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ELEMENTARY PROOFS OF THE DIAMETER BOUNDS FOR POWER GRAPHS

MARCO BARBIERI and KAMILLA REKVÉNYI✉

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Abstract

We give a simplified version of the proofs that, outside of their isolated vertices, the complement of the enhanced power graph and of the power graph are connected and have diameter at most 3.

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1. Introduction

Graphs arising from groups have become an important topic in the last fifteen years (see for example [4–6, 8]). In this note, we give a simplified proof of the following results.

THEOREM 1.1. *Let G be a finite group. Then, outside of its isolated vertices, the complement of the enhanced power graph of G is connected and its diameter is at most* 3.

COROLLARY 1.2. *Let G be a finite group. Then, outside of its isolated vertices, the complement of the power graph of G is connected and its diameter is at most* 3.

The problem of the connectedness of the complement of the enhanced power graph was posed in [2, Question 20]. We are aware of two distinct solutions of this problem: [1, Proposition 3.2] and [7, Theorem 1.6], both also showing that the unique connected component of such a graph has diameter at most 3. Our proof identifies the largest independent set of the complement of an enhanced power graph, and exploits its properties to show connectedness and bounded diameter. We would also like to point out that [1] predates [2], but their contribution has remained unnoticed because their name for the complement of the enhanced power graph is different.

The connectedness of the complement of the power graph is proved in [2, Theorem 9.9], while [2, Question 19] asks whether or not its diameter is bounded, and an answer

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in the affirmative has been given in [7, Theorem 1.5]. The novelty of our proof of Corollary 1.2 is that it is an easy consequence of Theorem 1.1.

We conclude this introduction by pointing out that both bounds in Theorem 1.1 and Corollary 1.2 are sharp. Indeed, for two distinct primes p and r, the complements of the power graph and of the enhanced power graph of $C_{pr} \times C_{p^2}$ have diameter 3 (see [7, Lemmas 2.6 and 3.3]).

2. Proof of Theorem 1.1

Let G be a finite group. The *enhanced power graph of* G is defined as the (undirected) graph $\Gamma_{\mathrm{EP}}(G)$ of vertex set G, where two vertices are declared adjacent if they are contained in a common cyclic subgroup of G. We are interested in its complement, which we denote by $C\Gamma_{\mathrm{EP}}(G)$.

Proof of Theorem 1.1. Let G be a finite group and let $g \in G$ be an element of maximal order. (In choosing g, we have already used the finiteness of G.) Observe that cliques in $\Gamma_{\mathrm{EP}}(G)$ are in one-to-one correspondence with maximal cyclic subgroups of G (see [2, Proposition 2.4]). It follows that independent sets in $C\Gamma_{\mathrm{EP}}(G)$ are in one-to-one correspondence with maximal cyclic subgroups of G. In particular, $\langle g \rangle$ is an independent set of maximal size in $C\Gamma_{\mathrm{EP}}(G)$, and hence, all the elements $h \in G$ which are not powers of G are adjacent to g.

Let g^ℓ be a power of g and suppose that g^ℓ is not an isolated vertex (that is, it is not contained in every maximal cyclic subgroup of G). Then, there exists a maximal cyclic subgroup $C \leq G$ such that $g^\ell \notin C$. If we denote by h_ℓ a generator of C, g^ℓ and h_ℓ are adjacent. Therefore, g^ℓ is part of the connected component that contains all the $h \in G$ which are not powers of g. This proves that $C\Gamma_{\mathrm{EP}}(G)$, outside of its isolated vertices, has exactly one connected component.

We now focus on the upper bound on the diameter of the connected component of $C\Gamma_{\mathrm{EP}}(G)$. Observe that our proof already shows that the diameter is bounded from above by 4, and the vertices that might meet the maximal distance are distinct powers of g. Aiming for a contradiction, we suppose that, for some integers a and b, g^a and g^b are at distance 4. We suppose that

$$g^a \sim h_a \sim g \sim h_b \sim g^b$$

is a path of minimal distance connecting g^a and g^b. We point out that for $i \in \{a, b\}$, h_i is the vertex adjacent to g^i built in the previous paragraph. By minimality of this path, h_a and h_b are not adjacent. By the definition of $C\Gamma_{\mathrm{EP}}(G)$, $\langle h_a, h_b \rangle$ is a cyclic subgroup of G and, for $i \in \{a, b\}$, by construction, h_i generates a maximal cyclic subgroup of G. Therefore,

$$\langle h_a, h_b \rangle = \langle h_a \rangle = \langle h_b \rangle .$$

It follows that g^a is adjacent to h_b and g^b is also adjacent to h_a. Hence, g^a and g^b are at distance 2, which is a contradiction. This proves that the diameter of $C\Gamma_{\mathrm{EP}}(G)$ is at most 3. □

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3. Proof of Corollary 1.2

The adjective *enhanced* suggest that there exists an original version of the *power graph*. Indeed, for a finite group G, the power graph $\Gamma_{\mathrm{P}}(G)$ is defined as the graph on the vertex set G such that two elements are declared adjacent if one is a power of the other. We denote its complement by $C\Gamma_{\mathrm{P}}(G)$.

