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Shadowing and structural stability for operators

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Abstract. A well-known result in the area of dynamical systems asserts that any invertible hyperbolic operator on any Banach space is structurally stable. This result was originally obtained by Hartman in 1960 for operators on finite-dimensional spaces. The general case was independently obtained by Palis and Pugh around 1968. We will exhibit a class of examples of structurally stable operators that are not hyperbolic, thereby showing that the converse of the above-mentioned result is false in general. We will also prove that an invertible operator on a Banach space is hyperbolic if and only if it is expansive and has the shadowing property. Moreover, we will show that if a structurally stable operator is expansive, then it must be uniformly expansive. Finally, we will characterize the weighted shifts on the spaces $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$ $(1 \leq p < \infty)$ that satisfy the shadowing property.

Key words: linear operators, shadowing, expansivity, hyperbolicity, structural stability
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1. *Introduction*

Given a metric space M with metric d and a homeomorphism $h : M \to M$, recall that a sequence $(x_n)_{n\in\mathbb{Z}}$ in M is called a δ-*pseudotrajectory* of h, where $\delta > 0$, if

$$d(h(x_n), x_{n+1}) \leq \delta \quad \text{for all } n \in \mathbb{Z}.$$

The homeomorphism h is said to have the *shadowing property* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudotrajectory $(x_n)_{n\in\mathbb{Z}}$ of h is ε-*shadowed* by a real trajectory of h, that is, there exists $x \in M$ such that

$$d(x_n, h^n(x)) < \varepsilon \quad \text{for all } n \in \mathbb{Z}.$$

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This notion, which comes from the works of Sinaĭ [**30**] and Bowen [**11**], plays a fundamental role in several branches of the area of dynamical systems. We refer the reader to the books [**19**, **25**, **26**] for nice expositions of the shadowing property and its applications.

Another fundamental notion in the area of dynamical systems is that of *hyperbolicity*. For (continuous linear) operators T on complex Banach spaces, this notion can be defined by requiring that the spectrum $\sigma(T)$ of T does not intersect the unit circle $\mathbb{T}$. In the case of real Banach spaces, it is required that $\sigma(T_{\mathbb{C}}) \cap \mathbb{T} = \emptyset$, where $T_{\mathbb{C}}$ denotes the complexification of T. For invertible operators on Banach spaces, it was well known that hyperbolicity implies the shadowing property and that the converse holds on finite-dimensional spaces [**23**] and for normal operators on Hilbert spaces [**20**]. It remained open for a while whether or not this converse was always true. This problem was finally solved by Bernardes *et al* [**10**], where examples of non-hyperbolic invertible operators with the shadowing property were obtained. In view of this fact, it is natural to ask which additional condition one has to add to shadowing in order to get hyperbolicity. The answer is *expansivity*. Indeed, we will establish the following result in §3. *An invertible operator T on a Banach space X is hyperbolic if and only if it is expansive and has the shadowing property*.

Yet another fundamental notion in the area of dynamical systems is that of *structural stability*, which was introduced by Andronov and Pontrjagin [**2**]. Actually, there are many variations of this notion. Here we will follow Pugh [**27**]. Given a Banach space X, we denote by $C_b(X)$ the Banach space of all bounded continuous maps $\varphi : X \to X$ endowed with the supremum norm

$$\|\varphi\|_\infty = \sup_{x \in X} \|\varphi(x)\|.$$

Recall that a map $\varphi : X \to X$ is said to be *Lipschitz* if the constant

$$\mathrm{Lip}(\varphi) = \sup_{x \neq y} \frac{\|\varphi(x) - \varphi(y)\|}{\|x - y\|}$$

is finite. Recall also that two maps $\varphi : X \to X$ and $\psi : X \to X$ are *topologically conjugate* if there is a homeomorphism $h : X \to X$ such that $h \circ \varphi = \psi \circ h$. An invertible operator T on X is said to be *structurally stable* if there exists $\varepsilon > 0$ such that $T + \varphi$ is topologically conjugate to T whenever $\varphi \in C_b(X)$ is a Lipschitz map with $\|\varphi\|_\infty \leq \varepsilon$ and $\mathrm{Lip}(\varphi) \leq \varepsilon$. Moreover, T is said to be *structurally stable relative to* $\mathrm{GL}(X)$ (the group of all invertible operators on X) if there exists $\varepsilon > 0$ such that S is topologically conjugate to T whenever $S \in \mathrm{GL}(X)$ and $\|S - T\| \leq \varepsilon$ (here $\|\cdot\|$ denotes the operator norm). The following result is classical. *Every invertible hyperbolic operator T on a Banach space X is*:

(a) *structurally stable*;

(b) *structurally stable relative to* $\mathrm{GL}(X)$.

Conclusion (a) is often called Hartman's theorem. The original version was obtained by Hartman [**17**] in the finite-dimensional setting. The general case was independently obtained by Palis [**24**] and Pugh [**27**], motivated by an argument in Moser [**22**].

We will prove in §3 that there is a class $\mathcal{C}$ of invertible operators on certain Banach spaces X such that each operator in $\mathcal{C}$ is structurally stable, but it is not hyperbolic and it is

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not structurally stable relative to $\mathrm{GL}(X)$. This shows that the notion of structural stability for operators does not coincide with the notion of hyperbolicity and does not coincide with structural stability relative to $\mathrm{GL}(X)$. It seems still to be an open problem whether or not hyperbolicity and structural stability relative to $\mathrm{GL}(X)$ coincide. Nevertheless, it is well known that these three notions coincide in the finite-dimensional setting (see [**28**, Theorem 2.4]).

Surprisingly enough, the structurally stable operators in the class $\mathcal{C}$ have non-trivial fixed points and satisfy the so-called *frequent hypercyclicity criterion* [**16**, §9.2], which is a very strong notion of chaos in linear dynamics. It simultaneously implies frequent hypercyclicity, mixing, Devaney chaos, dense distributional chaos and dense mean Li–Yorke chaos [**7**, **9**, **16**].

We will also prove the following result in §3. *Suppose that an invertible operator T on a complex Banach space X is structurally stable or structurally stable relative to* $\mathrm{GL}(X)$. *Then the following facts hold.*

(a) *If T is expansive, then T is uniformly expansive.*

(b) *If T is positively expansive, then T is hyperbolic.*

Moreover, we will show that if an invertible weighted shift on a space $\ell_p(\mathbb{Z})$ $(1 \leq p < \infty)$ or $c_0(\mathbb{Z})$ is expansive and strongly structurally stable, then it must be hyperbolic.

Due to the importance of *weighted shifts* in the area of operator theory and its applications, many researchers have extensively studied the dynamics of this class of operators (see the books [**5**, **16**] and the papers [**4**, **6**–**8**, **10**, **13**, **15**, **29**], for instance). In §4 we will contribute to this line of investigation by establishing the following characterization of the weighted shifts that satisfy the shadowing property. *Let $X = \ell_p(\mathbb{Z})$ $(1 \leq p < \infty)$ or $X = c_0(\mathbb{Z})$, let $w = (w_n)_{n\in\mathbb{Z}}$ be a bounded sequence of scalars with* $\inf_{n\in\mathbb{Z}} |w_n| > 0$, *and consider the bilateral weighted backward shift*

$$B_w : (x_n)_{n\in\mathbb{Z}} \in X \mapsto (w_{n+1}x_{n+1})_{n\in\mathbb{Z}} \in X.$$

Then B_w has the shadowing property if and only if one of the following conditions holds:

(A) $\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} |w_k w_{k+1} \cdot \ldots \cdot w_{k+n}|^{1/n} < 1$;

(B) $\lim_{n\to\infty} \inf_{k\in\mathbb{Z}} |w_k w_{k+1} \cdot \ldots \cdot w_{k+n}|^{1/n} > 1$;

(C) $\lim_{n\to\infty} \sup_{k\in\mathbb{N}} |w_{-k} w_{-k-1} \cdot \ldots \cdot w_{-k-n}|^{1/n} < 1$ *and* $\lim_{n\to\infty} \inf_{k\in\mathbb{N}} |w_k w_{k+1} \cdot \ldots \cdot w_{k+n}|^{1/n} > 1$.

We observe that (A) and (B) are exactly the cases in which the operator B_w is hyperbolic [**10**, Remark 35]. In these cases, B_w does not exhibit chaotic behaviour, since hyperbolic operators are not even Li–Yorke chaotic [**10**, Theorem C], which is a very weak notion of chaos. On the other hand, we will see later that condition (C) implies the frequent hypercyclicity criterion, and so, in this case, B_w exhibits several types of chaotic behaviours.

