansion of the solution.* For purposes of computation, the series derived in the preceding section converge much too slowly, unless the slit is extremely wide. Nevertheless, by exploiting the manner in which successive terms are formed, the asymptotic expansions of the current densities in inverse power of the ratio of slit-width to wavelength are readily obtained, and these are quite useful, even for only moderately large values of kh. It will be found that each of the functions $I_n^{\pm}$ gives rise to an asymptotic series, the leading term of which is of order $(kh)^{-\frac{1}{2}}$ relative to the leading term of $I_{n-1}^{\pm}$. The asymptotic expansion of the series solution to any finite order in $(kh)^{-1}$ is therefore obtained by the consideration of only a finite number of terms in the original series (equations (9)).

To exhibit more explicitly the dependence on kh, equations (10) may be written in the following normalized form:

$$I_n^{\pm}(hY;\alpha) = -\frac{1}{\pi}e^{-i\kappa}\frac{e^{-i\kappa Y}}{\sqrt{Y}}\int_0^\infty I_{n-1}^{\mp}(hZ;\alpha)\,\frac{e^{-i\kappa Z}}{Y+Z+1}(1+Z)^{\frac{1}{2}}\,dZ \tag{11}$$

for $n \geqslant 1$, where

$$y = hY, \quad z = hZ, \tag{12}$$

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and

$$I_0^+(hY;\alpha) = \frac{e^{\frac{1}{4}\pi i}}{\sqrt{\pi}}\cos\alpha\, e^{i\kappa Y\sin\alpha} f[\sqrt{(2\kappa Y)}\sin(\tfrac{1}{4}\pi+\tfrac{1}{2}\alpha)] + \frac{e^{-\frac{1}{4}\pi i}}{\sqrt{(2\pi)}}\frac{e^{-i\kappa Y}}{\sqrt{(\kappa Y)}}\sin(\tfrac{1}{4}\pi-\tfrac{1}{2}\alpha), \tag{13}$$

$$I_0^-(hY;\alpha) = e^{-i\kappa\sin\alpha} I_0^+(hY;-\alpha), \tag{14}$$

where

$$f(x) = \int_0^x e^{-i\lambda^2} d\lambda = \tfrac{1}{2}\sqrt{\pi}\, e^{-\frac{1}{4}\pi i} - F(x). \tag{15}$$

In these equations, and henceforth, κ is written in place of kh.

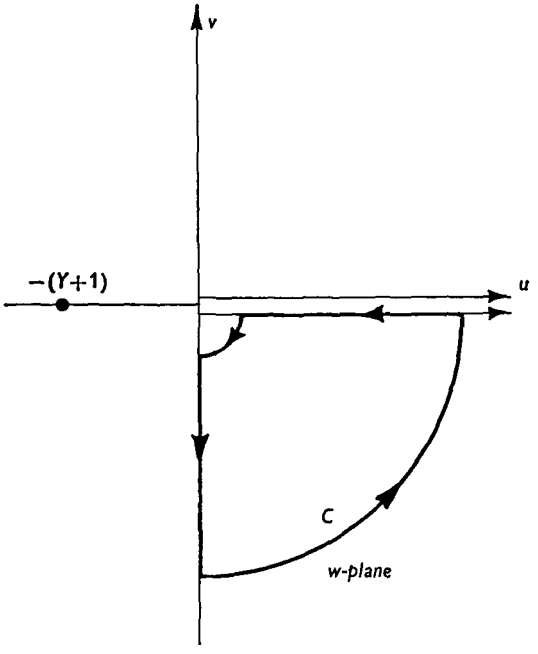


Fig. 2. Path of integration, C, in the complex plane of w which has a branch cut along the positive axis of u.

It is necessary to digress at this point to discuss the asymptotic expansions of two integrals which arise in the subsequent analysis. The following functions will therefore be defined:

$$K_1(Y) = \int_0^\infty \frac{e^{-i2\kappa Z}}{\sqrt{(\kappa Z)}}\frac{(1+Z)^{\frac{1}{2}}}{Y+Z+1}\,dZ, \tag{16}$$

$$K_2(Y;p,q) = \int_0^\infty f[(\kappa q Z)^{\frac{1}{2}}]\frac{e^{-i\kappa pZ}}{Y+Z+1}(1+Z)^{\frac{1}{2}}\,dZ, \tag{17}$$

where

$$p = 1+\sin\alpha, \quad q = 1-\sin\alpha. \tag{18}$$

These integrals may be evaluated for large κ by an application of Watson's lemma (see (9) or (24)), or a variation thereof in the case of K_2. Before the lemma can be applied, it is first necessary to deform the path of integration into one on which the exponent takes real negative values.

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Consider first $\int_C \frac{e^{-i2\kappa w}}{\sqrt{(\kappa w)}} \frac{(1+w)^{\frac{1}{2}}}{Y+w+1}\,dw$ taken around the contour C in the complex w-plane in which a branch cut is made along the positive real axis from $u = 0$ to $u = \infty$ (see Fig. 2). The branch of $\sqrt{w}$ is chosen which takes the value $i\,|w|^{\frac{1}{2}}$ when w is real and negative. The integrand is analytic in the fourth quadrant, and is exponentially small on the large circular arc of C given by $w = R\,e^{i\phi}$ ($\frac{3}{2}\pi \leqslant \phi \leqslant 2\pi$). Hence the integral over the large circular arc tends to zero as R approaches infinity. Likewise the integral over the small arc near $w = 0$ tends to zero with the radius of the circle, and

$$K_1(Y) = \frac{e^{-\frac{1}{4}\pi i}}{\kappa^{\frac{1}{2}}} \int_0^\infty \frac{e^{-2\kappa v}}{\sqrt{v}} \frac{(1-iv)^{\frac{1}{2}}}{Y+1-iv}\,dv.$$

Then, upon the assumption that

$$\frac{(1-iv)^{\frac{1}{2}}}{Y+1-iv} = \sum_{n=0}^{\infty} a_n(Y)\,v^n \tag{19}$$

for sufficiently small $|v|$, Watson's lemma states that

$$K_1(Y) \sim e^{-\frac{1}{4}\pi i} \sum_{n=0}^{\infty} \frac{a_n(Y)}{\kappa^{n+1}} \frac{\Gamma(n+\frac{1}{2})}{2^{n+\frac{1}{2}}}, \tag{20}$$

where

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt \quad (\Re x > 0). \tag{21}$$

