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Schmidt games and non-dense forward orbits of certain partially hyperbolic systems

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Abstract. Let $f : M \to M$ be a partially hyperbolic diffeomorphism with conformality on unstable manifolds. Consider a set of points with non-dense forward orbit: $E(f, y) := \{z \in M : y \notin \overline{\{f^k(z), k \in \mathbb{N}\}}\}$ for some $y \in M$. Define $E_x(f, y) := E(f, y) \cap W^u(x)$ for any $x \in M$. Following a method of Broderick, Fishman and Kleinbock [Schmidt's game, fractals, and orbits of toral endomorphisms. *Ergod. Th. & Dynam. Sys.* **31** (2011), 1095–1107], we show that $E_x(f, y)$ is a winning set for Schmidt games played on $W^u(x)$ which implies that $E_x(f, y)$ has Hausdorff dimension equal to dim $W^u(x)$. Furthermore, we show that for any non-empty open set $V \subset M$, $E(f, y) \cap V$ has full Hausdorff dimension equal to dim M, by constructing measures supported on $E(f, y) \cap V$ with lower pointwise dimension converging to dim M and with conditional measures supported on $E_x(f, y) \cap V$. The results can be extended to the set of points with forward orbit staying away from a countable subset of M.

1. *Introduction*

1.1. *Schmidt games, Diophantine approximation and bounded orbits.*

In this paper we shall study the Hausdorff dimension of a set of points with non-dense forward orbit under certain partially hyperbolic diffeomorphisms. The main tool that we use is Schmidt games, which were first introduced by Schmidt in [**24**] in 1966. A winning set for such games is large in the following sense: it is dense in the metric space, and its intersection with any non-empty open subset has full Hausdorff dimension. Moreover, the winning property is stable with respect to countable intersections. See §3 or [**24**] for more details. Recall that a real number x is called badly approximable if $\|x - p/q\| > c(x)/q^2$ for any rational p/q. The classical theorem of Jarnik and Schmidt states that the set of badly approximable numbers has full Hausdorff dimension 1. Schmidt proved in [**24**] that the set of badly approximable numbers is a winning set for Schmidt games and hence has full Hausdorff dimension 1. Moreover, Schmidt proved the set of badly approximable systems

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of linear forms is a winning set for Schmidt games and hence has full Hausdorff dimension (cf. [**25**]).

There is a well-known connection between Diophantine approximation and bounded orbits of flows on homogeneous spaces. Let $G = SL(2, \mathbb{R})$, $\Gamma = SL(2, \mathbb{Z})$, and $g_t = \left(\begin{smallmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{smallmatrix}\right)$. Then α is badly approximable if and only if the orbit $\{g_t \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)\Gamma \mid t \geq 0\}$ is bounded in G/Γ. The Jarnik–Schmidt theorem can be reformulated as follows: the set of points with bounded orbit under g_t in G/Γ has full Hausdorff dimension 3. Dani applied Schmidt games to generalize the above result to the following two cases.

THEOREM 1.1. (Cf. [**5**]) *Let* $G = SL(n, \mathbb{R})$, $\Gamma = SL(n, \mathbb{Z})$, *and*

$$g_t = \operatorname{diag}(e^{-t}, \ldots, e^{-t}, e^{\lambda t}, \ldots, e^{\lambda t}),$$

where λ *is such that the determinant of* g_t *is* 1. *Then the set of points with bounded orbit has full Hausdorff dimension equal to* $\dim G$.

THEOREM 1.2. (Cf. [**6**]) *Let* G *be a connected semisimple Lie group of* $\mathbb{R}$*-rank one,* Γ *a lattice in* G, *and* $\{g_t | t \in \mathbb{R}\}$ *a one-parameter non-quasiunipotent subgroup of* G, *that is,* $\operatorname{Ad} g_1$ *has an eigenvalue with modulus other than* 1. *Then the set of points with bounded orbit has full Hausdorff dimension equal to* $\dim G$.

Theorem 1.1 is equivalent to saying that the set of badly approximable systems of linear forms has full Hausdorff dimension. Theorem 1.2 implies that for a rank-one locally symmetric space of non-compact type with finite volume, the set of unit vectors tangent to a bounded geodesic has full Hausdorff dimension equal to the dimension of the unit tangent bundle. The result was strengthened by Aravinda and Leuzinger (cf. [**1**]) to state that the set of unit vectors on a non-constant C^1 curve in the unit tangent sphere at a point for which the corresponding geodesic is bounded is of Hausdorff dimension one by applying Schmidt games.

Theorems 1.1 and 1.2 motivate a conjecture of Margulis which was solved by Kleinbock and Margulis.

THEOREM 1.3. (Cf. [**16**]) *Let* G *be a connected semisimple Lie group without compact factors,* Γ *an irreducible lattice in* G, $F = \{g_t | t \in \mathbb{R}\}$ *a one-parameter non-quasiunipotent subgroup of* G, *and* $Y \subset G/\Gamma$ *a closed* F*-invariant set of Haar measure* 0. *Then for any non-empty open subset* V *of* G/Γ*:*

$$\dim(\{x \in V | Fx \text{ is bounded and } \overline{Fx} \cap Y = \emptyset\}) = \dim(G/\Gamma).$$

Recently Kleinbock and Weiss developed a type of modified Schmidt game in [**17**]. Under the same conditions as in Theorem 1.3, they proved in [**18**] for any $x \in G/\Gamma$ that the set $\{h \in H^+ | F^+hx \text{ is bounded and } \overline{F^+hx} \cap Y = \emptyset\}$ is a winning set for a certain modified Schmidt game on H^+, where $F^+ = \{g_t | t \geq 0\}$, H^+ is the expanding horospherical subgroup corresponding to F^+, and Y is a countable subset of G/Γ. It strengthens Theorem 1.3 since winning sets for modified Schmidt games on H^+ have stronger properties than having Hausdorff dimension equal to $\dim H^+$. On the other hand, Kleinbock and Weiss showed in [**17**] that the set of weighted badly approximable vectors is winning and has full Hausdorff dimension. Recently, Dani and Shah [**8**] used Schmidt

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games to show that a large class of Cantor-like sets of $\mathbb{R}^d$ contains uncountably many badly approximable numbers ($d = 1$) or badly approximable vectors ($d \geq 2$).

1.2. *Non-dense orbits.* Given a dynamical system with hyperbolic behavior on a smooth compact manifold M, it is natural to ask whether the set of points with orbit staying away from some point has full Hausdorff dimension. Let $f : M \to M$ be a smooth diffeomorphism, and $y \in M$. Denote

$$E(f, y) := \{z \in M : y \notin \overline{\{f^k(z), k \in \mathbb{N}\}}\}. \tag{1}$$

By definition, any point in $E(f, y)$ has a non-dense forward orbit in M.

Suppose that f is a partially hyperbolic diffeomorphism, and denote

$$E_x(f, y) := E(f, y) \cap W^u(x), \tag{2}$$

for any $x \in M$. We shall prove the following two main theorems in this paper; see §2 for more details and other results.

MAIN THEOREM. *Assume either:*

(1) *f has one-dimensional unstable distribution E^u; or*

(2) *if $\dim E^u \geq 2$, f is conformal on unstable manifolds, i.e. for each $x \in M$, the derivative map $T_x f|_{E^u_x}$ is a scalar multiple of an isometry.*

Then $E_x(f, y)$ is a winning set for Schmidt games played on $W^u(x)$.

Under the same conditions as in the above main theorem, one has the following result.

MAIN THEOREM. *For any non-empty open subset V of M, $\dim_H(E(f, y) \cap V) = n$, where $n = \dim M$.*

There are some previous results on the Hausdorff dimension of the set of points with non-dense orbit. By applying Schmidt games, Dani proved the following theorem for toral endomorphisms.

THEOREM 1.4. (Cf. [**7**]) *If f is a semisimple toral endomorphism on $\mathbb{T}^n$, $y \in \mathbb{Q}^n/\mathbb{Z}^n$, then $E(f, y)$ is a winning set for Schmidt games.*

This result was strengthened by Broderick, Fishman, and Kleinbock.

THEOREM 1.5. (Cf. [**3**]) *For any toral endomorphism f on $\mathbb{T}^n$, any $y \in M$, $E(f, y)$ is a winning set for Schmidt games.*

In fact, a more general result than Theorem 1.5 is proved in [**3**]: the lifting of $E(f, y)$ to the universal covering $\mathbb{R}^n$ is a winning set for Schmidt games played on K, where K is an absolutely friendly subset of $\mathbb{R}^n$. A main application is to prove that the set of badly approximable systems of affine forms has full Hausdorff dimension; see [**3**, Corollary 1.4] for more details.

However, it is hard to apply Schmidt games to the non-algebraic dynamical systems. If f is expanding or transitive Anosov, Urbański proved the following theorem by a symbolic approach since there exists the Markov partition for the system.

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THEOREM 1.6. (Cf. [**29**]) *Let f be a C^2 expanding endomorphism or a transitive C^2 Anosov diffeomorphism. Then the set of points with non-dense forward orbit or full orbit, respectively, has full Hausdorff dimension equal to* $\dim M$.

We can let Y be a subset of M, and $E(f, Y)$ be the set of points with forward orbit staying away from Y. Dolgopyat has shown in [**9**] that if f is a piecewise expanding map of an interval or an Anosov diffeomorphism of the two-dimensional torus and if Y has sufficiently small Hausdorff dimension, then $E(f, Y)$ has full Hausdorff dimension by a symbolic approach.

1.3. *Invariant measures and a conjecture of Katok.* Another motivation of the present paper is a related conjecture of Katok. If f is volume preserving and ergodic, it follows that the set $E(f, y)$ has Lebesgue measure zero. Nevertheless, the above results show that the set is still large in the sense of having full Hausdorff dimension. Indeed, for a toral endomorphism or an Anosov diffeomorphism, there exists an abundance of invariant measures, and a Lebesgue measure zero set could be very large under other measures. Katok has conjectured that any smooth diffeomorphism has an abundance of invariant measures with intermediate metric entropies between 0 and $h_{\text{top}}(f)$, the topological entropy of f.