2. *Preliminaries*

The letter $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Moreover, $\mathbb{D}$ and $\mathbb{T}$ denote the open unit disc and the unit circle in the complex plane $\mathbb{C}$, respectively.

Given a Banach space X, we denote by B_X and S_X the closed unit ball and the unit sphere of X, respectively. By an *operator* on X we mean a bounded linear map T from

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X into itself. $\mathcal{B}(X)$ denotes the space of all operators on X with the operator norm. If $T \in \mathcal{B}(X)$, then $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote the spectrum, the point spectrum and the approximate point spectrum of T. Recall that $\sigma_p(T)$ is the set of all eigenvalues of T and that $\lambda \in \sigma_a(T)$ if and only if there is a sequence $(x_n)_{n\in\mathbb{N}}$ in S_X such that $\|Tx_n - \lambda x_n\| \to 0$ as $n \to \infty$. Recall also that $\sigma_p(T) \subset \sigma_a(T) \subset \sigma(T)$ and that $\sigma_a(T)$ contains the boundary of $\sigma(T)$, which is a compact set. We denote by $r(T)$ the spectral radius of T. Recall that the spectral radius formula asserts that $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$ (see [**32**], for instance).

Let $c_0(\mathbb{Z})$ be the Banach space of all sequences $(x_n)_{n\in\mathbb{Z}}$ of scalars such that $\lim_{|n|\to\infty} x_n = 0$ endowed with the supremum norm. Let $\ell_p(\mathbb{Z})$ be the Banach space of all absolutely p-summable sequences $(x_n)_{n\in\mathbb{Z}}$ of scalars endowed with the p-norm, where $1 \le p < \infty$. Moreover, let $(e_n)_{n\in\mathbb{Z}}$ denote the sequence of canonical unit vectors in $c_0(\mathbb{Z})$ and in $\ell_p(\mathbb{Z})$.

Recall that a homeomorphism $f : M \to M$, where M is a metric space, is said to be *expansive* if there exists a constant $e > 0$ such that for every pair x, y of distinct points in M, there exists $n \in \mathbb{Z}$ with $d(f^n(x), f^n(y)) \ge e$. If T is an invertible operator on a Banach space X, this is equivalent to saying that there exists a constant $c > 1$ such that for each point $x \in S_X$, there exists $n \in \mathbb{Z}$ with $\|T^n x\| \ge c$. Recall also that T is said to be *uniformly expansive* if there exist $c > 1$ and $m \in \mathbb{N}$ such that

$$x \in S_X \implies \|T^m x\| \ge c \quad \text{or} \quad \|T^{-m} x\| \ge c.$$

Actually, in the notions of expansivity and uniform expansivity for operators, we can choose any $c > 1$ we want. We also recall that T is expansive if and only if $\sup_{n\in\mathbb{Z}} \|T^n x\| = \infty$ whenever $x \neq 0$. It is known that T is uniformly expansive if and only if $\sigma_a(T) \cap \mathbb{T} = \emptyset$ [**14**, **18**]. In particular, every invertible hyperbolic operator is uniformly expansive.

The following fact will be used a few times: *if T is an invertible operator on a Banach space X and $\varphi \in C_b(X)$ is a Lipschitz map with $\|\varphi\|_\infty$ and* $\mathrm{Lip}(\varphi)$ *sufficiently small, then $T + \varphi$ is a homeomorphism* (see [**27**, Lemma 1], for instance).

3. *Hyperbolicity, shadowing and structural stability*

We begin this section by recalling some notions from topological dynamics. Let M be a metric space with metric d and $h : M \to M$ a homeomorphism. For each $x \in M$ and $\varepsilon > 0$, we define the *ε-stable set*, the *ε-unstable set*, the *stable set* and the *unstable set of h at x* by

$$\begin{aligned}
W^s_\varepsilon(x, h) &= \{y \in M : d(h^n(x), h^n(y)) \le \varepsilon \text{ for all } n \ge 0\},\\
W^u_\varepsilon(x, h) &= \{y \in M : d(h^{-n}(x), h^{-n}(y)) \le \varepsilon \text{ for all } n \ge 0\},\\
W^s(x, h) &= \{y \in M : d(h^n(x), h^n(y)) \to 0 \text{ as } n \to \infty\},\\
W^u(x, h) &= \{y \in M : d(h^{-n}(x), h^{-n}(y)) \to 0 \text{ as } n \to \infty\},
\end{aligned}$$

respectively. The homeomorphism h is said to have *canonical coordinates* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) \le \delta$ implies $W^s_\varepsilon(x, h) \cap W^u_\varepsilon(y, h) \neq \emptyset$. Moreover, we say that h has *hyperbolic coordinates* if h has canonical coordinates and there exist constants $c \ge 1$, $0 < \beta < 1$ and $\gamma > 0$ such that the following properties hold for each $x \in M$:

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- $y \in W^s_\gamma(x, h) \implies d(h^n(x), h^n(y)) \le c\beta^n d(x, y)$ for all $n \in \mathbb{N}$;
- $y \in W^u_\gamma(x, h) \implies d(h^{-n}(x), h^{-n}(y)) \le c\beta^n d(x, y)$ for all $n \in \mathbb{N}$.

The original notion of 'canonical coordinates' came from Smale's investigations on Axiom A diffeomorphisms [**31**]. Here we are following the terminology of [**3**]. We will also consider the sets

$$W^{b,s}(x, h) = \{y \in M : (d(h^n(x), h^n(y)))_{n\in\mathbb{N}} \text{ is bounded}\},$$
$$W^{b,u}(x, h) = \{y \in M : (d(h^{-n}(x), h^{-n}(y)))_{n\in\mathbb{N}} \text{ is bounded}\}.$$

If T is an invertible operator on a Banach space X, we have that

$$W^s_\varepsilon(x, T) = x + W^s_\varepsilon(0, T), \quad W^s(x, T) = x + W^s(0, T),$$
$$W^{b,s}(x, T) = x + W^{b,s}(0, T),$$

and similarly with 'u' instead of 's'. Moreover, $W^{b,s}(0, T)$ and $W^{b,u}(0, T)$ are vector subspaces of X satisfying

$$T(W^{b,s}(0, T)) = W^{b,s}(0, T) \quad \text{and} \quad T(W^{b,u}(0, T)) = W^{b,u}(0, T).$$

THEOREM 1. *For any invertible operator T on any Banach space X, the following assertions are equivalent:*

(i) *T is hyperbolic;*
(ii) *T has hyperbolic coordinates;*
(iii) *T is expansive and has the shadowing property.*

For the proof we will need the following result.

PROPOSITION 2. *Let T be an invertible operator on a Banach space X. If T is uniformly expansive, then there are constants $c \ge 1$ and $0 < \beta < 1$ such that*

$$W^{b,s}(0, T) = W^s(0, T) = \{x \in X : \|T^n x\| \le c\beta^n \|x\| \text{ for all } n \in \mathbb{N}\}, \tag{1}$$

$$W^{b,u}(0, T) = W^u(0, T) = \{x \in X : \|T^{-n} x\| \le c\beta^n \|x\| \text{ for all } n \in \mathbb{N}\}. \tag{2}$$

In particular, the subspaces $W^{b,s}(0, T)$ and $W^{b,u}(0, T)$ are closed in X.

Proof. Since T is uniformly expansive, there exists $m \in \mathbb{N}$ such that $S_X = A \cup B$, where

$$A = \{x \in S_X : \|T^m x\| \ge 2\} \quad \text{and} \quad B = \{x \in S_X : \|T^{-m} x\| \ge 2\}.$$

Note that

$$x \in A \implies \frac{T^m x}{\|T^m x\|} \in A, \tag{3}$$

for otherwise we would get $1 = \|x\| \ge 2\|T^m x\| \ge 4$, a contradiction. Similarly,

$$x \in B \implies \frac{T^{-m} x}{\|T^{-m} x\|} \in B. \tag{4}$$

Let

$$\beta = \left(\frac{1}{2}\right)^{1/m} \quad \text{and} \quad c = \max\left\{\frac{\|T^j\|}{\beta^{|j|}} : -m < j < m\right\}.$$

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Let W denote the set on the right-hand side of (1). It is clear that $W \subset W^s(0, T) \subset W^{b,s}(0, T)$. Take $x \in W^{b,s}(0, T)\backslash\{0\}$. Given $n \in \mathbb{N}$, we define two sequences $(y_k)_{k\in\mathbb{N}}$ and $(z_k)_{k\in\mathbb{N}}$ as follows: $y_1 = z_1 = T^n x/\|T^n x\|$, $y_k = T^m y_{k-1}/\|T^m y_{k-1}\|$ and $z_k = T^{-m} z_{k-1}/\|T^{-m} z_{k-1}\|$ $(k \geq 2)$. Note that, for every $k \geq 2$,

$$y_k = \frac{T^{(k-1)m+n}x}{\|T^m y_1\| \cdot \ldots \cdot \|T^m y_{k-1}\|\|T^n x\|} \quad \text{and} \quad z_k = \frac{T^{-(k-1)m+n}x}{\|T^{-m} z_1\| \cdot \ldots \cdot \|T^{-m} z_{k-1}\|\|T^n x\|}.$$