Consideration of

$$\int_C f[-(\kappa q w)^{\frac{1}{2}}] \frac{e^{-i\kappa p w}}{Y+w+1} (1+w)^{\frac{1}{2}}\,dw$$

leads to

$$K_2(Y; p, q) = -i \int_0^\infty f[-(\kappa q v)^{\frac{1}{2}} e^{\frac{1}{4}\pi i}] \frac{e^{-\kappa p v}}{Y+1-iv} (1-iv)^{\frac{1}{2}}\,dv;$$

and since

$$f(-x\,e^{\frac{1}{4}\pi i}) = e^{-\frac{1}{4}\pi i} G(x) \quad (x \geqslant 0),$$

where

$$G(x) = \int_0^x e^{-\lambda^2}\,d\lambda,$$

it follows that

$$K_2(Y; p, q) = -e^{\frac{1}{4}\pi i} \int_0^\infty G[(\kappa q v)^{\frac{1}{2}}] \frac{e^{-\kappa p v}}{Y+1-iv} (1-iv)^{\frac{1}{2}}\,dv. \tag{22}$$

Since κp is not of necessity very much greater than unity, the form of the integrand in equation (22) is not such as to lead directly to the desired expansion of K_2. This may be effected with the aid of a subsidiary function $h(v, Y; \kappa p)$ defined by

$$h(v, Y; \kappa p) = \int_v^\infty \frac{e^{-\kappa p x}}{Y+1-ix} (1-ix)^{\frac{1}{2}}\,dx. \tag{23}$$

The change of variable $x = v + t$ removes the exponential dependence on v from under the sign of integration, and permits equation (23) to be written as

$$h(v, Y; \kappa p) = e^{-\kappa p v} H(v, Y; \kappa p),$$

where

$$H(v, Y; \kappa p) = \int_0^\infty \frac{e^{-\kappa p t}}{Y+1-i(v+t)} [1-i(v+t)]^{\frac{1}{2}}\,dt.$$

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Then an integration by parts in (22) leads to

$$K_2(Y;p,q) = -\tfrac{1}{2}e^{\frac{1}{4}\pi i}(\kappa q)^{\frac{1}{2}}\int_0^\infty \frac{e^{-2\kappa v}}{\sqrt{v}}H(v,Y;\kappa p)\,dv.$$

The function H is analytic near $v=0$, so that

$$H(v,Y;\kappa p) = \sum_{n=0}^{\infty}\frac{v^n}{n!}H^{(n)}(0,Y;\kappa p)$$

for sufficiently small values of $|v|$, where

$$H^{(n)}(0,Y;\kappa p) \equiv \left[\frac{\partial^n}{\partial v^n}H(v,Y;\kappa p)\right]_{v=0}.$$

If
$$\frac{(1-ix)^{\frac{1}{2}}}{Y+1-ix} = g(x,Y),$$

then
$$H(v,Y;\kappa p) = \int_0^\infty e^{-\kappa p t}g(v+t,Y)\,dt$$

and
$$\frac{\partial H}{\partial v} = \int_0^\infty e^{-\kappa p t}\frac{\partial g}{\partial t}\,dt = -g(v,Y)+\kappa p H(v,Y;\kappa p).$$

By repetition of this process, it may be seen that

$$\frac{\partial^n}{\partial v^n}H(v,Y;\kappa p) = (\kappa p)^n H(v,Y;\kappa p) - \sum_{\nu=0}^{n-1}\frac{\partial^\nu}{\partial v^\nu}g(v,Y)\,(\kappa p)^{n-\nu-1}$$

and
$$H^{(n)}(0,Y;\kappa p) = (\kappa p)^n\left[H(0,Y;\kappa p) - \sum_{\nu=0}^{n-1}\frac{a_\nu(Y)\,\nu!}{(\kappa p)^{\nu+1}}\right]$$

for $n \geqslant 1$, and
$$H^{(0)}(0,Y;\kappa p) = H(0,Y;\kappa p).$$

Then

$$K_2(Y;p,q) \sim -\tfrac{1}{2}\sqrt{\pi}\,e^{\frac{1}{4}\pi i}H(0,Y;\kappa p) + \frac{e^{\frac{1}{4}\pi i}q^{\frac{1}{2}}}{2\sqrt{2}}\sum_{n=1}^{\infty}\left(\frac{p}{2}\right)^n\frac{\Gamma(n+\frac{1}{2})}{n!}\sum_{\nu=0}^{n-1}\frac{a_\nu(Y)\,\nu!}{(\kappa p)^{\nu+1}}. \quad (24)$$

If the order of summation is changed, and use is made of the fact that

$$\sum_{\nu=n+1}^{\infty}\left(\frac{p}{2}\right)^\nu\frac{\Gamma(\nu+\frac{1}{2})}{\nu!} = \frac{\sqrt{(2\pi)}}{q^{\frac{1}{2}}} - \sum_{\nu=0}^{n}\left(\frac{p}{2}\right)^\nu\frac{\Gamma(\nu+\frac{1}{2})}{\nu!},$$

then
$$K_2(Y;p,q) \sim -\frac{\sqrt{\pi}}{2}e^{\frac{1}{4}\pi i}H(0,Y;\kappa p) + \frac{\sqrt{\pi}}{2}e^{\frac{1}{4}\pi i}\sum_{n=0}^{\infty}a_n(Y)\rho_n(p,q;\kappa), \quad (25)$$

where
$$\rho_n(p,q;\kappa) = \frac{n!}{(\kappa p)^{n+1}}\left[1 - \frac{q^{\frac{1}{2}}}{\sqrt{(2\pi)}}\sum_{\nu=0}^{n}\left(\frac{p}{2}\right)^\nu\frac{\Gamma(\nu+\frac{1}{2})}{\nu!}\right]. \quad (26)$$

Although it might appear that $\rho_n(p,q;\kappa)$ is unbounded as $p\to 0$, it may be verified that this is not so, and that $\rho_n(p,q;\kappa)$ is continuous for $-\frac{1}{2}\pi \leqslant \alpha \leqslant \frac{1}{2}\pi$.