CONJECTURE 1.7. (Katok, cf. [**28**]) *Let f be a $C^r (r > 1)$ diffeomorphism on a smooth compact manifold. Then for any $\beta \in [0, h_{\text{top}}(f))$, there is an ergodic invariant measure μ such that $h_\mu(f) = \beta$.*

Katok proved the conjecture for $C^r (r > 1)$ diffeomorphisms on compact surfaces (see [**14**, **15**]). Sun proved the conjecture for certain skew product diffeomorphisms in [**26**, **27**], and a weak form of the conjecture for linear toral automorphisms in [**28**]. A recent result of Quas and Soo on ergodic universality implies the conjecture for linear toral automorphisms, and other topological dynamical systems with almost weak specification, asymptotic entropy expansiveness, and the small boundary property (see [**23**]). The conjecture remains open for partially hyperbolic systems.

In this paper we consider the case when f is a partially hyperbolic diffeomorphism with conformality on unstable manifolds. In §5, to show that $E(f, y)$ has full Hausdorff dimension, we shall construct measures on M, with conditional measures on unstable manifolds originated from the process of Schmidt games. The measures constructed in Schmidt games might be used to construct invariant measures under f, and we hope that these invariant measures can possess at least most of intermediate entropies between 0 and $h_{\text{top}}(f)$, which is an attempt toward Katok's conjecture in the given special case.

2. *Formulation of results*

Let M be a n-dimensional smooth, connected, compact Riemannian manifold without boundary and let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism. f is *partially hyperbolic* (see, for example, [**13**]) if there exists a non-trivial Tf-invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$ into so called stable, center, and unstable distributions, such that all

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unit vectors $v^\sigma \in E^\sigma_x$ $(\sigma = c, s, u)$ with $x \in M$ satisfy

$$\|T_x f v^s\| < \|T_x f v^c\| < \|T_x f v^u\|,$$

and

$$\|T_x f|_{E^s_x}\| < 1 \quad \text{and} \quad \|T_x f^{-1}|_{E^u_x}\| < 1,$$

for some suitable Riemannian metric on M.

The distributions E^s, E^c, E^u are Hölder continuous over M but in general not Lipschitz continuous. The stable distribution E^s and unstable distribution E^u are integrable: there exist so-called stable and unstable foliations W^s and W^u respectively such that $TW^s = E^s$ and $TW^u = E^u$. It is well known that the foliations W^u and W^s are transversally absolutely continuous with bounded Jacobians (see [**2**, **4**, **22**]).

For any $y \in M$, the sets $E(f, y)$ and $E_x(f, y)$ are defined as in (1) and (2), where $x \in M$ is an arbitrary point and $W^u(x)$ is the (global) unstable manifold through x. We shall mainly work on unstable manifolds. The set $E(f, y)$ has been shown to be large in the sense of having full Hausdorff dimension, when f is a toral endomorphism or a transitive Anosov diffeomorphism (see [**3**, **7**, **29**]). This is not surprising for hyperbolic toral automorphisms, since there are many non-dense invariant sets, such as the periodic orbits, and other proper invariant sets studied in [**12**, **19**, **21**].

There are many known results on applying Schmidt games to study the Hausdorff dimension of $E(f, y)$ under algebraic dynamical systems (see [**3**, **7**, **18**]). In the first part of the present paper we are able to apply Schmidt games to non-algebraic dynamical systems, i.e. partially hyperbolic diffeomorphisms (PHDs) with conformality on the unstable manifolds.

MAIN THEOREM 1. *Assume either:*

(1) *f has one-dimensional unstable distribution E^u; or*
(2) *if $\dim E^u \geq 2$, f is conformal on unstable manifolds, i.e. for each $x \in M$, the derivative map $T_x f|_{E^u_x}$ is a scalar multiple of an isometry.*

Then $E_x(f, y)$ is a winning set for Schmidt games played on $W^u(x)$.

Remark 2.1. The assumption in the above theorem holds throughout the paper. Clearly, case (1) is contained in case (2). However, it is easier to illustrate the idea of the proof in case (1). So we shall prove for case (1) first, and then sketch the proof for case (2).

As a consequence of Main Theorem 1 and well-known results regarding Schmidt games, one has the following corollary.

COROLLARY 2.2. *$E_x(f, y)$ is dense in $W^u(x)$ and, for any non-empty open subset U of $W^u(x)$, $\dim_H(E_x(f, y) \cap U) = \dim W^u(x)$.*

Let Y be a countable subset of M and define similarly:

$$E(f, Y) := \{z \in M : Y \cap \overline{\{f^k(z), k \in \mathbb{N}\}} = \emptyset\},$$
$$E_x(f, Y) := E(f, Y) \cap W^u(x).$$

Due to a result of Schmidt [**24**], restated in Proposition 3.1, we have the following corollary.

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COROLLARY 2.3. *$E_x(f, Y)$ is a winning set for Schmidt games played on $W^u(x)$. Consequently, it is dense in $W^u(x)$ and for any non-empty open subset U of $W^u(x)$, $\dim_H(E_x(f, Y) \cap U) = \dim W^u(x)$.*

However, it is hard to play Schmidt games on the whole manifold M. A symbolic dynamics approach was developed by Urbański to prove that $E(f, y)$ has full Hausdorff dimension on M, when f is an expanding endormorphism or a transitive Anosov diffeomorphism (see [**29**]). But if f is a partially hyperbolic diffeomorphism, we do not have Markov partition or a symbolic representation of the system. In the second part of this paper, we develop another approach to prove the following theorem.

MAIN THEOREM 2. *Under the same conditions as in Main Theorem 1,*

$$\dim_H(E(f, y) \cap V) = n$$

for any non-empty open subset V of M, where $n = \dim M$.

COROLLARY 2.4. *For any non-empty open subset V of M, $\dim_H(E(f, Y) \cap V) = n$.*

In §3 we describe Schmidt games and the properties of a winning set. In §4 we modify a method from [**3**] and prove Main Theorem 1, with Schmidt games played on an unstable manifold. Corollaries 2.2 and 2.3 are also proved there. In §5 we prove Main Theorem 2 by constructing measures supported on $E(f, y)$ with lower pointwise dimension converging to n. We construct conditional measures supported on $E_x(f, y)$ in a way that has appeared in the proof of a result of McMullen (see [**20**, **29**]). Corollary 2.4 is proved at the end.

For simplicity, we denote $\sigma = \dim E^\sigma$, where $\sigma = u, c, s$. If W is a submanifold of M, then $B^W(x, r)$ denotes the open ball with center x and radius r in W, with respect to the induced Riemannian metric on W. Similarly $B^u(x, r)$ denotes an open ball in $W^u(x)$. We always denote by ν the volume measures on various manifolds if it does not cause confusion.

Remark 2.5. The referee pointed out that two possible generalizations can be made for Main Theorem 1.

(1) Conditions (1) and (2) in Main Theorem 1 are stronger than needed to draw the conclusion. In fact, all one needs is the existence of a Hölder continuous foliation W^u such that $\|T_x f^{-1}|_{E^u_x}\| < 1$ for all $x \in M$, and on which f is conformal. To draw the conclusion in Main Theorem 2, it is sufficient to assume in addition that the foliation W^u is transversally absolutely continuous with bounded Jacobians. Only the properties of W^u mentioned above are used in the proof of this paper.

(2) We should be able to draw a stronger conclusion in Main Theorem 1: $E_x(f, y)$ is $\mathcal{H}$-absolute winning for $\mathcal{H}$-absolute games played on $W^u(x)$, where $\mathcal{H} = \{\{z\} : z \in W^u(x)\}$ (see [**11**, Appendix C]). Absolute winning implies winning for Schmidt games and many properties are enjoyed by absolute winning sets that are not true for winning sets, e.g. the absolute winning property is preserved under quasisymmetric maps. Nevertheless, we only work on Schmidt games in this paper for coherence as the measure produced in Schmidt games is essential to the proof of Main Theorem 2.

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3. *Schmidt games*

Let (X, d) be a complete metric space. We denote by $B(x, r)$ the ball of radius r with center x. If $\omega = (x, r) \in X \times \mathbf{R}_+$, we also denote $B(\omega) := B(x, r)$.

Schmidt games are played by two players, Alice and Bob. Fix $0 < \alpha, \beta < 1$ and a subset $S \subset X$ (the target set). Bob starts the game by choosing $x_1 \in X$ and $r_1 > 0$, thus specifying a pair $\omega_1 = (x_1, r_1)$. Then Alice chooses a pair $\omega_1' = (x_1', r_1')$ such that $B(\omega_1') \subset B(\omega_1)$ and $r_1' = \alpha r_1$. In the second turn, Bob chooses a pair $\omega_2 = (x_2, r_2)$ such that $B(\omega_2) \subset B(\omega_1')$ and $r_2 = \beta r_1'$, and so on. In the kth turn, Bob and Alice choose $\omega_k = (x_k, r_k)$ and $\omega_k' = (x_k', r_k')$ respectively such that

$$B(\omega_k') \subset B(\omega_k) \subset B(\omega_{k-1}'), \quad r_k = \beta r_{k-1}', \quad r_k' = \alpha r_k.$$

Thus we have a nested sequence of balls in X:

$$B(\omega_1) \supset B(\omega_1') \supset \cdots \supset B(\omega_k) \supset B(\omega_k') \supset \cdots. \tag{3}$$

The intersection of all these balls consists of a unique point $x_\infty \in X$. We call Alice the winner if $x_\infty \in S$, and Bob the winner otherwise. S is called an (α, β)-winning set if Alice has a strategy to win regardless of how well Bob plays, and we call such a strategy a $(\alpha, \beta; S)$-winning strategy. S is called α-winning if it is (α, β)-winning for any $0 < \beta < 1$. S is called a winning set if it is α-winning for some $0 < \alpha < 1$.

The following nice properties of a winning set are proved in [**24**].

PROPOSITION 3.1. *The intersection of countably many α-winning sets is α-winning.*

PROPOSITION 3.2. *If the game is played on $X = \mathbb{R}^n$ with the Euclidean metric, then any winning set is dense and has full Hausdorff dimension n.*

For a more general metric space (X, d) than $\mathbb{R}^n$, we suppose that X supports a Federer measure.

Definition 3.3. Let μ be a locally finite Borel measure on a metric space (X, d) and $D > 0$. We say that μ is D-Federer if there exists $\rho_0 > 0$ such that

$$\mu(B(x, 2\rho)) < D\mu(B(x, \rho)), \quad \text{for all } x \in \operatorname{supp} \mu, \text{ for all } 0 < \rho < \rho_0.$$

Recall that the *lower pointwise dimension* of a measure μ at $x \in \operatorname{supp} \mu$ is defined as

$$\underline{d}_\mu(x) := \liminf_{\rho \to 0} \frac{\log \mu(B(x, \rho))}{\log \rho},$$

and for an open set $U \subset X$,

$$\underline{d}_\mu(U) := \inf_{x \in \operatorname{supp} \mu \cap U} \underline{d}_\mu(x).$$

The following proposition [**17**, Proposition 5.1] shows that any winning set on supports of a Federer measure has a lower bound for Hausdorff dimension.