If $y_1 \in A$, then (3) implies that $y_k \in A$ for all $k \in \mathbb{N}$, and so

$$\|T^{km+n}x\| = \|T^m y_1\| \cdot \ldots \cdot \|T^m y_k\| \cdot \|T^n x\| \geq 2^k \|T^n x\| \quad (k \in \mathbb{N}).$$

This contradicts the fact that $(T^j x)_{j\in\mathbb{N}}$ is bounded. Thus, we must have $z_1 = y_1 \in B$. By (4), $z_k \in B$ for all $k \in \mathbb{N}$, and so

$$\|T^{-km+n}x\| = \|T^{-m} z_1\| \cdot \ldots \cdot \|T^{-m} z_k\| \cdot \|T^n x\| \geq 2^k \|T^n x\| \quad (k \in \mathbb{N}).$$

Since $n \in \mathbb{N}$ is arbitrary, we may choose $n = km$ and obtain

$$\|T^{km}x\| \leq \frac{1}{2^k}\|x\| = \beta^{km}\|x\| \quad (k \in \mathbb{N}).$$

Now, an easy computation shows that $x \in W$. This proves (1). By applying (1) with T^{-1} instead of T we obtain (2). □

Remark 3. We cannot replace the hypothesis of uniform expansivity by expansivity in Proposition 2. For instance, let $X = \ell_p(\mathbb{Z})$ $(1 \leq p < \infty)$ or $X = c_0(\mathbb{Z})$, and let

$$B_w : (x_n)_{n\in\mathbb{Z}} \in X \mapsto (w_{n+1}x_{n+1})_{n\in\mathbb{Z}} \in X$$

be the bilateral weighted backward shift on X whose weight sequence $w = (w_n)_{n\in\mathbb{Z}}$ is given by $w_n = 2$ if $n < 0$, $w_n = 1$ if $n \geq 0$. Then B_w is positively expansive (for each $x \in S_X$, there exists $n \in \mathbb{N}$ such that $\|B_w^n x\| \geq 2$) and, in particular, it is expansive. However, $W^{b,u}(0, B_w) = X$ and $W^u(0, B_w) = \{0\}$.

Proof of Theorem 1. (i) ⇒ (iii). It is well known that every invertible hyperbolic operator on a Banach space is expansive (even uniformly expansive) [**14**] and has the shadowing property [**23**].

(iii) ⇒ (ii). Let us show that T has canonical coordinates. Fix $\varepsilon > 0$ arbitrary. Since T has the shadowing property, there exists $\delta \in (0, \varepsilon/2)$ such that every δ-pseudotrajectory of T is $\varepsilon/2$-shadowed by a real trajectory of T. Given $x, y \in X$ with $\|x - y\| \leq \delta$, we define

$$z_n = \begin{cases} T^n x & \text{if } n \geq 0, \\ T^n y & \text{if } n < 0. \end{cases}$$

By shadowing, there exists a point $z \in X$ such that

$$\|z_n - T^n z\| < \varepsilon/2 \quad \text{for all } n \in \mathbb{Z}.$$

By the definition of (z_n), we see that $z \in W_\varepsilon^s(x, T) \cap W_\varepsilon^u(y, T)$. Now, since T is expansive and has the shadowing property, T is uniformly expansive. Indeed, if this is not the case, then there exists $\lambda \in \sigma_a(T) \cap \mathbb{T}$ [**18**] and we can get a point $x_0 \in S_X$ such that

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$\|Tx_0 - \lambda x_0\| \leq \delta$. Since $(\lambda^n x_0)_{n\in\mathbb{Z}}$ is a δ-pseudotrajectory of T, there exists $y \in X$ such that $\|\lambda^n x_0 - T^n y\| < \varepsilon$ for all $n \in \mathbb{Z}$. This implies that $1 - \varepsilon < \|T^n y\| < 1 + \varepsilon$ for all $n \in \mathbb{Z}$, which contradicts the hypothesis that T is expansive. Thus, T is uniformly expansive and it follows from Proposition 2 that T has hyperbolic coordinates.

(ii) $\Rightarrow$ (i). Let $E = W^{b,s}(0, T)$ and $F = W^{b,u}(0, T)$. We know that E and F are vector subspaces of X satisfying $T(E) = E$ and $T(F) = F$. Let us prove that X decomposes as the topological direct sum of E and F. By hypothesis, there exist constants $c \geq 1$, $0 < \beta < 1$ and $\gamma > 0$ such that:

- $x \in W^s_\gamma(0, T) \implies \|T^n x\| \leq c\beta^n \|x\|$ for all $n \in \mathbb{N}$;
- $x \in W^u_\gamma(0, T) \implies \|T^{-n} x\| \leq c\beta^n \|x\|$ for all $n \in \mathbb{N}$.

This implies that

$$E = \{x \in X : \|T^n x\| \leq c\beta^n \|x\| \text{ for all } n \in \mathbb{N}\} \tag{5}$$

and

$$F = \{x \in X : \|T^{-n} x\| \leq c\beta^n \|x\| \text{ for all } n \in \mathbb{N}\}. \tag{6}$$

In particular, E and F are closed in X. Moreover, if $x \in E \cap F$ then

$$\|x\| = \|T^n T^{-n} x\| \leq c\beta^n \|T^{-n} x\| \leq c^2 \beta^{2n} \|x\|,$$

for all $n \in \mathbb{N}$, and so $x = 0$. This proves that $E \cap F = \{0\}$. Now let us prove that $E + F = X$. Since T has canonical coordinates, there exists $\delta > 0$ such that $W^s_1(x, T) \cap W^u_1(y, T) \neq \emptyset$ whenever $\|x - y\| \leq \delta$. It is enough to show that the ball $B_X(0; \delta)$ is contained in $E + F$. So take a point x in this ball. Then there must exist a point $z \in W^s_1(x, T) \cap W^u_1(0, T)$. This implies that $x - z \in E$ and $z \in F$. And, of course, $x = (x - z) + z$. Thus, $X = E \oplus F$. Finally, it follows from (5) that $\|(T|_E)^n\| \leq c\beta^n$ for all $n \in \mathbb{N}$. Hence, the spectral radius formula gives $r(T|_E) \leq \beta < 1$. Analogously, by (6), $r(T^{-1}|_F) < 1$. This completes the proof that T is hyperbolic. □

Remark 4. Cirilo *et al* [**12**] have independently obtained the implication (iii) $\Rightarrow$ (i) of Theorem 1 by means of a different argument.

For not necessarily invertible maps, we can define the notion of *positive shadowing* simply by replacing the set $\mathbb{Z}$ by the set $\mathbb{N}_0$ in the definition of shadowing.

PROPOSITION 5. *For any operator T on any Banach space X, the following assertions are equivalent:*

(i) *T is hyperbolic with spectrum contained in $\mathbb{C}\backslash\overline{\mathbb{D}}$;*

(ii) *T is positively expansive and has the positive shadowing property.*

Proof. (i) $\Rightarrow$ (ii). Clear.

(ii) $\Rightarrow$ (i). Suppose that T is not invertible. Since T is positively expansive, it follows that T is injective and not surjective. Let $\delta > 0$ be associated to $\varepsilon = 1$ in the definition of positive shadowing. Fix $z \in X\backslash T(X)$ with $\|z\| = \delta$. Consider the δ-pseudotrajectory $(y_n)_{n\in\mathbb{N}_0}$ of T defined by

$$y_0 = 0, \quad y_1 = z \quad \text{and} \quad y_n = Ty_{n-1} = T^{n-1} z \quad \text{for } n \geq 2.$$

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By positive shadowing, there exists $y \in X$ such that

$$\|T^{n-1}(Ty - z)\| < 1 \quad \text{for all } n \in \mathbb{N}.$$

Since T is positively expansive, we must have $Ty = z$. This is a contradiction, since z does not belong to $T(X)$. Thus, the operator T is invertible.