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If now the value of $I_0^-(hY;\alpha)$ given by equation (14) is inserted into equation (11) for $n = 1$, it is found that

$$I_1^+(hY;\alpha) = -\frac{e^{-i\kappa p}}{2\pi^{\frac{3}{2}}}\frac{e^{-i\kappa Y}}{\sqrt{Y}}\left[e^{-\frac{1}{4}\pi i}p^{\frac{1}{2}}K_1(Y) + 2e^{\frac{1}{4}\pi i}\cos\alpha K_2(Y;p,q)\right]$$

and

$$I_1^+(hY;\alpha) \sim \frac{i\,e^{-i\kappa p}}{2\pi^{\frac{3}{2}}}\cos\alpha\frac{e^{-i\kappa Y}}{\sqrt{Y}}\left\{\sqrt{\pi}\,H(0,Y;\kappa) + \sum_{n=0}^{\infty}a_n(Y)\left[\frac{\sqrt{2}}{q^{\frac{1}{2}}}\frac{\Gamma(n+\frac{1}{2})}{(2\kappa)^{n+1}} - \sqrt{\pi}\,\rho_n(p,q;\kappa)\right]\right\}. \tag{27}$$

If

$$\chi_n(\alpha;\kappa) = \frac{\sqrt{2}}{q^{\frac{1}{2}}}\frac{\Gamma(n+\frac{1}{2})}{(2\kappa)^{n+1}} - \sqrt{\pi}\,\rho_n(p,q;\kappa), \tag{28}$$

it follows that

$$I_1^+(hY;\alpha) \sim \frac{i\,e^{-i\kappa p}}{2\pi^{\frac{3}{2}}}\cos\alpha\frac{e^{-i\kappa Y}}{\sqrt{Y}}\left[\sqrt{\pi}\,H(0,Y;\kappa) + \sum_{n=0}^{\infty}a_n(Y)\,\chi_n(\alpha;\kappa)\right] \tag{29}$$

and

$$I_1^-(hY;\alpha) = e^{-i\kappa\sin\alpha}\,I_1^+(hY;-\alpha). \tag{30}$$

The expressions (29) and (30) may next be inserted into equations (11) to yield, after a rotation of the path of integration in the complex plane, the asymptotic developments of $I_{\frac{1}{2}}^{\pm}(hY;\alpha)$. Repetition of this process leads to terms of any desired order. If I_n^* designates I_n^+ when n is even and I_n^- when n is odd, then, when $n \geqslant 2$,

$$I_n^*(hY;\alpha) \sim \frac{e^{\frac{1}{4}(3n-1)\pi i}\cos\alpha}{\pi^{n+\frac{1}{2}}\,2^{\frac{1}{2}(n+1)}}\frac{e^{-in\kappa}}{\kappa^{\frac{1}{2}(n-1)}}\frac{e^{-i\kappa Y}}{\sqrt{Y}}\sum_{\nu_1+\nu_2+\ldots+\nu_n=0}{}'\frac{\Gamma(\nu_2+\frac{1}{2})\,\Gamma(\nu_3+\frac{1}{2})\ldots\Gamma(\nu_n+\frac{1}{2})}{(2\kappa)^{\nu_2+\nu_3+\ldots+\nu_n}} \times\left[a_{\nu_1\nu_2\ldots\nu_n}(Y)\,\chi_{\nu_1}(-\alpha;\kappa) + b_{\nu_2\nu_3\ldots\nu_n}(Y;\kappa q)\right], \tag{31}$$

and

$$I_n^-(hY;\alpha) = e^{-i\kappa\sin\alpha}\,I_n^+(hY;-\alpha) \tag{32}$$

for all $n \geqslant 0$.

In the above expression, the primed Σ implies that the terms independent of ν_1 are to be included once only. The coefficients in the expansions may be determined from equation (19) and

$$\frac{(1-iv)^{\frac{1}{2}}}{Y+1-iv}\,a_{\nu_1\nu_2\ldots\nu_{n-1}}(-iv) = \sum_{\nu_n=0}^{\infty}a_{\nu_1\nu_2\ldots\nu_n}(Y)\,v^{\nu_n}, \tag{33}$$

$$\sqrt{\pi}\,H(0,-iv;\kappa q)\frac{(1-iv)^{\frac{1}{2}}}{Y+1-iv} = \sum_{\nu_2=0}^{\infty}b_{\nu_2}(Y;\kappa q)\,v^{\nu_2}, \tag{34}$$

$$\frac{(1-iv)^{\frac{1}{2}}}{Y+1-iv}\,b_{\nu_2\nu_3\ldots\nu_{n-1}}(-iv;\kappa q) = \sum_{\nu_n=0}^{\infty}b_{\nu_2\nu_3\ldots\nu_n}(Y;\kappa q)\,v^{\nu_n}. \tag{35}$$

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The first few of the a's are readily obtained:

$$\left.\begin{aligned} a_0(Y) &= (Y+1)^{-1}, \quad a_1(Y) = i(Y+1)^{-2} - \tfrac{1}{2}i(Y+1)^{-1}, \\ &\qquad a_2(Y) = -(Y+1)^{-3} + \tfrac{1}{2}(Y+1)^{-2} + \tfrac{1}{8}(Y+1)^{-1}, \\ a_{00}(Y) &= (Y+1)^{-1}, \quad a_{01}(Y) = i(Y+1)^{-2} + \tfrac{1}{2}i(Y+1)^{-1}, \\ &\qquad a_{10}(Y) = \tfrac{1}{2}i(Y+1)^{-1}, \\ a_{000}(Y) &= (Y+1)^{-1}, \quad a_{001}(Y) = i(Y+1)^{-2} + \tfrac{1}{2}i(Y+1)^{-1}, \\ &\qquad a_{010}(Y) = i(Y+1)^{-3} + \tfrac{1}{2}i(Y+1)^{-2}, \quad a_{100}(Y) = \tfrac{1}{2}i(Y+1)^{-2}, \\ a_{0000}(Y) &= (Y+1)^{-1}. \end{aligned}\right\} \tag{36}$$

Certain of the b's will be found later, as required. For the present it is sufficient to notice that they are linear combinations of the a's, the coefficients being functions of κq.

With the knowledge of the asymptotic form of the induced current density on both halves of the screen, the problem is in effect solved, for the total field may be found according to equation (2).

It is perhaps worth while to remark that the asymptotic series obtained above depend explicitly on the normalized coordinate Y rather than the coordinate y. Of course, the dependence on y could be reintroduced, but at present this would serve no useful purpose, and would, in fact, tend to complicate the dependence on h.

Before proceeding with the calculation of the aperture field and the far-field distribution, it is of interest to examine the behaviour of the current density as the angle of incidence approaches $\frac{1}{2}\pi$ (grazing incidence). Although it is not immediately evident from the form of the series, the current densities must vanish (and the total field become null) for $\alpha = \frac{1}{2}\pi$. (That this is the case may be seen most readily by an application of Babinet's principle to the complementary problem which concerns an H-polarized wave at grazing incidence on a strip.)

In the present treatment, it is apparent that the only term in the series for the current densities which tends to zero as α approaches $\frac{1}{2}\pi$ is $I_0^+(hY;\alpha)$. However, with little difficulty it may be shown that $I_0^-(hY;\frac{1}{2}\pi)$ is cancelled by $I_1^-(hY;\frac{1}{2}\pi)$, and that the term of lowest order in $I_1^+(hY;\frac{1}{2}\pi)$ is balanced by a corresponding term in $I_2^+(hY;\frac{1}{2}\pi)$. Although it is difficult to consider terms of higher order in κ^{-1}, it may be inferred that if $\alpha = \frac{1}{2}\pi$, cancellation will occur for terms of all orders.