PROPOSITION 3.4. *Let X be a complete metric space which is the support of a Federer measure μ. If S is a winning set on X, then for any non-empty open set $U \subset X$, one has*

$$\dim_H(S \cap U) \geq \underline{d}_\mu(U).$$

We shall play Schmidt games on unstable manifolds. In §5, we restate Proposition 3.4 as Lemma 5.3 with X being a manifold, and μ being the volume measure on X. We shall prove Lemma 5.3 there, since it is essential to the proof of Main Theorem 2. The proof of Proposition 3.4 is essentially the same, with small modifications.

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4. *Schmidt games on unstable manifolds*

4.1. *Unstable manifold as a playground.* We shall play Schmidt games on $W^u(x)$, where x is an arbitrary point on M. Consider the Riemannian metric on $W^u(x)$ induced from that of M, and let d^u denote the distance on $W^u(x)$. In this subsection, we study a nice measure supported on $W^u(x)$ which can enable us to estimate the Hausdorff dimension of a winning set on $W^u(x)$, namely, the volume measure ν on $W^u(x)$.

Definition 4.1. We say a measure μ satisfies a power law if there exist positive numbers δ, c_1, c_2, ρ_0 such that

$$c_1\rho^\delta \le \mu(B(z,\rho)) \le c_2\rho^\delta \quad \text{for all } z \in \operatorname{supp}\mu, \text{ for all } 0<\rho<\rho_0.$$

LEMMA 4.2. *The volume measure ν on $W^u(x)$ satisfies a power law, with $\delta = u$.*

Proof. Since $\operatorname{supp}\nu = W^u(x)$, let us fix an arbitrary $p \in W^u(x)$. Consider the exponential map

$$\exp_p : \tilde{B}(0, 2\rho_0) \to B^u(p, 2\rho_0),$$

where $\tilde{B}(0, 2\rho_0) \subset T_pW^u(x)$, $B^u(p, 2\rho_0) \subset W^u(x)$, and ρ_0 is chosen such that $\exp_p$ is a diffeomorphism with $\|T_y \exp_p - \mathrm{Id}\| \le \epsilon_0$ for all $y \in \tilde{B}(0, 2\rho_0)$ and some $\epsilon_0 > 0$ small enough. Then it is easy to see that for any $z \in B^u(p, \rho_0)$, $0 < \rho < \rho_0$,

$$c_1\rho^u \le \nu(B^u(z,\rho)) \le c_2\rho^u,$$

for some $c_1, c_2 > 0$. We can choose ρ_0 to be independent of $p \in W^u(x)$. □

LEMMA 4.3. *The volume measure ν is a Federer measure on $W^u(x)$.*

Proof. By Lemma 4.2, there exists $\rho_0 > 0$, such that for all $0 < 2\rho < \rho_0$,

$$\nu(B(z, 2\rho)) \le c_2(2\rho)^u = \frac{c_2 2^u}{c_1} c_1\rho^u \le D\nu(B(z,\rho)),$$

where $D = c_2 2^u/c_1 > 0$. □

In [**3**], the authors define a so-called absolutely decaying measure on $\mathbb{R}^n$ so that we can estimate the measure of a neighborhood of a hyperplane in $\mathbb{R}^n$. Here we can weaken the condition, and the following lemma is enough for our purpose.

LEMMA 4.4. *Let ν be the volume measure on $W^u(x)$. Then there exist $\rho_0 > 0$ and some $C > 0$ such that*

$$\nu(B(x_1,\rho) \cap B(x_2, \epsilon\rho)) \le C\epsilon^u \nu(B(x_1,\rho))$$

for any $x_1, x_2 \in W^u(x)$, for all $0 < \rho < \rho_0$, $0 < \epsilon < 1$.

Proof. Let ρ_0 be as in Lemma 4.2. Then for all $0 < \rho < \rho_0$,

$$\nu(B(x_1,\rho) \cap B(x_2,\epsilon\rho)) \le \nu(B(x_2,\epsilon\rho)) \le c_2(\epsilon\rho)^u \le C\epsilon^u c_1\rho^u \le C\epsilon^u \nu(B(x_1,\rho)),$$

where $C = c_2/c_1 > 0$. □

Remark 4.5. In the next subsection, we consider a partially hyperbolic system with one-dimensional unstable manifolds (i.e. $u = 1$). Since the volume measure is the length, we have a more specific choice for the constants in the previous lemmas: $c_1 = c_2 = 2$, $D = 2$, $C = 1$.

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The next lemma is similar to [**3**, Lemma 3.4] which is crucial to our proof of Main Theorem 1. It guarantees that while playing Schmidt games on $W^u(x)$, at each turn Alice can stay away from at least some of the specified balls.

LEMMA 4.6. *Let C, D, ρ_0 be as in the previous lemmas, and*

$$0 < \alpha < \frac{1}{2}\left(\frac{1}{CD}\right)^{1/u}. \tag{4}$$

There exists $\epsilon = \epsilon(C, D) \in (0, 1)$ such that if $x_1 \in W^u(x)$, $0 < \rho < \rho_0$, and $y_1, y_2, \ldots, y_N$ are N points in $W^u(x)$, there exists $x_2 \in W^u(x)$ such that

$$B(x_2, \alpha\rho) \subset B(x_1, \rho)$$

and

$$B(x_2, \alpha\rho) \cap B(y_i, \alpha\rho) = \emptyset$$

for at least $\lceil \epsilon N \rceil$ (the smallest integer $\geq \epsilon N$) of the points y_i, $1 \leq i \leq N$.

Proof. Let $A_i := B(x_1, (1-\alpha)\rho) \backslash B(y_i, 2\alpha\rho)$, $1 \leq i \leq N$. Then

$$\begin{aligned} \frac{\nu(A_i)}{\nu(B(x_1, \rho))} &\geq \frac{\nu(B(x_1, (1-\alpha)\rho))}{\nu(B(x_1, \rho))} - \frac{\nu(B(y_i, 2\alpha\rho))}{\nu(B(x_1, \rho))} \\ &\geq \frac{1}{D} - C(2\alpha)^u \\ &:= \epsilon > 0. \end{aligned} \tag{5}$$

Thus,

$$\int_{B(x_1,\rho)} \sum_{i=1}^{N} \chi_{A_i}(x)\, d\nu(x) \geq N\epsilon\nu(B(x_1, \rho)).$$

Hence there exists x_2 with $\sum_{i=1}^{N} \chi_{A_i}(x_2) \geq N\epsilon$, i.e. there exist $j_1, \ldots, j_k$, where $k \geq \lceil \epsilon N \rceil$, such that $x_2 \in \bigcap_{i=1}^{k} A_{j_i}$ which implies $B(x_2, \alpha\rho) \subset B(x_1, \rho)$ and $B(x_2, \alpha\rho) \cap B(y_{j_i}, \alpha\rho) = \emptyset$. □

Remark 4.7. If we choose $C = c_2/c_1$, $D = (c_2 2^u)/c_1$, then $0 < \alpha < \frac{1}{4}(c_1/c_2)^{2/u}$. In the case where $\dim W^u(x_0) = 1$, we choose $c_1 = c_2 = 2$, so $0 < \alpha < \frac{1}{4}$.

Remark 4.8. The estimation in the above proof is a slight modification of the proof of [**3**, Lemma 3.4] which is for an absolutely friendly measure on $\mathbb{R}^n$. In fact we can do a better estimation than (5) by using Lemma 4.2. That is,

$$\begin{aligned} \frac{\nu(A_i)}{\nu(B(x_1, \rho))} &\geq \frac{\nu(B(x_1, (1-\alpha)\rho))}{\nu(B(x_1, \rho))} - \frac{\nu(B(y_i, 2\alpha\rho))}{\nu(B(x_1, \rho))} \\ &\geq \frac{c_1(1-\alpha)^u}{c_2} - \frac{c_2(2\alpha)^u}{c_1} \\ &= \frac{c_1}{c_2}\left((1-\alpha)^u - (2\alpha)^u\left(\frac{c_2}{c_1}\right)^2\right). \end{aligned}$$

So we can pick any α with $(1-\alpha)^u > (2\alpha)^u(c_2/c_1)^2$, i.e. $0 < \alpha < 1/(1 + 2(c_2/c_1)^{2/u})$ $\leq \frac{1}{3}$. If in Lemma 4.2 we choose ρ_0 very small, then c_1 and c_2 are very close to each other. Hence we can pick arbitrary $0 < \alpha < \frac{1}{3}$.

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We will restate and prove Main Theorem 1 according to the two different cases in Remark 2.1 separately in the next two subsections. The strategy to prove Main Theorem 1 is similar to that of [**3**, Theorem 4.1]. The difficulty here is that $f|_{W^u(x)}$ is nonlinear and hence has different expanding rates in different directions and at different points. Let W be a local manifold passing through y transversally to the foliation W^u ($\dim W = n - u$). We call the set

$$\Pi(c) := \Pi(y, W, c) := \bigcup_{z \in B^W(y, c/2)} B^u(z, c/2)$$

an open c-rectangle at y (c is very small). Let $I_k = I_k(c)$ denote a connected component of $f^{-k}(\Pi(c)) \cap W^u(x)$ on $W^u(x)$, $k \geq 0$. Note that there may exist more than one connected component for a single k.

4.2. *Partially hyperbolic diffeomorphism with one-dimensional unstable manifolds.* In this subsection we consider a PHD f with one-dimensional unstable manifolds. Since M is compact and E^u is a Hölder continuous distribution on M, we can suppose that

$$1 < \sigma_1 \leq \|T_z f|_{E^u_z}\| \leq \sigma_2 \quad \text{for all } z \in M.$$

We denote by $\|I_k\|$ the length of I_k on the unstable manifold. For simplicity we also write f for $f|_{W^u(x)}$ and $f'(z)$ for $\|T_z f|_{E^u_z}\|$. A useful tool is the following bounded distortion property.

LEMMA 4.9. (Bounded distortion) *For any $z_1, z_2 \in I_k(c)$,*

$$\frac{1}{K} \leq \frac{(f^k)'(z_1)}{(f^k)'(z_2)} \leq K,$$

for some $K = K(c)$, and $K \to 1$ as $c \to 0$.