Now we claim that

$$\sigma_a(T) \subset \mathbb{C}\backslash\overline{\mathbb{D}}. \tag{7}$$

Indeed, suppose that this is false and take a $\lambda \in \sigma_a(T) \cap \overline{\mathbb{D}}$. Let $x_0 \in S_X$ be such that $\|Tx_0 - \lambda x_0\| \leq \delta$. Since $(\lambda^n x_0)_{n\in\mathbb{N}_0}$ is a δ-pseudotrajectory of T, there exists $x \in X$ such that $\|\lambda^n x_0 - T^n x\| < 1$ for all $n \in \mathbb{N}_0$. This implies that $x \neq 0$ and $\|T^n x\| < 2$ for all $n \in \mathbb{N}$, contradicting the hypothesis that T is positively expansive. Thus, (7) holds.

Finally, since $0 \notin \sigma(T)$ and $\sigma_a(T)$ contains the boundary of $\sigma(T)$, (7) implies that $\sigma(T) \subset \mathbb{C}\backslash\overline{\mathbb{D}}$. □

THEOREM 6. *Suppose that an invertible operator T on a complex Banach space X is structurally stable or structurally stable relative to* $\mathrm{GL}(X)$. *Then, the following facts hold.*

(a) *If T is expansive, then T is uniformly expansive.*

(b) *If T is positively expansive, then T is hyperbolic.*

For the proof we will need the following simple fact.

LEMMA 7. *If $\alpha : X \to X$ is a Lipschitz map, then there exists a bounded Lipschitz map $\varphi : X \to X$ which extends $\alpha|_{B_X}$ and satisfies $\|\varphi\|_\infty \leq \|\alpha|_{2B_X}\|_\infty$ and* $\mathrm{Lip}(\varphi) \leq 3\,\mathrm{Lip}(\alpha)$.

Many variations of this lemma have appeared elsewhere, but we shall present a proof below for the sake of completeness.

Proof. Let $\rho : \mathbb{R} \to [0, 1]$ be given by $\rho = 1$ on $(-\infty, 1]$, $\rho(t) = 2 - t$ for $t \in [1, 2]$, and $\rho = 0$ on $[2, +\infty)$. Define

$$\varphi(x) = \alpha(0) + \rho(\|x\|)(\alpha(x) - \alpha(0)) \quad (x \in X).$$

Clearly, $\varphi = \alpha$ on B_X and $\|\varphi\|_\infty \leq \|\alpha|_{2B_X}\|_\infty$. Let $x, y \in X$. If $x \in 2B_X$ then

$$\begin{aligned}\|\varphi(x) - \varphi(y)\| &= \|(\rho(\|x\|) - \rho(\|y\|))(\alpha(x) - \alpha(0)) + \rho(\|y\|)(\alpha(x) - \alpha(y))\| \\ &\leq 3\,\mathrm{Lip}(\alpha)\|x - y\|.\end{aligned}$$

The same holds if $y \in 2B_X$ and the inequality is trivial if both x and y lie outside $2B_X$. □

Proof of Theorem 6. (a) Assume that T is expansive but not uniformly expansive. Then there exists $\lambda \in \sigma_a(T) \cap \mathbb{T}$ [**18**]. Given any $\varepsilon > 0$, there exists $y \in S_X$ such that $\|Ty - \lambda y\| \leq \varepsilon$. By the Hahn–Banach extension theorem, there is a linear functional ψ on X such that $\|\psi\| = 1$ and $\psi(ay) = a$ for all $a \in \mathbb{C}$. Let $S \in \mathcal{B}(X)$ be defined by

$$Sx = Tx + \psi(x)(\lambda y - Ty).$$

Then $\|S - T\| \leq \varepsilon$ and $Sy = \lambda y$.

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If T is structurally stable relative to $\mathrm{GL}(X)$, by choosing $\varepsilon > 0$ small enough, there is a homeomorphism $h : X \to X$ such that $T \circ h = h \circ S$. Hence,

$$T^n(h(y)) = h(\lambda^n y) \quad \text{for all } n \in \mathbb{Z}.$$

Since the sequence $(h(\lambda^n y))_{n\in\mathbb{Z}}$ is bounded (because it is contained in the compact set $\{h(\beta y) : \beta \in \mathbb{T}\}$) and $h(y) \neq 0$, this contradicts the expansivity of T.

If T is structurally stable, we apply Lemma 7 to obtain a Lipschitz map $\varphi : X \to X$ such that $\varphi = S - T$ on B_X, $\|\varphi\|_\infty \leq 2\varepsilon$ and $\mathrm{Lip}(\varphi) \leq 3\varepsilon$. By choosing $\varepsilon > 0$ small enough, there is a homeomorphism $h : X \to X$ such that $T \circ h = h \circ (T + \varphi)$. Since $T + \varphi = S$ on B_X, we have that $(T + \varphi)^{-1} = S^{-1}$ on $S(B_X)$. Let $\gamma \in (0, 1]$ be such that $S(B_X) \supset \gamma B_X$. Then

$$T^n(h(\gamma y)) = h(\lambda^n \gamma y) \quad \text{for all } n \in \mathbb{Z}.$$

As before, this contradicts the expansivity of T.

(b) The proof of (b) is a simple modification of the proof of (a). Indeed, assume that T is positively expansive but not hyperbolic. Then $\sigma(T) \cap \mathbb{T} \neq \emptyset$. Since $0 \notin \sigma(T)$, the boundary of $\sigma(T)$ must intersect $\overline{\mathbb{D}}$. Thus, there exists $\lambda \in \sigma_a(T) \cap \overline{\mathbb{D}}$. By arguing as in the proof of (a), we obtain a homeomorphism $h : X \to X$ such that

$$T^n(h(y)) = h(\lambda^n y) \quad \text{for all } n \in \mathbb{N},$$

no matter if T is structurally stable or structurally stable relative to $\mathrm{GL}(X)$. Since the sequence $(h(\lambda^n y))_{n\in\mathbb{N}}$ is bounded and $h(y) \neq 0$, we have a contradiction with the hypothesis that T is positively expansive. □

Remark 8. For invertible operators T with the property that $\sigma_a(T) = \sigma(T)$, the above proof shows that if T is expansive and structurally stable (or structurally stable relative to $\mathrm{GL}(X)$), then T is hyperbolic. In particular, this applies to invertible normal operators on Hilbert spaces.

A well-known result in the area of dynamical systems, from the 1960s, asserts that any invertible hyperbolic operator on any Banach space is structurally stable [**17**, **24**, **27**]. It is also known that these operators are structurally stable relative to $\mathrm{GL}(X)$ [**1**, **28**]. Our goal now is to establish the result below, which shows the existence of structurally stable operators that are not hyperbolic and are not structurally stable relative to $\mathrm{GL}(X)$. Our examples will be invertible bilateral weighted backward shifts

$$B_w : (x_n)_{n\in\mathbb{Z}} \in X \mapsto (w_{n+1}x_{n+1})_{n\in\mathbb{Z}} \in X,$$

where $X = \ell_p(\mathbb{Z})$ $(1 \leq p < \infty)$ or $X = c_0(\mathbb{Z})$. We recall that $w = (w_n)_{n\in\mathbb{Z}}$ is a bounded sequence of scalars with $\inf_{n\in\mathbb{Z}} |w_n| > 0$. The boundedness of w is a necessary and sufficient condition for B_w to be a well-defined operator on X and the condition $\inf_{n\in\mathbb{Z}} |w_n| > 0$ means that B_w is invertible.

THEOREM 9. *Let $X = \ell_p(\mathbb{Z})$ $(1 \leq p < \infty)$ or $X = c_0(\mathbb{Z})$, and let $w = (w_n)_{n\in\mathbb{Z}}$ be a bounded sequence of scalars with $\inf_{n\in\mathbb{Z}} |w_n| > 0$. If*

(C) $\lim_{n\to\infty} \sup_{k\in\mathbb{N}} |w_{-k}w_{-k-1} \cdot \ldots \cdot w_{-k-n}|^{1/n} < 1$ *and* $\lim_{n\to\infty} \inf_{k\in\mathbb{N}} |w_k w_{k+1} \cdot \ldots \cdot w_{k+n}|^{1/n} > 1$

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then B_w is structurally stable, but it is not hyperbolic and it is not structurally stable relative to $\mathrm{GL}(X)$.

Hence, simple concrete examples of non-hyperbolic structurally stable B_w can be obtained by putting $w_n = 1/a$ if $n < 0$ and $w_n = a$ if $n \geq 0$, where a is any real number greater than 1.

The following simple fact concerning hypercyclic operators (i.e., operators that have a dense orbit) will be used in the proof of the above theorem.