2·4. *The field in the slit.* It is a well-known consequence of the symmetry of Maxwell's equations that the field scattered by current flowing in a thin plane screen has vanishing tangential magnetic and normal electric components in any aperture cut in the screen. Consequently, only the tangential electric and normal magnetic components are unknown in the aperture, and the latter may be deduced from the former by Maxwell's equations.

In the present section, the tangential electric field distribution in the slit will be determined. It is convenient to rewrite equation (2) for the field at a general point (x_0, y_0) in a somewhat different form

$$E_z(x_0, y_0) = \sum_{n=0}^{\infty} E_n(x_0, y_0), \tag{37}$$

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where $$E_n(x_0, y_0) = -\kappa \int_0^\infty I_n^+(hY;\alpha)\, H_0^{(2)}(\kappa r')\, dY - \kappa \int_0^\infty I_n^-(hY;\alpha)\, H_0^{(2)}(\kappa r'')\, dY \tag{38}$$

for $n \geqslant 1$ and

$$E_0(x_0, y_0) = e^{ik(x_0\cos\alpha + y_0\sin\alpha)} - k\int_0^\infty I_0^+(y;\alpha)\, H_0^{(2)}(\kappa r')\, dy - k\int_0^\infty I_0^-(y;\alpha)\, H_0^{(2)}(\kappa r'')\, dy. \tag{39}$$

In the above expressions,

$$(hr')^2 = x_0^2 + (y-y_0)^2, \quad (hr'')^2 = x_0^2 + (h+y+y_0)^2, \tag{40}$$

where $y \geqslant 0$ (see Fig. 1). The integrals in (39) represent the field scattered by the two half-planes when interaction between them is neglected. It is consequently unnecessary to perform explicitly these integrations, for $E_0(x_0, y_0)$ may be written down immediately

$$\begin{aligned} E_0(x_0, y_0) &= -e^{ik(x_0\cos\alpha + y_0\sin\alpha)} \\ &+ e^{\frac{1}{4}\pi i}\{e^{i\kappa r_0\cos(\beta-\alpha)}\, F[(2\kappa r_0)^{\frac{1}{2}}\cos\tfrac{1}{2}(\beta-\alpha)] - e^{-i\kappa r_0\cos(\beta+\alpha)}\, F[(2\kappa r_0)^{\frac{1}{2}}\sin\tfrac{1}{2}(\beta+\alpha)] \\ &+ e^{-i\kappa\sin\alpha}(e^{i\kappa r_1\cos(\gamma+\alpha)}\, F[(2\kappa r_1)^{\frac{1}{2}}\cos\tfrac{1}{2}(\gamma+\alpha)] - e^{-i\kappa r_1\cos(\gamma-\alpha)}\, F[(2\kappa r_1)^{\frac{1}{2}}\sin\tfrac{1}{2}(\gamma-\alpha)])\}, \end{aligned} \tag{41}$$

where $\frac{1}{2}\pi \leqslant \beta \leqslant \frac{5}{2}\pi$, $\frac{1}{2}\pi \leqslant \gamma \leqslant \frac{5}{2}\pi$ and r_0, r_1 are the normalized distances from (x_0, y_0) to the points $(0, 0)$ and $(0, -h)$, respectively.

The tangential electric field in the aperture now is given by equations (38) and (41) when specialized to $x_0 = 0$, $-h \leqslant y_0 \leqslant 0$.

If $y_0 = -hY_0$, where $0 \leqslant Y_0 \leqslant 1$, then the terms in the series (37) for $E_z(0, -hY_0)$ are

$$\left.\begin{aligned} E_0(0, -hY_0) &= e^{-i\kappa Y_0\sin\alpha} - \frac{2e^{\frac{1}{4}\pi i}}{\sqrt{\pi}}[A_0(Y_0, -\alpha) + e^{-i\kappa\sin\alpha}\, A_0(1-Y_0, \alpha)], \\ E_n(0, -hY_0) &= -\frac{e^{\frac{1}{4}(3n-1)\pi i}}{\pi^{n+\frac{1}{2}}2^{\frac{1}{2}(n+1)}}\cos\alpha\,\frac{e^{-in\kappa}}{\kappa^{\frac{1}{2}(n-3)}} \\ &\quad \times \begin{cases} A_n(Y_0, -\alpha) + e^{-i\kappa\sin\alpha}\, A_n(1-Y_0, \alpha) & (n = 2, 4, 6, \ldots), \\ e^{-i\kappa\sin\alpha}\, A_n(Y_0, \alpha) + A_n(1-Y_0, -\alpha) & (n = 1, 3, 5, \ldots), \end{cases} \end{aligned}\right\} \tag{42}$$

where

$$\left.\begin{aligned} A_0(Y_0, -\alpha) &= e^{-i\kappa Y_0\sin\alpha}\, F[(\kappa q Y_0)^{\frac{1}{2}}], \\ A_n(Y_0, \alpha) &= H_n(Y_0, \alpha) + C_n(Y_0, \alpha) \quad (n \geqslant 1), \end{aligned}\right\} \tag{43}$$

with

$$\left.\begin{aligned} H_1(Y_0, \alpha) &\sim \sqrt{\pi}\int_0^\infty \frac{e^{-i\kappa Y}}{\sqrt{Y}}\, H(0, Y; \kappa p)\, H_0^{(2)}[\kappa(Y+Y_0)]\, dY, \\ C_1(Y_0, \alpha) &\sim \sum_{\nu_1=0}^{\infty} \chi_{\nu_1}(\alpha;\kappa)\int_0^\infty \frac{e^{-i\kappa Y}}{\sqrt{Y}}\, a_{\nu_1}(Y)\, H_0^{(2)}[\kappa(Y+Y_0)]\, dY \end{aligned}\right\} \tag{44}$$

and

$$\begin{aligned} H_n(Y_0, \alpha) &\sim \sum_{\nu_2+\ldots+\nu_n=0}^{\infty} \frac{\Gamma(\nu_2+\frac{1}{2})\ldots\Gamma(\nu_n+\frac{1}{2})}{(2\kappa)^{\nu_2+\ldots+\nu_n}}\int_0^\infty \frac{e^{-i\kappa Y}}{\sqrt{Y}}\, b_{\nu_2\ldots\nu_n}(Y;\kappa p)\, H_0^{(2)}[\kappa(Y+Y_0)]\, dY, \\ C_n(Y_0, \alpha) &\sim \sum_{\nu_1+\ldots+\nu_n=0}^{\infty} \frac{\Gamma(\nu_2+\frac{1}{2})\ldots\Gamma(\nu_n+\frac{1}{2})}{(2\kappa)^{\nu_2+\ldots+\nu_n}}\,\chi_{\nu_1}(\alpha;\kappa)\int_0^\infty \frac{e^{-i\kappa Y}}{\sqrt{Y}}\, a_{\nu_1\ldots\nu_n}(Y)\, H_0^{(2)}[\kappa(Y+Y_0)]\, dY \end{aligned} \tag{45}$$

for $n \geqslant 2$.