Proof. Since E^u and $z \mapsto T_z f$ are both Hölder continuous, $\log f'$ is as well, and thus there exist $l > 0$ and $0 < \theta < 1$ such that $\|\log f'(z_1) - \log f'(z_2)\| \leq l(d^u(z_1, z_2))^\theta$ for nearby z_1 and z_2. Recall that $\sigma_1 \leq f'(z) \leq \sigma_2$ for any $z \in W^u(x_0)$. For any $z_1, z_2 \in I_k$, since $d^u(f^k(z_1), f^k(z_2)) \leq c$, one has

$$d^u(f^i(z_1), f^i(z_2)) \leq \frac{c}{(\sigma_1)^{k-i}} \quad \text{for all } 0 \leq i \leq k. \tag{6}$$

Thus,

$$\begin{aligned} \left\| \log \frac{(f^k)'(z_1)}{(f^k)'(z_2)} \right\| &\leq \sum_{i=0}^{k-1} \|\log f'(f^i(z_1)) - \log f'(f^i(z_2))\| \\ &\leq \sum_{i=0}^{k-1} l(d^u(f^i(z_1), f^i(z_2)))^\theta \\ &\leq \sum_{i=0}^{k-1} \frac{lc^\theta}{(\sigma_1)^{\theta(k-i)}} \leq \frac{lc^\theta}{(\sigma_1)^\theta - 1}. \end{aligned} \tag{7}$$

Hence

$$\frac{1}{K} \leq \frac{(f^k)'(z_1)}{(f^k)'(z_2)} \leq K,$$

where $K = \exp(lc^\theta/((\sigma_1)^\theta - 1))$. □

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THEOREM 4.10. *Suppose that* $\dim E^u = 1$. *Let* α *be as in (4). Then* $E_x(f, y)$ *is* α*-winning on* $W^u(x)$.

Proof. Pick an arbitrary $0 < \beta < 1$. Let ϵ be as in (5). Choose $r \in \mathbf{N}$ large enough such that

$$(1-\epsilon)^r N < 1 \quad \text{where } N = \left\lfloor \frac{\log 2 + r \log(1/(\alpha\beta))}{\log \sigma_1} \right\rfloor + 3. \tag{8}$$

Fix $L > 0$ small. Regardless of Bob's initial move, Alice can make arbitrary moves waiting until Bob chooses a ball of radius $\rho \le \min\{\rho_0, L/100\}$. Hence without loss of generality, we may assume that $B(\omega_1)$ has radius

$$\rho \le \min\left\{\rho_0, \frac{L}{100}\right\}. \tag{9}$$

Choose $c' > 0$ small enough such that:

(1) $1 < K = K(c') \le 1 + \eta$ where $\eta > 0$ is very small;
(2) for any $z \in \Pi(c')$, $B^u(z, L) \cap \Pi(c')$ has only one connected component, which contains z.

Now choose $0 < c \ll c'$ such that

$$c \le \frac{\alpha c' (\alpha\beta)^{2r-1}}{100K} \tag{10}$$

and

$$c < \alpha\rho(\alpha\beta)^{2r-1}. \tag{11}$$

Note that the choice of c depends heavily on ρ, i.e. Bob's initial move.

Now we describe a strategy for Alice to win the (α, β)-Schmidt games on $W^u(x_0)$ with target set $S = E_x(f, y)$. We claim that for each $j \in \mathbb{N}$, Alice can ensure that for any $x \in B(\omega'_{r(j+1)})$ and any $I_k = I_k(c)$ with $\|I_k\| \ge \alpha\rho(\alpha\beta)^{(j+2)r-1}$, she has $x \notin I_k$. This will imply $\bigcap_i B(\omega'_i) \subset (\bigcup_k I_k)^c \subset E_x(f, y)$, and finish the proof.

We prove the claim by induction on j. At step $j = 0$, by (11) one has, for any $k \in \mathbb{N}$,

$$\|I_k\| \le c < \alpha\rho(\alpha\beta)^{2r-1}.$$

So there is no I_k for Alice to avoid and she can play arbitrarily at the first r turns.

Assume that the claim is true for $0, 1, \ldots, j-1$. Now we consider the jth step. Suppose that Bob has already picked $B(\omega_{jr+1})$. In this step (containing r turns of play), Alice only needs to avoid those I_k satisfying

$$\alpha\rho(\alpha\beta)^{(j+2)r-1} \le \|I_k\| < \alpha\rho(\alpha\beta)^{(j+1)r-1} \tag{12}$$

and

$$I_k \cap B(\omega_{jr+1}) \ne \emptyset. \tag{13}$$

In the following two lemmas we consider the I_k satisfying

$$I_k \subset B(\omega_{jr+1}). \tag{14}$$

We will prove there are at most $(N-2)$ I_k satisfying (12) and (14). Then considering the intersection at two endpoints of $B(\omega_{jr+1})$, there are at most N I_k satisfying (12) and (13).

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LEMMA 4.11. *For each k, there exists at most one I_k satisfying both (12) and (14).*

Proof. Assume that $I_k, I_k' \subset B(\omega_{jr+1})$ are two different intervals satisfying both (12) and (14). Then there exists an interval J_k such that $I_k \subset J_k \subset B(\omega_{jr+1})$, $f^k(J_k) \subset \Pi(c')$ and

$$\|f^k(J_k)\| \geq c'/2.$$

Reall that

$$\|I_k\| \geq \alpha\rho(\alpha\beta)^{(j+2)r-1}.$$

By the bounded distortion lemma (Lemma 4.9) and the mean value theorem,

$$\frac{\|J_k\|}{\|I_k\|} \geq \frac{\|f^k(J_k)\|}{K\|f^k(I_k)\|} \geq \frac{c'}{2Kc}.$$

Thus by (12) and (10),

$$\begin{aligned}\|J_k\| \geq \frac{c'}{2Kc}\|I_k\| &\geq \frac{c'}{2Kc}\alpha\rho(\alpha\beta)^{(j+2)r-1} \\ &\geq \frac{50}{\alpha(\alpha\beta)^{2r-1}} \cdot \alpha\rho(\alpha\beta)^{(j+2)r-1} = 50\rho(\alpha\beta)^{jr}. \end{aligned} \tag{15}$$

But $\|J_k\| \leq \|B(\omega_{jr+1})\| = \rho(\alpha\beta)^{jr}$, in contradiction to (15). This finishes the proof of the lemma. □

LEMMA 4.12. *There are at most $(N-2)$ choices of k satisfying both (12) and (14).*

Proof. Let k_1 and k_2 be the minimal and the maximal ones respectively among all k's satisfying both (12) and (14). Then $k_1 \leq k_2$. It is clear that

$$\|f^{k_1}(I_{k_1})\| = \|f^{k_2}(I_{k_2})\| = c, \tag{16}$$

and

$$\|f^{k_1}(I_{k_2})\| \leq \frac{\|f^{k_2}(I_{k_2})\|}{(\sigma_1)^{k_2-k_1}} = \frac{c}{(\sigma_1)^{k_2-k_1}}. \tag{17}$$

Moreover,

$$f^{k_1}(I_{k_2}) \subset f^{k_1}(B(\omega_{jr+1})) \subset \Pi(c').$$

The latter inclusion can be seen from the proof of Lemma 4.11. Indeed, if $f^{k_1}(B(\omega_{jr+1})) \subset \Pi(c')$ is not true, then there exists an interval J_k with $I_k \subset J_k \subset B(\omega_{jr+1})$ and $\|f^k(J_k)\| \geq c'/2$. This is impossible from the proof of Lemma 4.11. The inclusion guarantees that we can apply the bounded distortion Lemma 4.9 to obtain

$$\frac{\|I_{k_2}\|}{\|I_{k_1}\|} \leq \frac{K\|f^{k_1}(I_{k_2})\|}{\|f^{k_1}(I_{k_1})\|} \leq \frac{K}{(\sigma_1)^{k_2-k_1}} \tag{18}$$

where the latter inequality follows from (16) and (17). Combining (12) and (18), one has

$$\begin{aligned}\alpha\rho(\alpha\beta)^{(j+2)r-1} &\leq \|I_{k_2}\| \\ &\leq \frac{K}{(\sigma_1)^{k_2-k_1}}\|I_{k_1}\| \\ &\leq \frac{K}{(\sigma_1)^{k_2-k_1}}\alpha\rho(\alpha\beta)^{(j+1)r-1},\end{aligned}$$

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which implies

$$(\sigma_1)^{k_2-k_1} \le \frac{K}{(\alpha\beta)^r}.$$

Hence

$$k_2 - k_1 \le \left\lfloor \frac{\log K + r\log(1/\alpha\beta)}{\log \sigma_1} \right\rfloor < N - 3$$

which finishes the proof of the lemma. □

Now Alice can apply Lemma 4.6 r times to choose $B(\omega'_{jr+1}), \ldots, B(\omega'_{(j+1)r})$ respectively, to avoid all the I_k satisfying both (12) and (13). Indeed, by (12),

$$I_k \subset B(y_i, \alpha\rho(\alpha\beta)^{(j+1)r-1})$$

for some y_i, $1 \le i \le N$. Since $N(1-\epsilon)^r < 1$ by (8), one can have

$$B(\omega'_{(j+1)r}) \cap I_k = \emptyset$$

for all I_k satisfying both (12) and (13), which implies the claim. □

4.3. *Partially hyperbolic diffeomorphism with conformality on unstable manifolds.* We generalize the above result to the case where f is a PHD with higher-dimensional unstable manifolds and $Tf|_{E^u}$ is conformal, i.e. for each $z \in M$, the derivative map $T_z f|_{E^u_z}$ is a scalar multiple of an isometry. We denote the magnitude of this scalar by $\|T^u_z f\|$ for simplicity; then obviously $\|T^u_z f\| > 1$. Suppose that $1 < \sigma_1 \le \|T^u_z f\| \le \sigma_2$ for any $z \in M$. Then we also have the following lemma.

LEMMA 4.13. (Bounded distortion) *For any $z_1, z_2 \in I_k(c)$,*

$$\frac{1}{K} \le \frac{\|T^u_{z_1} f^k\|}{\|T^u_{z_2} f^k\|} \le K,$$

for some $K = K(c)$, and $K \to 1$ as $c \to 0$.