LEMMA 10. *For every complex Banach space X, the set $HC(X)$ of all hypercyclic operators on X has empty interior in $\mathcal{B}(X)$.*

Proof. Fix $T \in HC(X)$ and $\varepsilon > 0$. Let T^* be the adjoint operator of T acting on the dual space X^* of X by $T^*x^* = x^* \circ T$. Take $\lambda \in \sigma_a(T^*)$ and let $y^* \in S_{X^*}$ be such that

$$\|T^*y^* - \lambda y^*\| \leq \varepsilon/2.$$

Choose $y \in S_X$ such that $|y^*(y)| \geq 1/2$. Let $R \in \mathcal{B}(X^*)$ be defined by

$$Rx^* = \frac{x^*(y)}{y^*(y)} \cdot (\lambda y^* - T^*y^*).$$

Since R is w^*–w^*-continuous, there exists $Q \in \mathcal{B}(X)$ with $Q^* = R$ [**21**, Theorem 3.1.11]. Let $S = T + Q \in \mathcal{B}(X)$. Then $\|S - T\| = \|Q\| = \|R\| \leq \varepsilon$ and $S^*y^* = \lambda y^*$. This second property implies that $S \notin HC(X)$, for otherwise there would exist $x \in X$ such that the points $y^*(S^n x) = ((S^*)^n y^*)(x) = \lambda^n y^*(x)$ $(n \in \mathbb{N}_0)$ form a dense set in $\mathbb{C}$, which is impossible. □

Proof of Theorem 9. We can rewrite condition (C) as follows:

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} |w_{-k}w_{-k-1}\cdots w_{-k-n}|^{1/n} < 1 \quad \text{and} \quad \lim_{n\to\infty} \sup_{k\in\mathbb{N}} |w_k w_{k+1}\cdots w_{k+n}|^{-1/n} < 1. \tag{8}$$

Our goal is to find a homeomorphism $h : X \to X$ such that

$$h \circ B_w = S \circ h,$$

where $S = B_w + \alpha$ with $\alpha \in C_b(X)$ a Lipschitz map such that $\|\alpha\|_\infty$ and $\mathrm{Lip}(\alpha)$ are sufficiently small. By putting $h = I + \varphi$, we have that the above equality is equivalent to

$$\varphi(B_w x) - B_w(\varphi(x)) = \alpha(x + \varphi(x)).$$

We will first solve the equation

$$\varphi(B_w x) - B_w(\varphi(x)) = \alpha(x). \tag{9}$$

For this purpose, for each $\varphi \in C_b(X)$, we write $\varphi(x) = \sum_{i\in\mathbb{Z}} \varphi_i(x)e_i$. Let

$$C_{b,0}(X) = \{\varphi \in C_b(X) : \varphi_0 = 0\},$$

which is a closed subspace of $C_b(X)$. We claim that the bounded linear map

$$F : \varphi \in C_{b,0}(X) \mapsto \varphi \circ B_w - B_w \circ \varphi \in C_b(X)$$

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is bijective. Indeed, fix $\alpha \in C_b(X)$. Equation (9) is equivalent to

$$\varphi_i(B_w x) - w_{i+1}\varphi_{i+1}(x) = \alpha_i(x) \quad \text{for all } i \in \mathbb{Z},$$

which implies that

$$\varphi_i(x) = \frac{\varphi_0(B_w^i x)}{w_1 \cdots w_i} - \sum_{j=0}^{i-1} \frac{\alpha_j(B_w^{i-j-1}x)}{w_{j+1}\cdots w_i} \tag{10}$$

and

$$\varphi_{-i}(x) = w_{-i+1}\cdots w_0\varphi_0(B_w^{-i}x) + \sum_{j=1}^{i} w_{-i+1}\cdots w_{-j}\alpha_{-j}(B_w^{-i+j-1}x) \tag{11}$$

(where in the case $j = i$ we consider the 'empty product' $w_{-i+1}\cdots w_{-j}$ to be 1), for every $i \in \mathbb{N}$. Hence, the only possible solution of (9) in $C_{b,0}(X)$ is given by

$$\varphi_0 = 0, \quad \varphi_i(x) = -\sum_{j=0}^{i-1} \frac{\alpha_j(B_w^{i-j-1}x)}{w_{j+1}\cdots w_i},$$

$$\varphi_{-i}(x) = \sum_{j=1}^{i} w_{-i+1}\cdots w_{-j}\alpha_{-j}(B_w^{-i+j-1}x).$$

On the other hand, by defining φ as above, we have that $\varphi \in C_{b,0}(X)$ and (9) holds, which proves our claim. Indeed, let us verify that $\varphi \in C_{b,0}(X)$ in the case $X = \ell_p(\mathbb{Z})$ (the case $X = c_0(\mathbb{Z})$ will be left to the reader). Note that $\varphi = f + g$, where

$$f(x) = \sum_{i\in\mathbb{N}} \varphi_{-i}(x)e_{-i} \quad \text{and} \quad g(x) = \sum_{i\in\mathbb{N}} \varphi_i(x)e_i.$$

Hence, it is enough to prove that $f \in C_b(X)$ and $g \in C_b(X)$. By (8) and Lemma 19, there exist $s \in (0, 1)$ and $\beta > 1$ such that

$$|w_{-j}w_{-j-1}\cdots w_{-j-k+1}| \le \beta s^k \quad \text{and} \quad \frac{1}{|w_j w_{j+1}\cdots w_{j+k-1}|} \le \beta s^k \quad \text{for all } j, k \in \mathbb{N}. \tag{12}$$

Let $g_n(x) = \sum_{i=1}^n \varphi_i(x)e_i$ ($n \in \mathbb{N}$). If we rearrange the double series that defines $g_n(x)$ by putting together all terms that have the same number t of w_j in the denominator (a sort of 'diagonal ordering'), then we obtain

$$g_n(x) = -\sum_{t=1}^{n}\left(\sum_{k=t}^{n} \frac{\alpha_{k-t}(B_w^{t-1}x)}{w_{k-t+1}\cdots w_k}e_k\right).$$

Hence, it follows from Minkowski's inequality that

$$\begin{aligned}(|\varphi_1(x)|^p + \cdots + |\varphi_n(x)|^p)^{1/p} = \|g_n(x)\| &\le \sum_{t=1}^{n}\left\|\sum_{k=t}^{n} \frac{\alpha_{k-t}(B_w^{t-1}x)}{w_{k-t+1}\cdots w_k}e_k\right\| \\ &= \sum_{t=1}^{n}\left(\sum_{k=t}^{n}\left|\frac{\alpha_{k-t}(B_w^{t-1}x)}{w_{k-t+1}\cdots w_k}\right|^p\right)^{1/p}\end{aligned}$$

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$$\leq \sum_{t=1}^{n} \beta \|\alpha(B_w^{t-1}x)\| s^t \leq \frac{\beta}{1-s}\|\alpha\|_\infty < \infty, \quad (13)$$

for all $x \in X$ and $n \in \mathbb{N}$. This shows that $g(\ell_p(\mathbb{Z})) \subset \ell_p(\mathbb{Z})$ and that g is bounded. Since

$$g(x) = -\sum_{t=1}^{\infty}\left(\sum_{k=t}^{\infty} \frac{\alpha_{k-t}(B_w^{t-1}x)}{w_{k-t+1}\cdots w_k} e_k\right),$$

we have that

$$\begin{aligned}\|g(x) - g(y)\| &\leq \sum_{t=1}^{\infty}\left(\sum_{k=t}^{\infty}\left|\frac{\alpha_{k-t}(B_w^{t-1}x) - \alpha_{k-t}(B_w^{t-1}y)}{w_{k-t+1}\cdots w_k}\right|^p\right)^{1/p} \\ &\leq \beta \sum_{t=1}^{\infty} \|\alpha(B_w^{t-1}x) - \alpha(B_w^{t-1}y)\| s^t,\end{aligned}$$

which easily implies the continuity of g. Thus, $g \in C_b(X)$. The proof that $f \in C_b(X)$ is analogous, but we have to use the first estimate in (12) instead of the second one.

Now choose $0 < \varepsilon < \|F^{-1}\|^{-1}$, let $\alpha \in C_b(X)$ be a Lipschitz map with $\|\alpha\|_\infty \leq \varepsilon$ and $\mathrm{Lip}(\alpha) \leq \varepsilon$, and put $S = B_w + \alpha$. We claim that there exists a unique $u \in C_{b,0}(X)$ such that the map $h = I + u$ satisfies

$$h \circ B_w = S \circ h. \quad (14)$$

Indeed, let $G : C_{b,0}(X) \to C_{b,0}(X)$ be the map given by

$$G(\varphi) = F^{-1}(\alpha \circ (I + \varphi)).$$

Since

$$\|G(\psi) - G(\varphi)\|_\infty \leq \mathrm{Lip}(\alpha)\|F^{-1}\|\|\psi - \varphi\|_\infty,$$

we have that G is a contraction, and so it has a unique fixed point in $C_{b,0}(X)$. Since (14) is equivalent to $G(u) = u$, our second claim is proved.