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It is evidently necessary to consider the asymptotic development of $H_1(Y_0, \alpha)$ and integrals of the form

$$P_n(Y_0) = \int_0^\infty \frac{e^{-i\kappa Y}}{Y^{\frac{1}{2}}(Y+1)^n} H_0^{(2)}[\kappa(Y+Y_0)]\,dY. \tag{46}$$

Since the arguments of the Hankel functions which occur in the above integrals are not of necessity very much greater than unity, and do, in fact, vanish at the lower limit of integration if $Y_0 = 0$, it is not sufficient merely to replace these functions by their asymptotic representations, although such a step is justifiable when the point of observation is in the far-field region. As an aid in performing the integrations, the following function is introduced

$$A(z) = \frac{2i}{\pi}\int_0^\infty e^{-iz\cosh t}\,dt = \frac{2i}{\pi}e^{-iz}\int_0^\infty \frac{e^{-izu}}{[u(u+2)]^{\frac{1}{2}}}\,du, \tag{47}$$

and $A(z) = H_0^{(2)}(z)$ for z real and positive. But $A(z)$ is also defined and analytic when $z = re^{i\theta}$, $\pi \leqslant \theta \leqslant 2\pi$, $r > 0$. Then

$$P_n(Y_0) = \int_0^\infty \frac{e^{-i\kappa Y}}{Y^{\frac{1}{2}}(Y+1)^n} A[\kappa(Y+Y_0)]\,dY. \tag{48}$$

Since $A[\kappa(Y+Y_0)]$ is an analytic function of Y which tends to zero as $|Y|$ tends to infinity in the lower half of the complex plane of Y, the integral in equation (48) vanishes when taken around the contour C of Fig. 2, and $P_n(Y_0)$ takes the following form:

$$P_n(Y_0) = \frac{2e^{\frac{1}{4}\pi i}}{\pi}e^{-i\kappa Y_0}\int_0^\infty \frac{e^{-2\kappa v}}{v^{\frac{1}{2}}(1-iv)^n}\,dv\int_0^\infty \frac{e^{-i\kappa(Y_0-iv)u}}{[u(u+2)]^{\frac{1}{2}}}\,du.$$

Because of the absolute convergence of this repeated integral, the order of integration may be inverted to give

$$P_n(Y_0) = \frac{2e^{\frac{1}{4}\pi i}}{\pi}e^{-i\kappa Y_0}\int_0^\infty \frac{e^{-i\kappa Y_0 u}}{[u(u+2)]^{\frac{1}{2}}}\,du\int_0^\infty \frac{e^{-\kappa(u+2)v}}{v^{\frac{1}{2}}(1-iv)^n}\,dv.$$

Now

$$\int_0^\infty \frac{e^{-\kappa(u+2)v}}{v^{\frac{1}{2}}(1-iv)^n}\,dv \sim \sum_{m=0}^\infty \frac{r_{nm}\Gamma(m+\frac{1}{2})}{[\kappa(u+2)]^{m+\frac{1}{2}}},$$

so that

$$P_n(Y_0) \sim \frac{2}{\pi\kappa^{\frac{1}{2}}}\sum_{m=0}^\infty \frac{r_{nm}\Gamma(m+\frac{1}{2})}{\kappa^m}F_{m+1}(\kappa Y_0), \tag{49}$$

where

$$F_{m+1}(u) = e^{\frac{1}{4}\pi i}e^{-iu}\int_0^\infty \frac{e^{-iuv}}{v^{\frac{1}{2}}(v+2)^{m+1}}\,dv. \tag{50}$$

A recurrence relationship between F_m and F_{m+1} is readily obtained, for

$$-\frac{d}{du}[e^{-iu}F_{m+1}(u)] = i\,e^{-iu}F_m(u).$$

Then, since $F_m(u) \to 0$ as $u \to \infty$ for all $m \geqslant 0$, it follows that

$$F_{m+1}(u) = i\,e^{iu}\int_u^\infty e^{-iv}F_m(v)\,dv \tag{51}$$

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and
$$\left.\begin{aligned} F_0(u) &= \sqrt{\pi}\,\frac{e^{-iu}}{\sqrt{u}}, \\ F_1(u) &= i\sqrt{(2\pi)}\,e^{iu}\,F[\sqrt{(2u)}], \\ F_2(u) &= \sqrt{(2\pi)}\,e^{iu}\left\{uF[\sqrt{(2u)}] + \frac{i\sqrt{u}}{2\sqrt{2}}\,e^{-i2u} + \frac{i}{4}F[\sqrt{(2u)}]\right\}. \end{aligned}\right\} \tag{52}$$

$H_1(Y_0, \alpha)$ may now be evaluated in an analogous manner.

$$H_1(Y_0, \alpha) \sim \frac{2}{\sqrt{\pi}\,\kappa^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{h_n(\kappa p)\,\Gamma(n+\frac{1}{2})}{\kappa^n} F_{n+1}(\kappa Y_0), \tag{53}$$

where
$$\int_0^\infty \frac{e^{-\kappa(u+2)v}}{\sqrt{v}} H(0, -iv; \kappa p)\,dv \sim \sum_{n=0}^{\infty} \frac{h_n(\kappa p)\,\Gamma(n+\frac{1}{2})}{[\kappa(u+2)]^{n+\frac{1}{2}}} \tag{54}$$

and
$$H(0, -iv; \kappa p) = \sum_{n=0}^{\infty} h_n(\kappa p)\,v^n \tag{55}$$

for $|v|$ sufficiently small. The first few coefficients r_{nm} and h_n are

$$r_{10} = r_{20} = r_{30} = 1, \quad r_{11} = i, \tag{56}$$

$$\left.\begin{aligned} h_0(\kappa p) &= \frac{2i\,e^{i\kappa p}}{(\kappa p)^{\frac{1}{2}}} F[(\kappa p)^{\frac{1}{2}}], \\ h_1(\kappa p) &= -2 + 4i(\kappa p)^{\frac{1}{2}}\,e^{i\kappa p} F[(\kappa p)^{\frac{1}{2}}], \\ h_2(\kappa p) &= -\tfrac{1}{3}i - \tfrac{2}{3}\kappa p + \tfrac{4}{3}i(\kappa p)^{\frac{3}{2}}\,e^{i\kappa p}\,F[(\kappa p)^{\frac{1}{2}}]. \end{aligned}\right\} \tag{57}$$