Proof. Note that $z \mapsto \log \|T^u_z f\|$ is Hölder continuous on M, and (6) is still true. Hence the same estimation in (7) is valid which implies the desired statement. □

For preimages of a small ball on unstable manifolds, the ratio of major radius R and minor radius r is bounded. For instance, suppose that $I_k = f^{-k}(B^u(z, c/2))$ for some $z \in B^W(y, c/2)$. Denote

$$R(I_k) := \max_{w \in \partial I_k} d^u(f^{-k}(z), w)$$

and

$$r(I_k) := \min_{w \in \partial I_k} d^u(f^{-k}(z), w).$$

Then we have the following results.

LEMMA 4.14.

$$\frac{R(I_k)}{r(I_k)} \le K.$$

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Proof. By the mean value theorem and the bounded distortion lemma (Lemma 4.13),

$$\frac{d^u(f^{-k}(z), w_1)}{d^u(f^{-k}(z), w_2)} \le \frac{\|T^u_{w_1'} f^k\|^{-1} d^u(z, f^k(w_1))}{\|T^u_{w_2'} f^k\|^{-1} d^u(z, f^k(w_2))} \le K \frac{c/2}{c/2} = K,$$

for any $w_1, w_2 \in \partial I_k$ and some $w_1', w_2' \in I_k$. Hence $R(I_k)/r(I_k) \le K$. □

THEOREM 4.15. *Suppose that* $\dim E^u \ge 2$ *and* $Tf|_{E^u}$ *is conformal. Let* α *be as in (4). Then* $E_x(f, y)$ *is* α*-winning on* $W^u(x)$.

Proof. We follow the scheme in the proof of Theorem 4.10. Instead of $\|I_k\|$ (the diameter of I_k), one has to estimate $R(I_k)$ and $r(I_k)$. It turns out that we do not need to modify anything about the choice of r, N, ρ, c' and c. They remain the same as in Theorem 4.10.

Similarly, we claim that for each $j \in \mathbb{N}$, Alice can ensure for any $x \in B(\omega'_{r(j+1)})$, and any $I_k = I_k(c)$ with $R(I_k) \ge \alpha\rho(\alpha\beta)^{(j+2)r-1}$, that she has $x \notin I_k$. Step $j = 0$ follows from (11) as well. In the jth step, Alice needs to avoid the I_k satisfying

$$\alpha\rho(\alpha\beta)^{(j+2)r-1} \le R(I_k) < \alpha\rho(\alpha\beta)^{(j+1)r-1} \tag{19}$$

and (13). Similarly, we have the following two lemmas which imply that there are at most N such I_k. Here we consider

$$I_k \subset B(\tilde{\omega}_{jr+1}) \tag{20}$$

where $B(\tilde{\omega}_{jr+1})$ denotes the ball concentric with $B(\omega_{jr+1})$ of radius $\rho(\alpha\beta)^{jr}(1 + 2\alpha(\alpha\beta)^{r-1})$. Then any I_k satisfying both (19) and (13) also satisfies (20).

LEMMA 4.16. *For each* k*, there exists at most one* I_k *satisfying both (19) and (20).*

Proof. Assume that there are two different $I_k, I_k' \subset B(\tilde{\omega}_{jr+1})$ satisfying both (19) and (20). Then there exists a ball J_k such that $I_k \subset J_k \subset B(\tilde{\omega}_{jr+1})$, $f^k(J_k) \subset \Pi(c')$ and

$$R(f^k(J_k)) \ge c'/4.$$

By Lemma 4.13 and the mean value theorem,

$$\frac{R(J_k)}{R(I_k)} \ge \frac{R(f^k(J_k))}{K R(f^k(I_k))} \ge \frac{c'}{4Kc}.$$

Thus, by (19) and (10),

$$\begin{aligned} R(J_k) &\ge R(I_k) \cdot \frac{c'}{4Kc} \ge \alpha\rho(\alpha\beta)^{(j+2)r-1} \cdot \frac{c'}{4Kc} \\ &\ge 25\rho(\alpha\beta)^{jr}. \end{aligned} \tag{21}$$

But $R(J_k) \le R(B(\tilde{\omega}_{jr+1})) = \rho(\alpha\beta)^{jr}(1 + 2\alpha(\alpha\beta)^{r-1})$, in contradiction to (21). □

LEMMA 4.17. *There are at most* N *choices of* k *satisfying both (19) and (20).*

Proof. One has

$$f^{k_1}(I_{k_2}) \subset f^{k_1}(B(\tilde{\omega}_{jr+1})) \subset \Pi(c').$$

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The latter inclusion can be seen from the proof of Lemma 4.16. Indeed, if $f^{k_1}(B(\tilde{\omega}_{jr+1})) \subset \Pi(c')$ is not true, then there exists a ball J_k with $I_k \subset J_k \subset B(\tilde{\omega}_{jr+1})$ and $R(f^k(J_k)) \geq c'/4$. This is impossible from the proof of Lemma 4.16. Hence by Lemma 4.13 and the mean value theorem,

$$\frac{R(I_{k_2})}{R(I_{k_1})} \leq \frac{K\,R(f^{k_1}(I_{k_2}))}{r(f^{k_1}(I_{k_1}))} \leq \frac{K}{r(f^{k_1}(I_{k_1}))} \cdot \frac{R(f^{k_2}(I_{k_2}))}{(\sigma_1)^{k_2-k_1}} = \frac{K}{(\sigma_1)^{k_2-k_1}}. \tag{22}$$

The last equality follows from $R(f^{k_2}(I_{k_2})) = r(f^{k_1}(I_{k_1})) = c/2$. Hence,

$$\begin{aligned} \alpha\rho(\alpha\beta)^{(j+2)r-1} &\leq R(I_{k_2}) \\ &\leq \frac{K}{(\sigma_1)^{k_2-k_1}} R(I_{k_1}) \\ &\leq \frac{K}{(\sigma_1)^{k_2-k_1}} \alpha\rho(\alpha\beta)^{(j+1)r-1}, \end{aligned}$$

which implies

$$(\sigma_1)^{k_2-k_1} \leq \frac{K}{(\alpha\beta)^r}.$$

Hence

$$k_2 - k_1 \leq \left\lfloor \frac{\log K + r\log(1/\alpha\beta)}{\log \sigma_1} \right\rfloor < N - 3$$

which finishes the proof of the lemma. □

The same argument of applying Lemma 4.6 finishes the proof of the theorem. □

Proof of Corollary 2.2. The volume measure ν is a Federer measure on $W^u(x)$ by Lemma 4.3. By Lemma 4.2,

$$\underline{d}_\nu(z) := \liminf_{\rho\to 0} \frac{\log \nu(B^u(z,\rho))}{\log \rho} = u.$$

Hence $\underline{d}_\nu(U) = u$. So by Proposition 3.4, one has $\dim_H(E_x(f, y) \cap U) = u$. Obviously as a winning set $E_x(f, y)$ is dense in $W^u(x)$. □

Proof of Corollary 2.3. By Proposition 3.1, $E_x(f, Y)$ is a winning set. Then the proof of Corollary 2.2 also applies to $E_x(f, Y)$. □

5. *Full Hausdorff dimension on the manifold*

Our aim in this section is to prove Main Theorem 2, that is, for any non-empty open subset V of M, $\dim_H(E(f, y) \cap V) = n$. Recall that for any $x \in M$, $\dim_H(E_x(f, y)) = u$. We cannot directly apply the Marstrand slicing theorem (see [**16**]) to get the estimate $\dim_H(E(f, y)) \geq \dim_H(W) + \min_{x\in W} \dim_H(E_x(f, y))$, as the unstable foliation is only absolutely continuous, but not Lipschitz continuous (cf. [**2**, **4**, **22**]). Instead we will apply the following easy half of Frostman's lemma (see [**10**, p. 60]).

LEMMA 5.1. *Let F be a Borel subset of a Riemannian manifold X. Let μ be a Borel measure on F, and let $0 < c < \infty$ be a constant. If*

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^h} < c \quad \textit{for all } x \in F, \tag{23}$$

then $\dim_H F \geq h$.

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In the remainder of this section, we shall construct a Borel measure μ on $E(f, y)$ satisfying the condition in (23). Fix $x_0 \in M$. Suppose that W is a local smooth foliation transverse to W^u near x_0. Recall that a δ-rectangle at x_0 is defined as

$$\Pi(x_0) := \Pi(x_0, W, \delta) := \bigcup_{x \in \overline{B^W(x_0, \delta/2)}} B^u(x, \delta/2). \tag{24}$$

The measure μ will be constructed to be supported on $\Pi(x_0) \cap E(f, y)$. At the first stage we construct the 'conditional' measures on local unstable manifolds.

5.1. *Construction of measure family* $\{\mu_x\}$. The following construction for producing fractal sets has been described in [**16**]. Let $A_0 \subset X$ be a compact subset of a Riemannian manifold X and let m be a Borel measure on X. For each $l \in \mathbb{N}_0$, let $\mathcal{A}_l$ denote a finite collection of compact subsets of A_0 satisfying the following conditions:

$$\mathcal{A}_0 = \{A_0\} \quad \text{and} \quad m(A_0) > 0; \tag{25}$$

$$\text{for all } l \in \mathbb{N}, \quad \text{if } A, B \in \mathcal{A}_l \text{ and } A \neq B, \text{ then } m(A \cap B) = 0; \tag{26}$$

$$\text{for all } l \in \mathbb{N}, \quad \text{every element } B \in \mathcal{A}_l \text{ is contained in an element } A \in \mathcal{A}_{l-1}. \tag{27}$$

Let $\mathcal{A}$ be the union of subcollections $\mathcal{A}_l$, $l \in \mathbb{N}_0$. Then $\mathcal{A}$ is called *tree-like* if it satisfies conditions (25)–(27). $\mathcal{A}$ is called *strongly tree-like* if it is tree-like and, in addition,

$$d_l(\mathcal{A}) := \sup\{\operatorname{diam}(A) : A \in \mathcal{A}_l\} \to 0 \quad \text{as } l \to \infty. \tag{28}$$

For each $l \in \mathbb{N}_0$, denote $\mathbf{A}_l = \bigcup_{A \in \mathcal{A}_l} A$. Then we can define the limit set of $\mathcal{A}$ to be

$$\mathbf{A}_\infty := \bigcap_{l \in \mathbb{N}_0} \mathbf{A}_l.$$

For any Borel $B \subset A_0$ with $m(B) > 0$ and $l \in \mathbb{N}$, define the lth-stage density $\delta_l(B, \mathcal{A})$ of $\mathcal{A}$ in B by

$$\delta_l(B, \mathcal{A}) = \frac{m(\mathbf{A}_l \cap B)}{m(B)}.$$

Assume that

$$\Delta_l(\mathcal{A}) := \inf_{B \in \mathcal{A}_l} \delta_{l+1}(B, \mathcal{A}) > 0. \tag{29}$$

The following lemma is proved in [**20**] and [**29**]. We present the proof from [**29**] here since it is essential to our construction of measures.