REMARK 3.1 [3, Theorem 2.12]. For any prime p and for any positive integer m, $\Gamma_{\mathrm{P}}(C_{p^m})$ is complete. The converse is also true: if $\Gamma_{\mathrm{P}}(G)$ is complete, then G is a cyclic p-group. Indeed, this follows from the fact that the series

$$C_{p^m} \geq (C_{p^m})^p \geq \cdots \geq (C_{p^m})^{p^{m-1}} \geq 1$$

contains all the subgroups of C_{p^m}. Hence, $C\Gamma_{\mathrm{P}}(C_{p^m})$ contains no edges.

PROOF OF COROLLARY 1.2. By construction, $\Gamma_{\mathrm{P}}(G)$ is a subgraph of $\Gamma_{\mathrm{EP}}(G)$, and hence the complement of the power graph $C\Gamma_{\mathrm{P}}(G)$ contains $C\Gamma_{\mathrm{EP}}(G)$ as a subgraph. We need to focus our attention on those isolated vertices of $C\Gamma_{\mathrm{EP}}(G)$ that have neighbours in $C\Gamma_{\mathrm{P}}(G)$. In the proof of Theorem 1.1, we have shown that all isolated vertices in $C\Gamma_{\mathrm{EP}}(G)$ are powers of an element of maximal order g.

Aiming for a contradiction, we suppose that there is an integer ℓ such that g^ℓ is contained in a connected component of $C\Gamma_{\mathrm{P}}(G)$ that does not contain g. It follows that g^ℓ is not adjacent to any $h \in G - \langle g \rangle$. In particular, g^ℓ is a power of every element of the group that is not a power of g. If two distinct primes divide the order of G, say p and r, then g^ℓ lies both in a Sylow p-subgroup and a Sylow r-subgroup, because g^ℓ is a power of both a p-element and an r-element. The only element with this property is the identity, and thus $g^\ell = 1$ is an isolated vertex. Therefore, G is a p-group, for a suitable prime p, and the neighbourhood of g^ℓ is contained in $\langle g \rangle$. Since g is a p-element, by Remark 3.1, $C\Gamma_{\mathrm{P}}(\langle g \rangle)$ has no edges. Hence, g^ℓ is an isolated vertex, which is a contradiction. This shows that $C\Gamma_{\mathrm{P}}(G)$, outside of its isolated vertices, has a single connected component.

We now focus on the diameter of $C\Gamma_{\mathrm{P}}(G)$. Since $C\Gamma_{\mathrm{P}}(G)$ is a subgraph of $C\Gamma_{\mathrm{EP}}(G)$, Theorem 1.1 implies that if two vertices are at distance 4 in $C\Gamma_{\mathrm{P}}(G)$, then they are distinct powers of g. Aiming for a contradiction, suppose that, for two suitable integers a and b, g^a and g^b are at distance 4 in $C\Gamma_{\mathrm{P}}(G)$. Let

$$g^a \sim x \sim y \sim z \sim g^b$$

be a path of minimal distance connecting g^a and g^b. By definition of $C\Gamma_{\mathrm{P}}(G)$ and by the minimality of the path, we have

$$g^a \notin \langle x \rangle, \quad g^b \notin \langle z \rangle, \quad x \in \langle z \rangle.$$

It might happen that $z \in \langle x \rangle$ is true instead: in this case, we just swap a with b and x with z. Therefore, $\langle x \rangle$ is a subgroup of $\langle z \rangle$. Hence, g^b cannot be an element of $\langle x \rangle$, and g^b and x are adjacent, against the minimality of the chosen path. This final contradiction completes our proof and our note. □

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We observe that the above proof actually never uses the assumption that g^a and g^b are powers of g.

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MARCO BARBIERI, Dipartimento di Matematica 'Felice Casorati',
University of Pavia, Via Ferrata 5, 27100 Pavia, Italy
e-mail: marco.barbieri07@universitadipavia.it

KAMILLA REKVÉNYI, Department of Mathematics,
University of Manchester, M13 9PL Manchester, UK
and
Heilbronn Institute for Mathematical Research, BS8 1UG Bristol, UK
e-mail: kamilla.rekvenyi@manchester.ac.uk

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