From equations (19), (34) and (55), it is seen that

$$b_\nu(Y; \kappa p) = \sqrt{\pi} \sum_{n+m=\nu} h_n(\kappa p)\,a_m(Y). \tag{58}$$

The first few functions $A_n(Y_0, \alpha)$ are then easily found. If $\psi_n(\alpha)$ is defined by

$$\psi_n(\alpha) = \kappa^{n+1}\chi_n(\alpha; \kappa), \tag{59}$$

then

$$\left.\begin{aligned} A_1(Y_0, \alpha) &\sim \frac{2h_0(\kappa p)}{\kappa^{\frac{1}{2}}} F_1(\kappa Y_0) + \frac{1}{\kappa^{\frac{3}{2}}}\left[\frac{2}{\sqrt{\pi}}\psi_0(\alpha)\,F_1(\kappa Y_0) + h_1(\kappa p)\,F_2(\kappa Y_0)\right] + O(\kappa^{-\frac{5}{2}}), \\ A_2(Y_0, \alpha) &\sim \frac{2\sqrt{\pi}\,h_0(\kappa p)}{\kappa^{\frac{1}{2}}} F_1(\kappa Y_0) + \frac{\sqrt{\pi}}{\kappa^{\frac{3}{2}}}\left\{ih_0(\kappa p)\,F_2(\kappa Y_0)\right. \\ &\qquad \left. + \left[\frac{2}{\sqrt{\pi}}\psi_0(\alpha) + \tfrac{1}{2}h_1(\kappa p) + \tfrac{1}{4}ih_0(\kappa p)\right] F_1(\kappa Y_0)\right\} + O(\kappa^{-\frac{5}{2}}), \\ A_3(Y_0, \alpha) &\sim \frac{2\pi h_0(\kappa p)}{\kappa^{\frac{1}{2}}} F_1(\kappa Y_0) + O(\kappa^{-\frac{3}{2}}), \\ A_4(Y_0, \alpha) &\sim \frac{2\pi^{\frac{3}{2}} h_0(\kappa p)}{\kappa^{\frac{1}{2}}} F_1(\kappa Y_0) + O(\kappa^{-\frac{3}{2}}). \end{aligned}\right\} \tag{60}$$

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The expressions (60), in conjunction with equations (42) and (43), then determine the electric field in the slit up to and including terms of order κ^{-1} for all angles of incidence, for it is readily verified that $E_5(0, -hY_0) = O(\kappa^{-\frac{3}{2}})$. A partial check on the above expressions may be obtained by allowing α to tend to $\frac{1}{2}\pi$. Then

$$E_0(0, -hY_0) = -\frac{2e^{\frac{1}{4}\pi i}}{\sqrt{\pi}} e^{-i\kappa Y_0} F\{[2\kappa(1-Y_0)]^{\frac{1}{2}}\},$$

$$E_1(0, -hY_0) \sim \frac{2e^{\frac{1}{4}\pi i}}{\sqrt{\pi}} e^{-i\kappa Y_0} F\{[2\kappa(1-Y_0)]^{\frac{1}{2}}\} - \frac{i e^{-i2\kappa}}{\pi^{\frac{3}{2}}\kappa^{\frac{1}{2}}} F_1(\kappa Y_0) + O(\kappa^{-\frac{3}{2}}),$$

$$E_2(0, -hY_0) \sim \frac{i e^{-i2\kappa}}{\pi^{\frac{3}{2}}\kappa^{\frac{1}{2}}} F_1(\kappa Y_0) + \frac{e^{\frac{1}{4}\pi i}}{\sqrt{2\pi^2}} \frac{e^{-i3\kappa}}{\kappa} F_1[\kappa(1-Y_0)] + O(\kappa^{-\frac{3}{2}}),$$

$$E_3(0, -hY_0) \sim -\frac{e^{\frac{1}{4}\pi i}}{\sqrt{2\pi^2}} \frac{e^{-i3\kappa}}{\kappa} F_1[\kappa(1-Y_0)] + O(\kappa^{-\frac{3}{2}}),$$

$$E_4(0, -hY_0) \sim O(\kappa^{-\frac{3}{2}}).$$

Thus it has been verified, to terms of order κ^{-1} inclusive, that the total field in the slit does indeed vanish for $\alpha = \frac{1}{2}\pi$.

2·5. *The far-field behind the slit.* The distant field behind the slit is of considerable interest since, in particular, the far-field in the direction of propagation of the incident wave determines the plane wave transmission coefficient (3).

When the angle β differs sufficiently from $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$, the far-field may be determined in the most simple manner with the aid of equations (37), (38) and (41) and the known expansions for the current density on the screen. Subject to these conditions, it may be assumed that $\kappa r' \gg 1$, $\kappa r'' \gg 1$. The Hankel functions may be replaced by the first term of their asymptotic representations, and the integrals evaluated asymptotically in inverse powers of κ and $\kappa(1 \pm \sin\beta)$. On the other hand, when β is near $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$, the above approximation to the Hankel functions is no longer justified, and the direct determination of the field through the current density on the screen is more difficult.

It will become evident that the determination of the transmission coefficient for all angles of incidence requires knowledge of the far-field for $\pi \leqslant \beta \leqslant \frac{3}{2}\pi$, where $\beta + \gamma = 2\pi$. Consequently, a method of calculation will be employed which differs from that outlined above, and which determines the far-field everywhere behind the screen. Nevertheless, certain results which follow more readily from the alternative method of calculation (and which have been obtained elsewhere, (14)) will be quoted later.

It is well known that the field is determined not only by the induced current density on the screen, but also by the distribution of tangential electric field in the aperture. If $E_z(0, -hY)$, where $0 \leqslant Y \leqslant 1$, represents the aperture field, then the total field $E_z(x_0, y_0)$ at a point where $x_0 \leqslant 0$ may be determined from the Rayleigh formula in its two-dimensional form

$$E_z(x_0, y_0) = -\frac{ih}{2} \frac{\partial}{\partial x_0} \int_0^1 E_z(0, -hY) H_0^{(2)}(\kappa r)\, dY, \tag{61}$$

where

$$r^2 = r_0^2 + Y^2 + 2r_0 Y \sin\beta. \tag{62}$$

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Now $\kappa r \gg 1$ for any points in the far-field region (since integration is only over the slit) including such as lie near or on the screen. Consequently, the Hankel function may be replaced by the first term of its asymptotic representation, or rather by the following approximation to this first term

$$H_0^{(2)}(\kappa r) \sim \frac{\sqrt{2}\, e^{\frac{1}{4}\pi i}}{\sqrt{\pi}} \frac{e^{-i\kappa r_0}}{(\kappa r_0)^{\frac{1}{2}}} e^{-i\kappa Y \sin\beta} + O(r_0^{-\frac{3}{2}}). \tag{63}$$