LEMMA 5.2. *Let $\mathcal{A}$ be defined as above, satisfying conditions (25)–(29). Assume that there exist constants $D > 0$ and $k > 0$ such that for any $z \in A_0$,*

$$m(B(z, r)) \leq D r^k.$$

Denote

$$\epsilon := \limsup_{l \to \infty} \frac{\sum_{i=0}^{l} \log(1/\Delta_i(\mathcal{A}))}{\log(1/d_l(\mathcal{A}))}.$$

Then there exists a sequence of measures $\mu^{(l)}$ supported on $\mathbf{A}_l$ such that:

(1) *the sequence $\mu^{(l)}$ has a unique limit measure $\bar{\mu}$, which is supported on $\mathbf{A}_\infty$;*

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(2) $\bar{\mu}(B(z, r)) \le Cr^{k-\epsilon}$ *for any* $z \in \mathbf{A}_\infty$, $r > 0$, *and some* $C > 0$;
(3) $\dim_H(\mathbf{A}_\infty) \ge k - \epsilon$.

Proof. We define a sequence of Borel measures $\{\mu^{(l)}\}_{l=0}^{\infty}$ on A_0 inductively as follows. Let $\mu^{(0)} := m|_{A_0}$, and for each Borel $B \subset A_0$,

$$\mu^{(l+1)}(B) := \sum_{A \in \mathcal{A}_l} \frac{m(B \cap A \cap \mathbf{A}_{l+1})}{m(A \cap \mathbf{A}_{l+1})} \mu^{(l)}(A). \tag{30}$$

Then by induction each $\mu^{(l)}$ is supported on $\mathbf{A}_l$. For any $A \in \mathcal{A}_l$, $\mu^{(l+1)}(A) = \mu^{(l)}(A)$, hence $\mu^{(i)}(A) = \mu^{(l)}(A)$ for any $i \ge l$ by (26) and (27). Since $\lim_{l\to\infty} d_l(\mathcal{A}) = 0$, there exists a unique measure $\bar{\mu}$ as the limit measure of $\{\mu^{(l)}\}$ on A_0 such that $\bar{\mu}$ is supported on $\mathbf{A}_\infty$. Moreover, for any $A \in \mathcal{A}_l$, we have $\bar{\mu}(A) = \mu^{(l)}(A)$ and $\bar{\mu}(A) \le m(A)/(\prod_{i=0}^{l-1} \Delta_i)$.

Next we prove part (2) of the lemma. Consider $y \in \mathbf{A}_\infty$, and small $r > 0$. Choose l such that $d_{l+1}(\mathcal{A}) \le r < d_l(\mathcal{A})$. Then $B(y, 2r)$ contains all the sets in $\mathbf{A}_{l+1}$ which meet $B(y, r)$. Thus,

$$\bar{\mu}(B(y, r)) \le \bar{\mu}(B(y, 2r)) \le \frac{m(B(y, 2r))}{\prod_{i=0}^{l} \Delta_i} \le Cr^{k-\epsilon'} \left(\frac{d_l^{\epsilon'}}{\prod_{i=0}^{l} \Delta_i} \right). \tag{31}$$

Whenever ϵ' is greater than $\epsilon = \limsup_{l\to\infty} ((\sum_{i=0}^{l} \log(1/\Delta_i(\mathcal{A})))/\log(1/d_l(\mathcal{A})))$, the term in the large brackets of (31) goes to zero as $l \to \infty$ and hence is bounded above. Note that the constant C in (31) is independent of ϵ'. So we have proved (2).

Part (3) of the lemma follows from (2) and Lemma 5.1. □

The following lemma, which is essentially proved in [**17**, Theorem 2.7], relates Lemma 5.2 and Schmidt games.

LEMMA 5.3. *Let X be a k-dimensional Riemannian manifold and let $S \subset X$ be an α-winning set for Schmidt games played on X. Then, for any open $U \subset X$,*

$$\dim_H(S \cap U) = k.$$

Proof. Consider (α, β)-Schmidt games. We will construct a family $\mathcal{A}$ satisfying (25)–(29) whose limit set $\mathbf{A}_\infty$ is a subset of $S \cap U$. It will be constructed by considering possible moves for Bob at each stage and Alice's corresponding counter-moves, which give us different sequences as in (3). We will take the measure m in Lemma 5.2 to be the volume measure ν on X. Let $\rho_0 > 0$ be small enough as in Definition 4.1 and have the following property: for any $0 < \rho < \rho_0$, any ball of radius ρ on X contains at least $N(\beta)$ disjoint balls of radius $\beta\rho$, where $N(\beta) \ge c\beta^{-k}$ for some constant $c > 0$.

Bob may begin the game by choosing a ball $B(\omega_1) \subset U$ with $r_1 < \rho_0$. Since S is winning, Alice can choose a ball $B(\omega_1')$ which has non-empty intersection with S. Take $A_0 := B(\omega_1')$; then (25) is satisfied.

By the choice of r_1, $B(\omega_1')$ contains $N(\beta)$ disjoint balls of radius $\beta r_1'$, say $B(\omega_2^{(1)}), \ldots, B(\omega_2^{(N(\beta))})$, and each of them could be chosen by Bob as $B(\omega_2)$. For each such choice $B(\omega_2^{(i)})$ of Bob, Alice can pick a ball $B((\omega')_2^{(i)}) \subset B(\omega_2^{(i)})$. Let $\mathcal{A}_1$ be the collection of $N(\beta)$ balls $B((\omega')_2^{(i)})$, i.e. the balls chosen by Alice. Repeating the same moves for each turn of the game, we obtain $\mathcal{A}_2$, $\mathcal{A}_3$, etc. Conditions (26) and (27) are

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immediate from the construction above. Equation (28) follows from $d_l(\mathcal{A}) = 2r_1\alpha(\alpha\beta)^l \to 0$ as $l \to \infty$.

Let us verify (29). By Lemma 4.2, for any $0 < \rho < \rho_0$,

$$\Delta_l(\mathcal{A}) = \frac{N(\beta)\nu(B(\omega'_{l+2}))}{\nu(B(\omega'_{l+1}))} \geq \frac{c\beta^{-k}c_1(r_1\alpha(\alpha\beta)^{l+1})^k}{c_2(r_1\alpha(\alpha\beta)^l)^k} \geq \frac{cc_1\alpha^k}{c_2} > 0.$$

By Lemma 5.2,

$$\begin{aligned} \dim_H(\mathbf{A}_\infty) &\geq k - \limsup_{l\to\infty} \frac{\sum_{i=0}^{l} \log(1/\Delta_i(\mathcal{A}))}{\log(1/d_l(\mathcal{A}))} \\ &\geq k - \frac{\log\,(c_2/(cc_1\alpha^k))}{\log\,(1/\alpha\beta)} \\ &\to k \quad \text{as } \beta \to 0. \end{aligned} \tag{32}$$

□

Remark 5.4. The family $\mathcal{A}$ in the construction above is obviously not unique. It depends on the choice of β, the choice of Bob's first move (in particular ρ_0), and the choice of all the possible following moves of Alice and Bob.

We now combine the constructions that appear in the proofs of Lemmas 5.2 and 5.3. Recall the definition of $\Pi(x_0)$ in (24), and let $x \in \overline{B^W(x_0, \delta/2)}$.

PROPOSITION 5.5. *For any $\epsilon > 0$, any $x \in \overline{B^W(x_0, \delta/2)}$, there exists a sequence of measures $\{\mu_x^{(l)}\}_{l=0}^{\infty}$ and $\{\mu_x\}$ such that:*

(1) *$\mu_x^{(l)}$ is supported on $B^u(x, \delta/2)$, and μ_x is supported on $B^u(x, \delta/2) \cap E_x(f, y)$;*
(2) *μ_x is the unique weak limit of $\mu_x^{(l)}$;*
(3) *for any $z \in B^u(x, \delta/2) \cap \operatorname{supp}(\mu_x)$, $r > 0$ small enough,*

$$\mu_x(B^u(z, r)) \leq Cr^{u-\epsilon}. \tag{33}$$

Proof. For any $\epsilon > 0$, we can choose β small enough as in the proof of Lemma 5.3 such that the second expression in (32) is greater than $u - \epsilon$ (now $k = u$). For each $x \in \overline{B^W(x_0, \delta/2)}$, we can construct a family $\mathcal{A}_x = \{\mathcal{A}_x^{(l)}\}_{l=0}^{\infty}$ of collections of subsets of $B^u(x, \delta/2)$ with limit set $\mathbf{A}_x^{\infty} \subset B^u(x, \delta/2) \cap E_x(f, y)$ as in the proof of Lemma 5.3. Following the proof of Lemma 5.2, we can construct $\mu_x^{(l)}$ supported on $\mathbf{A}_x^{(l)}$ and μ_x supported on $B^u(x, \delta/2) \cap E_x(f, y)$ such that μ_x is the unique weak limit of $\mu_x^{(l)}$. Moreover, (33) follows from (2) in Lemma 5.2. From the proof of (2) in Lemma 5.2, we know that the constant C in (33) depends only on the underlying manifold M, and hence can be chosen to be independent of x. This fact is important in the proof of Lemma 5.11 below. □

Remark 5.6. Since the construction of $\mu_x^{(l)}$ and μ_x in Proposition 5.5 relies on the construction of $\mathcal{A}_x^{(l)}$, they are not unique.

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5.2. *Measurability of* $\{\mu_x\}$. Recall that the choice of $\mu_x^{(l)}$ and μ_x is not unique for any $x \in \overline{B^W(x_0, \delta/2)}$. We shall show that there there exist choices of each $\mu_x^{(l)}$ and μ_x such that $x \mapsto \mu_x^{(l)}$ and $x \mapsto \mu_x$ are measurable with respect to the volume measure ν on W. So we need to specify the choice of the families $\mathcal{A}_x^{(l)}$ for any l and any $x \in \overline{B^W(x_0, \delta/2)}$.