We now claim that there is a unique $v \in C_{b,0}(X)$ such that the map $h' = I + v$ satisfies

$$h' \circ S = B_w \circ h'. \quad (15)$$

Indeed, (15) is equivalent to the equation

$$v \circ S - B_w \circ v = -\alpha.$$

With $v_0 = 0$, this equation implies that

$$v_i(x) = \sum_{j=0}^{i-1} \frac{\alpha_j(S^{i-j-1}x)}{w_{j+1}\cdots w_i} \quad \text{and} \quad v_{-i}(x) = -\sum_{j=1}^{i} w_{-i+1}\cdots w_{-j}\alpha_{-j}(S^{-i+j-1}x),$$

for every $i \in \mathbb{N}$. So, if there is a solution v in $C_{b,0}(X)$, then it must be unique. However, by defining v as above, we have that $v \in C_{b,0}(X)$ and $h' = I + v$ satisfies (15). This proves our third claim.

By (14) and (15),

$$h' \circ h \circ B_w = B_w \circ h' \circ h. \quad (16)$$

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Since $h' \circ h = I + \psi$ with $\psi \in C_{b,0}(X)$, (16) is equivalent to

$$\psi_i(B_w x) = w_{i+1}\psi_{i+1}(x) \quad \text{for all } i \in \mathbb{Z}.$$

Since $\psi_0 = 0$, this clearly implies that $\psi = 0$. Thus,

$$h' \circ h = I. \tag{17}$$

Let us now prove that

$$h \circ h' = I. \tag{18}$$

It is enough to show that the only solution to

$$(I + \theta) \circ S = S \circ (I + \theta), \tag{19}$$

with $\theta \in C_{b,0}(X)$, is $\theta = 0$. For this purpose, consider the bounded linear map

$$H : \gamma \in C_{b,0}(X) \mapsto \gamma \circ S - B_w \circ \gamma \in C_b(X).$$

We have that H is bijective. Indeed, given $\eta \in C_b(X)$, the equation

$$\gamma(Sx) - B_w(\gamma(x)) = \eta(x)$$

is equivalent to

$$\gamma_i(Sx) - w_{i+1}\gamma_{i+1}(x) = \eta_i(x) \quad \text{for all } i \in \mathbb{Z},$$

and so we can argue as in the case of the map F. Define $K : C_{b,0}(X) \to C_{b,0}(X)$ by

$$K(\varphi) = H^{-1}(\alpha \circ (I + \varphi) - \alpha).$$

We have that

$$\|K(\psi) - K(\varphi)\|_\infty \leq \mathrm{Lip}(\alpha)\|H^{-1}\|\|\psi - \varphi\|_\infty.$$

Moreover, we know that

$$H^{-1}(\eta)(x) = \sum_{i\in\mathbb{N}} \gamma_{-i}(x)e_{-i} + \sum_{i\in\mathbb{N}} \gamma_i(x)e_i,$$

where

$$\gamma_i(x) = -\sum_{j=0}^{i-1} \frac{\eta_j(S^{i-j-1}x)}{w_{j+1}\cdots w_i} \quad \text{and} \quad \gamma_{-i}(x) = \sum_{j=1}^{i} w_{-i+1}\cdots w_{-j}\eta_{-j}(S^{-i+j-1}x).$$

A computation analogous to (13) shows that

$$\|H^{-1}(\eta)\|_\infty \leq \frac{2\beta}{1-s}\|\eta\|_\infty.$$

This shows that, although H depends on ε, $\|H^{-1}\|$ is bounded by a constant which is independent of ε. Thus, by choosing $\varepsilon > 0$ small enough, we can guarantee that K is a contraction, and so it has a unique fixed point in $C_{b,0}(X)$. Since $K(0) = 0$ and (19) is equivalent to $K(\theta) = \theta$, we conclude that $\theta = 0$ is the only solution to (19) in $C_{b,0}(X)$.

By (17) and (18), h is a homeomorphism, which proves that B_w is structurally stable.

By the characterization of hyperbolic weighted shifts mentioned in [**10**, Remark 35], B_w is not hyperbolic. Moreover, we mentioned in the introduction and will prove in the next section that, in the present case, B_w satisfies the frequent hypercyclicity criterion. Hence, it cannot be structurally stable relative to $\mathrm{GL}(X)$ in view of Lemma 10. □

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Remark 11. An invertible operator T on a Banach space X is said to be *strongly structurally stable* [**28**] if for every $\gamma > 0$ there exists $\varepsilon > 0$ such that the following property holds: for any Lipschitz map $\varphi \in C_b(X)$ with $\|\varphi\|_\infty \leq \varepsilon$ and $\mathrm{Lip}(\varphi) \leq \varepsilon$, there is a homeomorphism $h : X \to X$ such that $h \circ T = (T + \varphi) \circ h$ and $\|h - I\|_\infty \leq \gamma$. So it is now required that the homeomorphism h conjugating T and $T + \varphi$ is close to the identity operator. The proof of the fact that invertible hyperbolic operators are structurally stable given in [**27**], for instance, shows that these operators are in fact strongly structurally stable. We also observe that the proof of Theorem 9 actually shows that B_w is strongly structurally stable. Indeed, recall that the homeomorphism h conjugating B_w and S constructed in the proof of Theorem 9 has the form $h = I + u$, where u is the unique fixed point of the map G. Hence,

$$\|u\|_\infty = \|G(u)\|_\infty = \|F^{-1}(\alpha \circ (I + \varphi))\|_\infty \leq \|F^{-1}\|\|\alpha\|_\infty \leq \|F^{-1}\|\varepsilon.$$

Hence, we can make $\|u\|_\infty$ as small as we want by choosing ε small enough.

Remark 12.

(a) Note that the homeomorphism h of the form $I + u$ constructed in the proof of Theorem 9 is not unique, contrary to what happens in the hyperbolic case. Indeed, if instead of the subspace $C_{b,0}(X)$ we consider the subspace $C_{b,j}(X) = \{\varphi \in C_b(X) : \varphi_j = 0\}$ for some $j \neq 0$, then we would obtain a different u and so a different h.

(b) The proof of Theorem 9 works as well if we replace the space $C_b(X)$ by the space $U_b(X)$ of all bounded uniformly continuous maps $\varphi : X \to X$ endowed with the supremum norm. Thus, the homeomorphism $h = I + u$ conjugating B_w and S can be chosen to be a uniform homeomorphism.

Remark 13. By combining Theorem 9 with the characterization of shadowing for weighted shifts that will be given in the next section, we see that for the class of invertible weighted shifts on the spaces $\ell_p(\mathbb{Z})$ $(1 \leq p < \infty)$ and $c_0(\mathbb{Z})$, shadowing implies structural stability. This is also true for invertible operators on finite-dimensional spaces. These remarks motivate the question below.

Question 14. Is it true that every invertible operator on a Banach space that has the shadowing property is structurally stable? How about the converse?

PROPOSITION 15. *Let $X = \ell_p(\mathbb{Z})$ $(1 \leq p < \infty)$ or $X = c_0(\mathbb{Z})$. If an invertible bilateral weighted backward shift B_w on X is expansive and strongly structurally stable, then it is hyperbolic.*

Proof. We assume that B_w is expansive, strongly structurally stable and not hyperbolic, and we will reach a contradiction. By Theorem 6(a), B_w is uniformly expansive. It was proved in [**10**, Theorem E] that B_w is uniformly expansive if and only if one of the following conditions holds:

(a) $\lim_{n\to\infty} \inf_{k\in\mathbb{Z}} |w_{k+1} \cdots w_{k+n}|^{-1} = \infty$;

(b) $\lim_{n\to\infty} \inf_{k\in\mathbb{Z}} |w_{k-n+1} \cdots w_k| = \infty$;

(c) $\lim_{n\to\infty} \inf_{k\in\mathbb{N}} |w_{k+1} \cdots w_{k+n}|^{-1} = \infty$ and $\lim_{n\to\infty} \inf_{k\in-\mathbb{N}} |w_{k-n+1} \cdots w_k| = \infty$.