If the series representation (37) for $E_z(x_0, y_0)$ is again employed, it is readily seen that

$$E_n(x_0, y_0) \sim -\frac{e^{\frac{1}{4}\pi i}}{\sqrt{(2\pi)}} \kappa \cos\beta \frac{e^{-i\kappa r_0}}{(\kappa r_0)^{\frac{1}{2}}} \int_0^1 E_n(0, -hY)\, e^{-i\kappa Y \sin\beta}\, dY + O(r_0^{-\frac{3}{2}})$$

for $n \geqslant 0$. Hence from (42) it follows that

$$\left.\begin{aligned} E_0(x_0, y_0) \sim{}& \frac{e^{-\frac{1}{4}\pi i}}{\sqrt{(2\pi)}} \kappa \cos\beta \frac{e^{-i\kappa r_0}}{(\kappa r_0)^{\frac{1}{2}}} \left[\frac{e^{-i\kappa(\sin\alpha + \sin\beta)} - 1}{\kappa(\sin\alpha + \sin\beta)}\right] \\ &+ \frac{\sqrt{2}\, i}{\pi} \kappa \cos\beta \frac{e^{-i\kappa r_0}}{(\kappa r_0)^{\frac{1}{2}}} [\phi_0(-\alpha, \beta) + e^{-i\kappa(\sin\alpha + \sin\beta)} \phi_0(\alpha, -\beta)] \end{aligned}\right\}$$

and, for $n \geqslant 1$,

$$\left.\begin{aligned} E_n(x_0, y_0) \sim{}& \frac{e^{\frac{3}{4}n\pi i}}{\pi^{n+1} 2^{\frac{1}{2}n+1}} \cos\alpha \cos\beta \frac{e^{-in\kappa}}{\kappa^{\frac{1}{2}(n-5)}} \frac{e^{-i\kappa r_0}}{(\kappa r_0)^{\frac{1}{2}}} \\ &\times \begin{cases} e^{-i\kappa \sin\alpha} \phi_n(\alpha, \beta) + e^{-i\kappa \sin\beta} \phi_n(-\alpha, -\beta) & (n = 1, 3, 5, \ldots), \\ \phi_n(-\alpha, \beta) + e^{-i\kappa(\sin\alpha + \sin\beta)} \phi_n(\alpha, -\beta) & (n = 2, 4, 6, \ldots), \end{cases} \end{aligned}\right\} \tag{64}$$

where

$$\phi_n(\alpha, \beta) = \int_0^1 A_n(Y, \alpha)\, e^{-i\kappa Y \sin\beta}\, dY \tag{65}$$

for $n \geqslant 0$.

When the expressions (43) and (60) for A_0 to A_4 are inserted into equation (65) it is found that

$$\left.\begin{aligned} \phi_0(-\alpha, \beta) &= \int_0^1 e^{-i\kappa(\sin\alpha + \sin\beta) Y} F[(\kappa q Y)^{\frac{1}{2}}]\, dY, \\ \phi_1(\alpha, \beta) &= \frac{2h_0(\kappa p)}{\kappa^{\frac{1}{2}}} I_1(\beta) + \frac{1}{\kappa^{\frac{3}{2}}} \left[\frac{2}{\sqrt{\pi}} \psi_0(\alpha) I_1(\beta) + h_1(\kappa p) I_2(\beta)\right] + O(\kappa^{-3}), \\ \phi_2(\alpha, \beta) &= \frac{2\sqrt{\pi}\, h_0(\kappa p)}{\kappa^{\frac{1}{2}}} I_1(\beta) + \frac{\sqrt{\pi}}{\kappa^{\frac{3}{2}}} \Big\{ i h_0(\kappa p) I_2(\beta) \\ &\qquad + \left[\frac{2}{\sqrt{\pi}} \psi_0(\alpha) + \tfrac{1}{2} h_1(\kappa p) + \tfrac{1}{4} i h_0(\kappa p)\right] I_1(\beta) \Big\} + O(\kappa^{-3}), \\ \phi_3(\alpha, \beta) &= 2\pi h_0(\kappa p)\, \kappa^{-\frac{1}{2}} I_1(\beta) + O(\kappa^{-2}), \\ \phi_4(\alpha, \beta) &= 2\pi^{\frac{3}{2}} h_0(\kappa p)\, \kappa^{-\frac{1}{2}} I_1(\beta) + O(\kappa^{-2}), \end{aligned}\right\} \tag{66}$$

where

$$I_n(\beta) = \int_0^1 e^{-i\kappa Y \sin\beta} F_n(\kappa Y)\, dY \quad (n \geqslant 1). \tag{67}$$

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The evaluation of the integrals in equations (66) and (67) is straightforward, and details will be omitted. It may be shown that

$$\phi_0(-\alpha,\beta) = \frac{i}{\kappa(\sin\alpha+\sin\beta)}\left\{(\kappa q)^{\frac{1}{2}}e^{-i\kappa n}\left(\frac{e^{i\kappa q}F[(\kappa q)^{\frac{1}{2}}]}{(\kappa q)^{\frac{1}{2}}}-\frac{e^{i\kappa n}F[(\kappa n)^{\frac{1}{2}}]}{(\kappa n)^{\frac{1}{2}}}\right)\right.$$
$$\left.+\frac{\sqrt{\pi}}{2}e^{-\frac{1}{4}\pi i}\left(\frac{q^{\frac{1}{2}}}{n^{\frac{1}{2}}}-1\right)\right\}, \tag{68}$$

$$I_1(\beta) = \frac{\sqrt{(2\pi)}}{\kappa m}\left\{(2\kappa)^{\frac{1}{2}}e^{-i\kappa n}\left(\frac{e^{i2\kappa}F[(2\kappa)^{\frac{1}{2}}]}{(2\kappa)^{\frac{1}{2}}}-\frac{e^{i\kappa n}F[(\kappa n)^{\frac{1}{2}}]}{(\kappa n)^{\frac{1}{2}}}\right)+\frac{\sqrt{\pi}}{2}e^{-\frac{1}{4}\pi i}\left(\frac{\sqrt{2}}{n^{\frac{1}{2}}}-1\right)\right\}, \tag{69}$$

$$I_2(\beta) = \frac{1}{m}I_1(\beta)-\frac{\pi}{4\sqrt{2}}\frac{e^{-\frac{1}{4}\pi i}}{\kappa m}-\frac{i\,e^{-i\kappa\sin\beta}}{\kappa m}F_2(\kappa), \tag{70}$$

where
$$m = 1-\sin\beta, \quad n = 1+\sin\beta. \tag{71}$$

Evidently both $I_1(\beta)$ and $I_2(\beta)$ are of order $\kappa^{-\frac{1}{2}}$ for all β and hence the first term in $E_5(x_0, y_0)$ is of order κ^{-1}. Thus the first five terms in the series (37) give $E_z(x_0, y_0)$ at a general point in the far-field region, correctly to terms of order $\kappa^{-\frac{1}{2}}$ inclusive.