PROPOSITION 5.7. *There exist a sequence of finite partitions* $\mathcal{P}^{(l)}$ *of* $\overline{B^W(x_0, \delta/2)}$*, a family* $\mathcal{A}_x = \{\mathcal{A}_x^{(l)}\}_{l=0}^{\infty}$ *of collections of subsets of* $B^u(x, \delta/2)$ *and a family of measures* $\{\mu_x^{(l)}\}$ *such that:*

(1) *for each element* $P_i^{(l)} \in \mathcal{P}^{(l)}$*, the interior of* $P_i^{(l)}$*, denoted as* $\mathrm{Int}(P_i^{(l)})$*, is an open and connected set, and* $\nu(\partial P_i^{(l)}) = 0$*;*

(2) $\mathcal{P}^{(l)} \le \mathcal{P}^{(l+1)}$*;*

(3) *for each* $x \in \overline{B^W(x_0, \delta/2)}$*, each* $l \in \mathbb{N}_0$*,* $\mu_x^{(l)}$ *is supported on* $\mathbf{A}_x^{(l)}$ *and obtained as in Proposition 5.5—and, moreover,* $\bigcup_{x \in \mathrm{Int}(P_i^{(l)})} \mathbf{A}_x^{(l)}$ *is a union of finitely many open and connected sets;*

(4) *for any* $l \in \mathbb{N}_0$*,* $x \mapsto \mu_x^{(l)}$ *is continuous on each* $\mathrm{Int}(P_i^{(l)})$*, and hence is measurable with respect to the volume measure* ν *on* $\overline{B^W(x_0, \delta/2)}$.

Proof. Our proof is by induction on l. First, consider $l = 0$. For each $x \in \overline{B^W(x_0, \delta/2)}$ let Bob choose a ball $B_x^u(\omega_1)$ satisfying (9) (where ρ_0 is as in the proof of Lemma 5.3) such that $\bigcup_{x \in \overline{B^W(x_0, \delta/2)}} B_x^u(\omega_1)$ is open and connected. Since there is no $I_x^{(k)}$, the kth preimages of $\Pi(c)$, for Alice to avoid at the first step, Alice can choose $B_x^u(\omega_1') \subset B_x^u(\omega_1)$ such that $\bigcup_{x \in \overline{B^W(x_0, \delta/2)}} B_x^u(\omega_1')$ is open and connected. Set $\mathbf{A}_x^{(0)} = B_x^u(\omega_1')$ and $\mu_x^{(0)} = \nu|_{B_x^u(\omega_1')}$. Then $\mathcal{P}^{(0)} = \{\overline{B^W(x_0, \delta/2)}\}$, and $x \mapsto \mu_x^{(0)}$ is obviously continuous on $B^W(x_0, \delta/2)$.

Suppose that the conclusion is true for $0, \ldots, l-1$; let us prove it for l. Since $\mathcal{A}_x^{(l)}$ will be constructed in the $(l+1)$th turn of Schmidt games, let us suppose that $jr < l + 1 \le (j+1)r$, i.e. we are in the jth step in the proof of Theorem 4.15. In the jth step, Alice only needs to avoid some of the I_k with $R(I_k) \ge \alpha\rho(\alpha\beta)^{(j+2)r-1}$. Since $R(I_k) \le c/\sigma_1^k$, we have

$$\alpha\rho(\alpha\beta)^{(j+2)r-1} \le \frac{c}{\sigma_1^k}.$$

Hence $k \le \log\left(c/(\alpha\rho(\alpha\beta)^{(j+2)r-1})\right)/\log\sigma_1 := K(j)$, that is, k is bounded above when j is fixed. Let $\pi : \Pi(x_0) \to \overline{B^W(x_0, \delta/2)}$ be the natural map such that $\pi(z)$ is the unique point in $B^u(z, \delta/2) \cap \overline{B^W(x_0, \delta/2)}$. Consider the boundaries of the projection to $\overline{B^W(x_0, \delta/2)}$ under π of all these kth preimages of $\Pi(c)$, i.e. $\bigcup_{k \le K(j)} \partial(\pi(f^{-k}(\Pi(c)) \cap \Pi(x_0)))$. Let $\mathcal{Q}^{(l)}$ be the partition of $\overline{B^W(x_0, \delta/2)}$ into connected sets such that

$$\bigcup_{Q \in \mathcal{Q}^{(l)}} \partial Q = \bigcup_{k \le K(j)} \partial(\pi(f^{-k}(\Pi(c)) \cap \Pi(x_0))) \bigcup \partial(B^W(x_0, \delta/2)).$$

Then by definition of $\mathcal{Q}^{(l)}$, for each $Q \in \mathcal{Q}^{(l)}$ the set $f^{-k}(\Pi(c)) \bigcap \bigcup_{x \in Q} B^u(x, \delta/2)$ is the union of finitely many open and connected sets. Since the number of choices of k is finite, the partition $\mathcal{Q}^{(l)}$ is finite and $\mathcal{Q}^{(l-1)} \le \mathcal{Q}^{(l)}$. Note that $\mathcal{Q}^{(l)}$ can be trivial if $\pi(f^{-k}(\Pi(c)) \cap \Pi(x_0)) \supset \overline{B^W(x_0, \delta/2)}$ for all $k \le K(j)$. Set $\mathcal{P}^{(l)} = \mathcal{P}^{(l-1)} \vee \mathcal{Q}^{(l)}$. We have proved (1) and (2), and in fact the $\mathcal{P}^{(l)}$ are same for all l in the jth step, i.e. with $jr < l + 1 \le (j+1)r$.

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We now prove (3) and (4). Consider $P_i^{(l)} \in \mathcal{P}^{(l)}$. Since $\mathcal{P}^{(l-1)} \le \mathcal{P}^{(l)}$, by the induction hypothesis $\bigcup_{x\in P_i^{(l)}} \mathbf{A}_x^{(l-1)}$ is a finite union of open and connected sets. So inside each of Alice's balls in the lth turn, we can let Bob choose the $N(\beta)$ balls $B_x^u(\omega_{l+1})$ on the $(l+1)$th turn such that each $\bigcup_{x\in P_i^{(l)}} B_x^u(\omega_{l+1})$ is open and connected. Then by the definition of $\mathcal{P}^{(l)}$, $f^{-k}(\Pi(c)) \bigcap \bigcup_{x\in P_i^{(l)}} B_x^u(\omega_{l+1})$ is a finite union of open and connected sets for each $k \le K(j)$. (In general the intersection of two connected sets may have infinitely many connected components. But here on each unstable manifold we have intersection of balls and ellipses which are almost like balls, so we should have finitely many connected components.) Note that in the $(l+1)$th turn, it is enough for Alice to avoid some of the I_k with $k \le K(j)$, and hence $\bigcup_{x\in P_i^{(l)}} (I_x^{(k)} \cap B_x^u(\omega_{l+1}))$ is open and connected for each such k. By avoiding the above open and connected sets, Alice can choose balls $B_x^u(\omega'_{l+1})$ such that $\bigcup_{x\in P_i^{(l)}} B_x^u(\omega'_{l+1})$ is open and connected. Alice can do so because of the continuity of the unstable foliation and because we can let δ be small enough in the definition of $\Pi(x_0)$ such that the position of $I_x^{(k)} \cap B_x^u(\omega_{l+1})$ on unstable manifolds varies only slightly as x varies in $P_i^{(l)}$. We have proved (3). Part (4) is immediate from (3) and the fact that $\mu_x^{(l)}$ is obtained by a rescaling of the volume measure in each of the l steps according to (30). □

PROPOSITION 5.8. *Fix an arbitrary small $\epsilon > 0$. For each $x \in \overline{B^W(x_0, \delta/2)}$, there exists a measure μ_x supported on $B^u(x, \delta/2) \cap E_x(f, y)$, such that:*

(1) *for any $z \in B^u(x, \delta/2)$, $\mu_x(B^u(z, r)) \le Cr^{u-\epsilon}$;*

(2) *$x \mapsto \mu_x$ is measurable with respect to the volume measure ν on $\overline{B^W(x_0, \delta/2)}$.*

Proof. As in the proof of Proposition 5.5, we take μ_x as the unique weak limit of $\mu_x^{(l)}$ in Proposition 5.7, and (1) is exactly (33). Since $x \mapsto \mu_x^{(l)}$ is measurable, so is $x \mapsto \mu_x$, and we get (2). □

5.3. *Measure μ.* Now we define the measure μ supported on $\Pi(x_0) \cap E(f, y)$.

Definition 5.9. For any Borel set $A \subset M$, define

$$\mu(A) := \int_{\overline{B^W(x_0,\delta/2)}} \int_{B^u(x,\delta/2)} \chi_A(z)\, d\mu_x(z)\, d\nu(x),$$

where χ_A is the characteristic function of A, ν is the volume measure on $\overline{B^W(x_0, \delta/2)}$, and μ_x is as in Proposition 5.8. μ is well-defined since $x \mapsto \mu_x$ is measurable with respect to ν.

Fix $0 < \tau < 1$. For any $z \in \Pi(x_0)$, define

$$B_l(z) := B(z, \tau^l)$$

and

$$C_l(z) := \bigcup_{u\in B^W(z,\tau^l)} B^u(u, \tau^l).$$

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LEMMA 5.10. *There exists an $l_0 > 0$ such that, for any $l \geq l_0$ and any $z \in \Pi(x_0)$,*

$$C_l(z) \subset B_{l-l_0}(z) \quad \textit{and} \quad B_l(z) \subset C_{l-l_0}(z).$$

Proof. Suppose that $w \in C_l(z)$ for some l large enough. Let $u \in W(z) \cap W^u(w)$. Then

$$d(z, w) \leq d(z, u) + d(u, w) \leq d^W(z, u) + d^u(u, w) \leq 2\tau^l \leq \tau^{l-l_0},$$

if we pick some l_0 with $2 \leq \tau^{-l_0}$. Hence $w \in B_{l-l_0}(z)$. We get the first inclusion.

For the second inclusion, suppose that $w \in B_l(z)$ and let $u \in W(z) \cap W^u(w)$. Since the angles between W^u and W are uniformly bounded away from zero, there exists a $C_1 > 0$ such that (see [**13**, Lemma B.9.1]):

$$d^W(z, u) \leq C_1 d(z, w), \quad \text{and} \quad d^u(u, w) \leq C_1 d(z, w).$$

Thus $d^W(z, u) \leq C_1\tau^l \leq \tau^{l-l_0}$ and $d^u(u, w) \leq C_1\tau^l \leq \tau^{l-l_0}$ for some l_0 with $C_1 \leq \tau^{-l_0}$. Therefore, $w \in C_{l-l_0}(z)$; hence $B_l(z) \subset C_{l-l_0}(z)$. □

LEMMA 5.11. *For any $r > 0$ small enough and any $z \in \Pi(x_0)$,*

$$\mu(B(z, r)) \leq Dr^{n-\epsilon}$$

for some constant $0 < D < \infty$.