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By the version of Lemma 19 with $\mathbb{Z}$ instead of $\mathbb{N}$, we see that (a) and (b) are equivalent to

$$\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} |w_{k+1}\cdots w_{k+n}|^{1/n} < 1 \quad \text{and} \quad \lim_{n\to\infty} \sup_{k\in\mathbb{Z}} |w_{k-n+1}\cdots w_k|^{-1/n} < 1,$$

respectively, which are exactly the cases in which B_w is hyperbolic [**10**, Remark 35]. Since we are assuming that B_w is not hyperbolic, we conclude that (c) must hold. In particular,

$$\lim_{i\to\infty} \frac{1}{|w_1\cdots w_i|} = \infty \quad \text{and} \quad \lim_{i\to\infty} |w_{-i+1}\cdots w_0| = \infty. \tag{20}$$

Let $\varepsilon > 0$ be associated to $\gamma = 1$ according to the definition of strong structural stability. Let $\alpha \in C_b(X)$ be the constant map given by $\alpha(x) = (\varepsilon/2)e_0$. There exists $\varphi \in C_b(X)$ with $\|\varphi\|_\infty \le 1$ such that $h = I + \varphi$ is a homeomorphism satisfying

$$h \circ B_w = (B_w + \alpha) \circ h.$$

This equation is equivalent to

$$\varphi(B_w x) - B_w(\varphi(x)) = \alpha(x + \varphi(x)),$$

which is the same as (9) since α is constant. Thus, φ must satisfy equations (10) and (11), and so

$$\varphi_i(x) = \frac{\varphi_0(B_w^i x)}{w_1\cdots w_i} - \frac{\varepsilon}{2w_1\cdots w_i} \quad \text{and} \quad \varphi_{-i}(x) = w_{-i+1}\cdots w_0\varphi_0(B_w^{-i}x),$$

for each $i \in \mathbb{N}$. Since φ is bounded, the second equality in (20) shows that

$$\varphi_0(x) = \frac{\varphi_{-i}(B_w^i x)}{w_{-i+1}\cdots w_0} \to 0 \quad \text{as } i \to \infty,$$

proving that φ_0 is identically zero. Hence,

$$\varphi_i(x) = -\frac{\varepsilon}{2w_1\cdots w_i} \quad \text{for all } i \in \mathbb{N}.$$

Now, the first equality in (20) leads to a contradiction with the boundedness of φ. This completes the proof. □

The previous results motivate the following question.

Question 16. If an invertible operator on a Banach space is expansive and structurally stable, is it necessarily hyperbolic?

Recall that the answer is positive if we replace ‘expansive’ by ‘positively expansive’ (Theorem 6(b)).

The following question also seems to be open.

Question 17. Do hyperbolicity and structural stability relative to $\mathrm{GL}(X)$ coincide?

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4. *Weighted shifts with the shadowing property*

THEOREM 18. *Let* $X = \ell_p(\mathbb{Z})$ $(1 \le p < \infty)$ *or* $X = c_0(\mathbb{Z})$. *Let* $w = (w_n)_{n\in\mathbb{Z}}$ *be a bounded sequence of scalars with* $\inf_{n\in\mathbb{Z}} |w_n| > 0$ *and consider the bilateral weighted backward shift*

$$B_w : (x_n)_{n\in\mathbb{Z}} \in X \mapsto (w_{n+1}x_{n+1})_{n\in\mathbb{Z}} \in X.$$

Then B_w *has the shadowing property if and only if one of the following conditions holds:*

(A) $\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} |w_k w_{k+1} \cdot \ldots \cdot w_{k+n}|^{1/n} < 1$;

(B) $\lim_{n\to\infty} \inf_{k\in\mathbb{Z}} |w_k w_{k+1} \cdot \ldots \cdot w_{k+n}|^{1/n} > 1$;

(C) $\lim_{n\to\infty} \sup_{k\in\mathbb{N}} |w_{-k} w_{-k-1} \cdot \ldots \cdot w_{-k-n}|^{1/n} < 1$ *and* $\lim_{n\to\infty} \inf_{k\in\mathbb{N}} |w_k w_{k+1} \cdot \ldots \cdot w_{k+n}|^{1/n} > 1$.

We recall that (A) and (B) are exactly the cases in which the operator B_w is hyperbolic [**10**, Remark 35]. In these cases, B_w does not exhibit chaotic behaviour, since hyperbolic operators are not Li–Yorke chaotic [**10**, Theorem C]. On the other hand, we will see after the proof of Theorem 18 that condition (C) implies the frequent hypercyclicity criterion. Hence, in this case, B_w exhibits several types of chaotic behaviours, including mixing, Devaney chaos and dense distributional chaos.

LEMMA 19. *If* $(w_n)_{n\in\mathbb{N}}$ *is a bounded sequence of scalars, then the following assertions are equivalent:*

(i) $\lim_{n\to\infty} \sup_{k\in\mathbb{N}} |w_k w_{k+1} \cdot \ldots \cdot w_{k+n}|^{1/n} < 1$;

(ii) $\sup_{k\in\mathbb{N}} |w_k w_{k+1} \cdot \ldots \cdot w_{k+n-1}| < 1$ *for some* $n \in \mathbb{N}$;

(iii) $\sup_{k\in\mathbb{N}} \sum_{n=0}^{\infty} |w_k w_{k+1} \cdot \ldots \cdot w_{k+n}| < \infty$;

(iv) $\sup_{k\in\mathbb{N}} \sum_{n=0}^{k-1} |w_k w_{k-1} \cdot \ldots \cdot w_{k-n}| < \infty$.

Proof. It is clear that (i) implies (ii). The fact that (ii) implies (iii) and (iv) follows easily by comparing the series in (iii) and (iv) with a suitable geometric series. Let us show that (iii) implies (i). For each $k \in \mathbb{N}$, let $R_k = \sum_{n=0}^{\infty} |w_k w_{k+1} \cdot \ldots \cdot w_{k+n}|$. By hypothesis, there is a constant $K > 0$ such that $R_k \le K$ for all $k \in \mathbb{N}$. Since $|w_k| = (R_k/(1 + R_{k+1}))$ $(k \in \mathbb{N})$,

$$|w_k w_{k+1} \cdot \ldots \cdot w_{k+n}| = \frac{R_k}{1 + R_{k+n+1}} \cdot \frac{R_{k+1}}{1 + R_{k+1}} \cdot \ldots \cdot \frac{R_{k+n}}{1 + R_{k+n}} \le K \cdot \left(\frac{K}{1 + K}\right)^n,$$

where we used the fact that $R_j/(1 + R_j) = 1 - 1/(1 + R_j) \le K/(1 + K)$ for all $j \in \mathbb{N}$. This implies (i). The proof that (iv) implies (i) is similar and is left to the reader. □

Proof of Theorem 18. Since invertible hyperbolic operators have the shadowing property, in order to prove sufficiency it is enough to show that (C) implies shadowing. Suppose that (C) holds. Let

$$M = \{(x_n)_{n\in\mathbb{Z}} \in X : x_n = 0 \text{ for all } n > 0\}$$

and

$$N = \{(x_n)_{n\in\mathbb{Z}} \in X : x_n = 0 \text{ for all } n \le 0\},$$

which are closed subspaces of X such that $X = M \oplus N$, $B_w(M) \subset M$ and $B_w^{-1}(N) \subset N$. Since, for each $n \in \mathbb{N}$,

$$\|(B_w|_M)^n\| = \sup_{k\in\mathbb{N}_0} |w_{-k} w_{-k-1} \cdot \ldots \cdot w_{-k-n+1}|$$

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and

$$\|(B_w^{-1}|_N)^n\| = \sup_{k\in\mathbb{N}} |w_{k+1}w_{k+2}\cdot\ldots\cdot w_{k+n}|,$$

we have that condition (C) is equivalent to saying that

$$\sigma(B_w|_M) \subset \mathbb{D} \quad \text{and} \quad \sigma(B_w^{-1}|_N) \subset \mathbb{D},$$

where $\mathbb{D}$ denotes the open unit disk in the complex plane. Thus, by [**10**, Theorem A], B_w has the shadowing property.

Conversely, assume that B_w has the shadowing property. We claim that one of the following conditions must be true:

(C1) $\lim_{n\to\infty} \sup_{k\in\mathbb{N}} |w_k w_{k+1}\cdot\ldots\cdot w_{k+n}|^{1/n} < 1$;

(C2) $\lim_{n\to\infty} \inf_{k\in\mathbb{N}} |w_k w_{k+1}\cdot\ldots\cdot w_{k+n}|^{1/n} > 1$.