2·6. *The transmission coefficient.* The expressions for the far-field deduced in the previous section may now be employed to determine the plane wave transmission coefficient per unit length of the slit for E-polarization, which will be denoted by $t_E(\alpha;\kappa)$. The two-dimensional form of the cross-section theorem (3) states that if the far-field in the direction of propagation of the incident wave is written in the form

$$E_z(x_0, y_0) \sim \frac{e^{\frac{1}{4}\pi i}}{\sqrt{(2\pi)}}\frac{e^{-i\kappa r_0}}{(\kappa r_0)^{\frac{1}{2}}}T_E+O(r_0^{-\frac{3}{2}}), \tag{72}$$

then
$$t_E(\alpha;\kappa) = \frac{1}{\kappa}\Re T_E. \tag{73}$$

Evidently T_E may be written as

$$T_E = \sum_{n=0}^{\infty} T_E^{(n)}, \tag{74}$$

where the first few terms in the series are determined by equations (64) to (70) in the limit as $\beta\to\pi+\alpha$:

$$\left.\begin{aligned}
&\kappa^{-1}T_E^{(0)} \sim \cos\alpha-2\pi^{-\frac{1}{2}}e^{\frac{1}{4}\pi i}\cos\alpha[\phi_0(-\alpha,\pi+\alpha)+\phi_0(\alpha,-\pi-\alpha)],\\
&\kappa^{-1}T_E^{(n)} \sim -\frac{e^{\frac{1}{2}(3n-1)\pi i}}{\pi^{n+\frac{1}{2}}2^{\frac{1}{2}(n+1)}}\cos^2\alpha\,\frac{e^{-in\kappa}}{\kappa^{\frac{1}{2}(n-3)}}\\
&\qquad\times\begin{cases}e^{-i\kappa\sin\alpha}\phi_n(\alpha,\pi+\alpha)+e^{i\kappa\sin\alpha}\phi_n(-\alpha,-\pi-\alpha) & (n=1,3,\ldots),\\ \phi_n(-\alpha,\pi+\alpha)+\phi_n(\alpha,-\pi-\alpha) & (n=2,4,\ldots),\end{cases}
\end{aligned}\right\} \tag{75}$$

and
$$\left.\begin{aligned}
&\phi_0(\alpha,-\pi-\alpha) = F[(\kappa p)^{\frac{1}{2}}]-\frac{i}{2\kappa p}\left(\frac{\sqrt{\pi}}{2}e^{-\frac{1}{4}\pi i}-F[(\kappa p)^{\frac{1}{2}}]\right)+\frac{i}{2(\kappa p)^{\frac{1}{2}}}e^{-i\kappa p},\\
&\phi_0(-\alpha,\pi+\alpha) = \phi_0(-\alpha,-\pi+\alpha).
\end{aligned}\right\} \tag{76}$$

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For near-normal incidence, $\kappa p \gg 1$, $\kappa q \gg 1$. Since

$$-e^{ix^2}F(x) \sim \frac{1}{(-2i)x} + \frac{1}{(-2i)^2x^3} + \frac{1.3}{(-2i)^3x^5} + \frac{1.3.5}{(-2i)^4x^7} + \ldots,$$

the following expression may be found for $t_E(\alpha; \kappa)$:

$$t_E(\alpha; \kappa) \sim \cos\alpha - \frac{\sqrt{2}}{\sqrt{\pi}\,\kappa^{\frac{3}{2}}}\left\{\frac{\cos[\kappa(1+\sin\alpha)-\frac{1}{4}\pi]}{1+\sin\alpha} + \frac{\cos[\kappa(1-\sin\alpha)-\frac{1}{4}\pi]}{1-\sin\alpha}\right\} + O(\kappa^{-2}). \quad (77)$$

When $\alpha = 0$, the expression (77) reduces to

$$t_E(0; \kappa) \sim 1 - \frac{2\sqrt{2}}{\sqrt{\pi}}\frac{\cos(\kappa-\frac{1}{4}\pi)}{\kappa^{\frac{3}{2}}} + O(\kappa^{-2}). \quad (78)$$

The expression (77) is identical with that obtained by Keller (12). Higher-order terms may be obtained in the above manner but, as remarked previously, for near-normal incidence the far-field in the direction of propagation may be obtained most readily by a different method. The following result for normal incidence is quoted from (14):

$$t_E(0; \kappa) \sim 1 - \frac{2\sqrt{2}}{\sqrt{\pi}}\frac{\cos(\kappa-\frac{1}{4}\pi)}{\kappa^{\frac{3}{2}}} + \frac{2}{\pi}\frac{\cos 2\kappa}{\kappa^2} - \frac{\sqrt{2}}{\pi^{\frac{3}{2}}}\left[\frac{\cos(3\kappa+\frac{1}{4}\pi) - \frac{7}{4}\pi\cos(\kappa+\frac{1}{4}\pi)}{\kappa^{\frac{5}{2}}}\right]$$
$$- \frac{1}{\pi^2}\left[\frac{\sin 4\kappa - \frac{5}{2}\pi\sin 2\kappa}{\kappa^3}\right] + O(\kappa^{-\frac{7}{2}}). \quad (79)$$

This latter result has been deduced previously by Clemmow (5), and to a lesser degree by Burger (4), Timman (23), Karp and Russek ((10), (11)) and Keller (12). The relationship expressed by equation (77) will hold if $\alpha \neq \pm\frac{1}{2}\pi$ for all sufficiently large values of κ. On the other hand, if $\kappa p \gg 1$ and $\kappa q \ll 1$, equation (77) is no longer valid and it may be shown with the aid of equations (75) and (76) that

$$t_E(\alpha; \kappa) \sim \frac{4\cos\alpha}{3\sqrt{(2\pi)}}\{(\kappa q)^{\frac{1}{2}} + O[(\kappa q)^{\frac{3}{2}}] + O(\kappa^{-1})\}. \quad (80)$$

If α is between 0 and $\frac{1}{2}\pi$, but of magnitude such that neither of equations (77), (80) is valid, it is necessary to employ the more general formula for the transmission coefficient, the terms of which are given by equations (75).

Some of the foregoing results will be considered in more detail in Part II of the present work.

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