Proof. There exists a $l > 0$ such that $\tau^{l+1} \leq r < \tau^l$. Let h^u be the holonomy map along W^u from $W(z)$ to $W(x_0)$. Since W^u is transversally absolutely continuous with bounded Jacobians (see [**2**, **4**, **22**]), there exists $C_2 > 1$ such that, for any Borel $A \subset W(z)$,

$$C_2^{-1}\nu(A) \leq \nu(h^u(A)) \leq C_2\nu(A). \tag{34}$$

Then

$$\begin{aligned}
\mu(B(z, r)) &\leq \mu(B(z, \tau^l)) \leq \mu(C_{l-l_0}(z)) \\
&= \int_{h^u(B^W(z,\tau^{l-l_0}))} \int_{B^u((h^u)^{-1}(x),\tau^{l-l_0})} d\mu_x(w)\, d\nu(x) \\
&\leq C \int_{h^u(B^W(z,\tau^{l-l_0}))} (\tau^{l-l_0})^{u-\epsilon}\, d\nu(x) \quad \text{(by Proposition 5.8(1))} \\
&\leq CC_2\nu(B^W(z, \tau^{l-l_0}))(\tau^{l-l_0})^{u-\epsilon} \quad \text{(by (34))} \\
&\leq CC_2C_3(\tau^{l-l_0})^{c+s}(\tau^{l-l_0})^{u-\epsilon} \\
&= CC_2C_3(\tau^{l-l_0})^{n-\epsilon} \\
&\leq CC_2C_3(\tau^{-l_0-1})^{n-\epsilon}r^{n-\epsilon} \quad (\text{by } \tau^{l+1} \leq r) \\
&:= Dr^{n-\epsilon}.
\end{aligned}$$

□

Proof of Main Theorem 2. For any non-empty open subset $V \subset M$, we can find some $x_0 \in V$, a local foliation W transversal to W^u near x_0, and $\delta > 0$ small enough such that $\Pi(x_0, W, \delta) \subset V$. Main Theorem 2 follows immediately from Lemmas 5.1 and 5.11 and letting $\epsilon \to 0$ in Proposition 5.8. □

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Proof of Corollary 2.4. Let $Y = \{y_t\}_{t=1}^{\infty}$. Then $E_x(f, Y) = \bigcap_{t=1}^{\infty} E_x(f, y_t)$ is also a winning set for Schmidt games played on $W^u(x)$. Let us recall a proof of this fact due to [**24**]. Let α be as in (4). We know that $E_x(f, y_t)$ is α-winning for all $t \in \mathbb{N}$, and we want to show that $E_x(f, Y)$ is (α, β)-winning for any $0 < \beta < 1$. Here is the strategy for Alice to win. At the first, third, fifth, ... turns, Alice uses an $(\alpha, \beta\alpha\beta; E_x(f, y_1))$-winning strategy which forces $\bigcap_{j=1}^{\infty} B(\omega_{1+2(j-1)}) \subset E_x(f, y_1)$. At the second, sixth, tenth, ... turns, Alice uses an $(\alpha, \beta(\alpha\beta)^3; E_x(f, y_2))$-winning strategy which forces $\bigcap_{j=1}^{\infty} B(\omega_{2+2^2(j-1)}) \subset E_x(f, y_2)$. In general, at the kth turn with $k \equiv 2^{t-1} (\mathrm{mod}\, 2^t)$, Alice uses an $(\alpha, \beta(\alpha\beta)^{2^t-1}; E_x(f, y_t))$-winning strategy which forces $\bigcap_{j=1}^{\infty} B(\omega_{2^{t-1}+2^t(j-1)}) \subset E_x(f, y_t)$. By this strategy, Alice can force the unique point in the intersection of all balls to be in $E_x(f, Y)$.

Now we construct the measure μ supported on $\Pi(x_0) \cap E(f, Y)$ as in Definition 5.9. We can construct $\mu_x^{(l)}$ and μ_x as in Proposition 5.5 supported on $B^u(x, \delta/2) \cap E_x(f, Y)$ since $E_x(f, Y)$ is a winning set for Schmidt games. So it is enough to specify a choice such that $x \mapsto \mu_x$ is measurable. The idea is same as in the proof of Proposition 5.7. The difference is that at the lth step of the induction (i.e. the $(l+1)$th turn of the game) with $l + 1 \equiv 2^{t-1} (\mathrm{mod}\, 2^t)$, Alice needs to avoid some $I_x^{(k)}(y_t)$, the preimages of $\Pi(y_t, c_t)$, where $\Pi(y_t, c_t)$ is the open rectangle neighborhood of y_t as in the proof of Theorem 4.15 for the $(\alpha, \beta(\alpha\beta)^{2^t-1}; E_x(f, y_t))$ Schmidt games. If $l + 1 = 2^{t-1} + 2^t(j-1)$, then k is bounded above by some number $K(j)$, and there are only finitely many such k. So the argument in Proposition 5.7 still works, i.e. at the $(l+1)$th turn, there exists a finite partition $\mathcal{P}^{(l)}$, and Alice can choose balls such that $\bigcup_{x \in P_i^{(l)}} \mathbf{A}_x^{(l)}$ is a finite union of open and connected sets. Hence we have that $x \mapsto \mu_x$ is measurable as before. □

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References

[**1**] C. S. Aravinda and E. Leuzinger. Bounded geodesics in rank-1 locally symmetric spaces. *Ergod. Th. & Dynam. Sys.* **15**(5) (1995), 813–820.
[**2**] M. Brin and Ja. B. Pesin. Partially hyperbolic dynamical systems. *Math. USSR-Izv.* **8**(1) (1974), 117–218.
[**3**] R. Broderick, L. Fishman and D. Y. Kleinbock. Schmidt's game, fractals, and orbits of toral endomorphisms. *Ergod. Th. & Dynam. Sys.* **31** (2011), 1095–1107.
[**4**] K. Burns and A. Wilkinson. On the ergodicity of partially hyperbolic systems. *Ann. of Math. (2)* **171** (2010), 451–489.
[**5**] S. G. Dani. Divergent trajectories of flows on homogeneous spaces and Diophantine approximation. *J. Reine Angew. Math.* **359** (1985), 55–89, 102.
[**6**] S. G. Dani. Bounded orbits of flows on homogeneous spaces. *Comment. Math. Helv.* **61**(1) (1986), 636–660.
[**7**] S. G. Dani. On orbits of endomorphisms of tori and the Schmidt game. *Ergod. Th. & Dynam. Sys.* **8**(4) (1988), 523–529.
[**8**] S. G. Dani and H. Shah. Badly approximable numbers and vectors in Cantor-like sets. *Proc. Amer. Math. Soc.* **140**(8) (2012), 2575–2587.

https://doi.org/10.1017/etds.2014.136 Published online by Cambridge University Press

[**9**] D. Dolgopyat. Bounded orbits of Anosov flows. *Duke Math. J.* **87**(1) (1997), 87–114.
[**10**] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. Wiley, Chichester, 2003.
[**11**] L. Fishman, D. S. Simmons and M. Urbański. Diophantine approximation and the geometry of limit sets in Gromov hyperbolic metric spaces (extended version). *Preprint*, 2013, arXiv:1301.5630.
[**12**] J. M. Franks. Invariant sets of hyperbolic toral automorphisms. *Amer. J. Math.* **99**(5) (1977), 1089–1095.
[**13**] F. R. Hertz, M. A. R. Hertz and R. Ures. Accessibility and stable ergodicity for partially hyperbolic diffeomorphisms with 1d-center bundle. *Invent. Math.* **172**(2) (2008), 353–381.
[**14**] A. Katok. *Nonuniform Hyperbolicity and Structure of Smooth Dynamical Systems (Proc. Int. Congress Math., Vol. 2, Warszawa, 1983)*. Polish Scientific Publishers PWN, Warsaw, 1984, pp. 1245–1254.
[**15**] A. Katok and L. Mendoza. Dynamical systems with non-uniformly hyperbolic behavior. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge, 1995, pp. 659–700.
[**16**] D. Y. Kleinbock and G. A. Margulis. Bounded orbits of non-quasiunipotent flows on homogeneous spaces. *Amer. Math. Soc. Transl.* **171** (1996), 141–172.
[**17**] D. Y. Kleinbock and B. Weiss. Modified Schmidt games and Diophantine approximation with weights. *Adv. Math.* **223**(4) (2010), 1276–1298.
[**18**] D. Y. Kleinbock and B. Weiss. Modified Schmidt games and a conjecture of Margulis. *Preprint*, 2010, arXiv:1001.5017.
[**19**] R. Mañé. Orbits of paths under hyperbolic toral automorphisms. *Proc. Amer. Math. Soc.* **73**(1) (1979), 121–125.
[**20**] C. McMullen. Area and Hausdorff dimension of Julia sets of entire functions. *Trans. Amer. Math. Soc.* **300**(1) (1987), 329–342.
[**21**] F. Przytycki. Construction of invariant sets for Anosov diffeomorphisms and hyperbolic attractors. *Studia Math.* **68**(2) (1980), 199–213.
[**22**] C. Pugh and M. Shub. Ergodicity of Anosov actions. *Invent. Math.* **15**(1) (1972), 1–23.
[**23**] A. Quas and T. Soo. Ergodic universality of some topological dynamical systems. *Preprint*, 2012, arXiv:1208.3501.
[**24**] W. M. Schmidt. On badly approximable numbers and certain games. *Trans. Amer. Math. Soc.* **123**(1) (1966), 178–199.
[**25**] W. M. Schmidt. Badly approximable systems of linear forms. *J. Number Theory* **1**(2) (1969), 139–154.
[**26**] P. Sun. Zero-entropy invariant measures for skew product diffeomorphisms. *Ergod. Th. & Dynam. Sys.* **30**(3) (2010), 923–930.
[**27**] P. Sun. Measures of intermediate entropies for skew product diffeomorphisms. *Discrete Contin. Dyn. Syst.* **27** (2010), 1219–1231.
[**28**] P. Sun. Density of metric entropies for linear toral automorphisms. *Dyn. Syst.* **27**(2) (2012), 197–204.
[**29**] M. Urbański. The Hausdorff dimension of the set of points with non-dense orbit under a hyperbolic dynamical system. *Nonlinearity* **4**(2) (1991), 385–397.

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