Indeed, assume that condition (C1) is false. We will prove that condition (C2) must be true. Take $\varepsilon = 1$ and let $\delta > 0$ be associated to this ε according to the definition of shadowing. Since (C1) is false, Lemma 19 guarantees that there exist $t \in \mathbb{N}$ and $m_0 \in \mathbb{N}$ such that

$$\sum_{n=0}^{m_0} |w_t w_{t+1}\cdot\ldots\cdot w_{t+n}| \geq \frac{1+\delta}{\delta^2}. \tag{21}$$

Given $m > m_0$, we construct a δ-pseudotrajectory of B_w in the following way:

$$x_0 = e^{i\theta_0}e_{t+m}, \quad x_k = B_w(x_{k-1}) + \delta e^{i\theta_k}e_{t+m-k}, \quad k = 1,\ldots,m,$$

and

$$x_n = B_w(x_{n-1}) \quad \text{for all } n \geq m+1,$$

where the angles $\theta_0, \theta_1, \ldots, \theta_m$ are chosen so that

$$e^{i\theta_k} w_t w_{t+1}\cdot\ldots\cdot w_{t+m-k} = |w_t w_{t+1}\cdot\ldots\cdot w_{t+m-k}| \quad \text{for each } 0 \leq k \leq m.$$

By shadowing, there exists $a = (a_n)_{n\in\mathbb{Z}} \in X$ such that

$$\|x_n - B_w^n(a)\| < 1 \quad \text{for all } n \in \mathbb{Z}. \tag{22}$$

Since $B_w e_k = w_k e_{k-1}$ for all $k \in \mathbb{Z}$, the $(t-1)$th coordinate of x_{m+1} is

$$e^{i\theta_0} w_t w_{t+1}\cdot\ldots\cdot w_{t+m} + (e^{i\theta_1} w_t w_{t+1}\cdot\ldots\cdot w_{t+m-1} + \cdots + e^{i\theta_{m-1}} w_t w_{t+1} + e^{i\theta_m} w_t)\delta,$$

which is equal to

$$|w_t w_{t+1}\cdot\ldots\cdot w_{t+m}| + (|w_t w_{t+1}\cdot\ldots\cdot w_{t+m-1}| + \cdots + |w_t w_{t+1}| + |w_t|)\delta.$$

By writing $a_{t+m} = e^{i\theta_0} + \gamma$, we have that $|\gamma| < 1$ and the $(t-1)$th coordinate of $B_w^{m+1}(a)$ is

$$w_t w_{t+1}\cdot\ldots\cdot w_{t+m}\cdot(e^{i\theta_0} + \gamma).$$

Hence, (22) gives

$$|(|w_t w_{t+1}\cdot\ldots\cdot w_{t+m-1}| + \cdots + |w_t w_{t+1}| + |w_t|)\delta - w_t w_{t+1}\cdot\ldots\cdot w_{t+m}\gamma| < 1. \tag{23}$$

By (21) and (23),

$$|w_t w_{t+1}\cdot\ldots\cdot w_{t+m}| > 1/\delta.$$

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Again by (23),

$$(|w_t w_{t+1} \cdot \ldots \cdot w_{t+m-1}| + \cdots + |w_t w_{t+1}| + |w_t|)\delta < 1 + |w_t w_{t+1} \cdot \ldots \cdot w_{t+m}||\gamma|.$$

By dividing both sides by $|w_t w_{t+1} \cdot \ldots \cdot w_{t+m}|\delta$, we obtain

$$\frac{1}{|w_{t+m}|} + \frac{1}{|w_{t+m} w_{t+m-1}|} + \cdots + \frac{1}{|w_{t+m} w_{t+m-1} \cdot \ldots \cdot w_{t+1}|} < 1 + \frac{1}{\delta}.$$

Since this is true for every $m > m_0$, it follows that $\inf_{n\in\mathbb{N}} |w_n| > 0$ and, by Lemma 19, condition (C2) holds.

Now the inverse of B_w is the bilateral weighted forward shift

$$F_{w'} : (x_n)_{n\in\mathbb{Z}} \in X \mapsto (w'_{n-1} x_{n-1})_{n\in\mathbb{Z}} \in X,$$

where $w'_n = 1/w_{n+1}$ for each $n \in \mathbb{Z}$. On the other hand, the isometric isomorphism $\phi : (x_n)_{n\in\mathbb{Z}} \in X \mapsto (x_{-n})_{n\in\mathbb{Z}} \in X$ establishes a conjugation between $F_{w'}$ and $B_{w''}$ (that is, $\phi \circ F_{w'} = B_{w''} \circ \phi$), where

$$w''_n = w'_{-n} = \frac{1}{w_{-n+1}} \quad \text{for each } n \in \mathbb{Z}.$$

Hence, $B_{w''}$ also has the shadowing property and, by the above reasoning, we conclude that one of the following properties must be true:
(D1) $\lim_{n\to\infty} \inf_{k\in\mathbb{N}} |w_{-k} w_{-k-1} \cdot \ldots \cdot w_{-k-n}|^{1/n} > 1$;
(D2) $\lim_{n\to\infty} \sup_{k\in\mathbb{N}} |w_{-k} w_{-k-1} \cdot \ldots \cdot w_{-k-n}|^{1/n} < 1$.
Thus, we have four possibilities. Note that condition (A) is equivalent to saying that (C1) and (D2) are true, condition (B) is equivalent to saying that (C2) and (D1) are true, and condition (C) means that (C2) and (D2) are true. Consequently, it remains to show that it is not possible for both (C1) and (D1) to be true. Indeed, assume that both (C1) and (D1) hold. Let δ and ε be as above. We construct a δ-pseudotrajectory of B_w as follows:

$$y_0 = e_0, \quad y_n = B_w(y_{n-1}) + \delta e_0 \quad \text{for } n \geq 1, \quad y_n = B_w^{-1}(y_{n+1} + \delta e_0) \quad \text{for } n \leq -1.$$

Note that

$$\begin{aligned} y_n = {} & w_0 \cdots w_{-n+1} e_{-n} + \delta w_0 \cdots w_{-n+2} e_{-n+1} \\ & + \delta w_0 \cdots w_{-n+3} e_{-n+2} + \cdots + \delta w_0 e_{-1} + \delta e_0 \end{aligned}$$

and

$$y_{-n} = \frac{\delta + 1}{w_1 \cdots w_n} e_n + \frac{\delta}{w_1 \cdots w_{n-1}} e_{n-1} + \frac{\delta}{w_1 \cdots w_{n-2}} e_{n-2} + \cdots + \frac{\delta}{w_1} e_1,$$

for each $n \in \mathbb{N}$. There exists $b = (b_n)_{n\in\mathbb{Z}} \in X$ such that

$$\|y_n - B_w^n(b)\| < 1 \quad \text{for every } n \in \mathbb{Z}. \tag{24}$$

For each $k \geq 1$ and each $n \geq 0$, since the $(-k-(n+1))$th coordinate of $B_w^{n+1}(b)$ is equal to $w_{-k-n} \cdot \ldots \cdot w_{-k-1} w_{-k} b_{-k}$, (24) implies that

$$|w_{-k-n} \cdot \ldots \cdot w_{-k-1} w_{-k} b_{-k}| < 1.$$

Hence, (D1) gives $b_{-k} = 0$ for all $k \geq 1$. Analogously, (C1) gives $b_k = 0$ for all $k \geq 1$. Thus, $b = b_0 e_0$, which contradicts (24). This completes the proof of Theorem 18. □

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Recall that an operator T on a separable Banach space Y is said to satisfy the *frequent hypercyclicity criterion* if there exist a dense subset Y_0 of Y and a map $S : Y_0 \to Y_0$ such that the following properties hold for every $y \in Y_0$:

(i) $\sum_{k=0}^{\infty} T^k y$ converges unconditionally;
(ii) $\sum_{k=0}^{\infty} S^k y$ converges unconditionally;
(iii) $TSy = y$.

Recall also that T is said to be *frequently hypercyclic* if for each non-empty open subset U of Y, the set $A = \{n \in \mathbb{N}_0 : T^n x \in U\}$ has positive lower density, that is,

$$\liminf_{N\to\infty} \frac{\operatorname{card}(A \cap [0, N])}{N+1} > 0.$$

As mentioned in the Introduction, the frequent hypercyclicity criterion implies not only frequent hypercyclicity, but also several other notions of chaotic behaviours in linear dynamics. Frequent hypercyclic weighted shifts on $\ell_p(\mathbb{Z})$ and $c_0(\mathbb{Z})$ were characterized by Bayart and Ruzsa [**6**] and their characterization shows that condition (C) implies the frequent hypercyclicity criterion.

Remark 20. We close this work by presenting an overview of the relations between hyperbolicity and the other notions considered in the paper for invertible operators on Banach spaces. We have that:

(a) hyperbolicity implies uniform expansivity, shadowing and structural stability, but each of the converses is false in general;
(b) hyperbolicity is equivalent to expansivity plus shadowing;
(c) positive expansivity plus structural stability implies hyperbolicity, but not conversely.

The basic open problem here is exactly our Question 16.

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