Ergod. Th. & Dynam. Sys. (2003), **23**, 1611–1621
DOI: 10.1017/S014338570200158X
© *2003 Cambridge University Press*
Printed in the United Kingdom

Non-wandering sets of topological dynamical systems and C^*-algebras

JUN TOMIYAMA

Department of Mathematics and Physics, Japan Women's University, Mejirodai 2-8-1, Bunkyo-ku, Tokyo 112-0015, Japan

(*Received* 12 *November* 1999 *and accepted in revised form* 10 *December* 2002)

Abstract. Let $\Sigma = (X, \sigma)$ be a topological dynamical system where σ is a homeomorphism in a compact metric space X. Denote by $A(\Sigma)$ the transformation group C^*-algebra associated with this system. We describe the shrinking steps of the non-wandering set $\Omega(\sigma)$ down to the Birkhoff center (depth of the center) in terms of a composition series of the particular ideal of type 1 in $A(\Sigma)$, which corresponds to the center. The result implies C^*-algebraic characterizations of the cases where the depth is 0 and 1. We also give the structure of dynamical systems for which the associated C^*-algebras become algebras with continuous traces.

1. *Introduction*

Let $\Sigma = (X, \sigma)$ be a topological dynamical system on a compact metric space X with a single homeomorphism σ. We then have a family of elementary sets associated with this dynamical system: the set of periodic points, $\mathrm{Per}(\sigma)$, the set of recurrent points, $c(\sigma)$, the set of non-wandering points, $\Omega(\sigma)$, and the set of chain recurrent points, $X(\sigma)$. One might count here the set of aperiodic points, $\mathrm{Aper}(\sigma)$, as another elementary set. However, if we let $C(X)$ be the algebra of all complex valued continuous functions on X and consider the automorphism α on $C(X)$ defined as $\alpha(f)(x) = f(\sigma^{-1}x)$, we then have, as an algebraic counterpart of the dynamical system Σ, the C^*-crossed product $A(\Sigma)$ of $C(X)$ with respect to this automorphism α (considered as an action of the integer group Z). In this context, we have been investigating the interplay between two objects, Σ and $A(\Sigma)$.

Now for most of these elementary sets of dynamical systems their meanings have been clarified from the viewpoint of C^*-algebras as found in [**3, 11**] and [**10**]. For instance, the difference between $c(\sigma)$ and $\mathrm{Per}(\sigma)$ shows how the obstruction differentiates the C^*-algebra $A(\Sigma)$ from a C^*-algebra of type 1 [**3**] and the outside of $X(\sigma)$ reveals how it differs from the class of quasi-diagonal C^*-algebras [**10**]. No C^*-algebraic result, however, has so far been obtained for the non-wandering set $\Omega(\sigma)$. The main purpose of this paper is, therefore, to provide the results to fill this last gap. In fact, we clarify (in the main

https://doi.org/10.1017/S014338570200158X Published online by Cambridge University Press

theorem) the shrinking steps of non-wandering sets down to the Birkhoff center, $\overline{c(\sigma)}$, in terms of the properties of a composition series of the ideal $J(\sigma)$ of type 1 in $A(\Sigma)$ corresponding to the center. Meanwhile, we also characterize the topological properties of the gaps of closed invariant sets that correspond to the composition series of $J(\sigma)$, all of whose quotient algebras become liminal C^*-algebras and, moreover, C^*-algebras with continuous trace. The gaps for the shrinking steps of non-wandering sets, however, seem to be more subtle and we have to introduce another class of type 1 C^*-algebras in between.

We then present related results such as the structure of the C^*-algebra $A(\Sigma)$ for a dynamical system with a finite set, Σ.

2. *C^*-algebraic structure of non-wandering sets*

Throughout this paper we assume that the space X is a compact metric Hausdorff space, while we impose no condition on the dynamical system $\Sigma = (X, \sigma)$ in general. Thus, we emphasize here that in the following all relevant C^*-algebras are separable.

Henceforth, we call the C^*-crossed product $A(\Sigma)$ a homeomorphism C^*-algebra, which is generated by $C(X)$ and a single unitary operator δ implementing the automorphism α on $C(X)$, $\delta f \delta^* = \alpha(f)$. We denote the canonical projection of norm 1 of $A(\Sigma)$ to $C(X)$ by E. The generalized Fourier coefficient of an element a of $A(\Sigma)$ of nth order is defined by $a(n) = E(a\delta^{\star n})$. In the following, all ideals of C^*-algebras are closed two sided ideals. A representation $\tilde{\pi}$ of $A(\Sigma)$ on a Hilbert space H is often written as $\tilde{\pi} = \pi \times u$, meaning a covariant representation $\{\pi, u\}$ of the system $\{C(X), \alpha\}$; i.e. π is the restriction of $\tilde{\pi}$ to $C(X)$ and u is a unitary operator on H with $\tilde{\pi}(\delta) = u$ and $u\pi(f)u^* = \pi(\alpha(f))$.

Recall that a point x of X is called a non-wandering point if for any neighborhood U of x there is a natural number n such that $\sigma^n(U) \cap U \neq \emptyset$. We call a point x recurrent if there exists a subsequence in $O(x)$, the orbit of x, converging to x and denote by $c(\sigma)$ the set of all recurrent points. Let $\Omega(\sigma)$ be the set of all non-wandering points for the dynamical system Σ. For a positive number ε, a sequence $\{x_n\}$ is called an ε-pseudo-orbit or ε-shadow if the distance between $\sigma(x_n)$ and x_{n+1} is less than ε for every n. A point x is then called a chain recurrent point if there exists a cyclic ε-pseudo-orbit for any positive number ε. Let $X(\sigma)$ be the set of all chain recurrent points. We then have the inclusions

$$\mathrm{Per}(\sigma) \subseteq c(\sigma) \subseteq \Omega(\sigma) \subseteq X(\sigma) \subseteq X.$$

Here $\Omega(\sigma)$ and $X(\sigma)$ are invariant closed sets. The closure of $c(\sigma)$, written as $\overline{c(\sigma)}$, is usually called the Birkhoff center.

Now in order to handle non-wandering sets, we must first observe the exceptional behavior of the set $\Omega(\sigma)$ compared with the other elementary sets. The other sets do not change when we restrict the domain of the homeomorphism σ; mostly, this follows from their definitions, but in the case of the set $X(\sigma)$, the fact that the set of chain recurrent points of the restriction of the map σ to $X(\sigma)$ remains the same is highly non-trivial (cf. [**1**, Theorem 7.14]). The non-wandering set of the restriction of σ to $\Omega(\sigma)$ is, however, not the same set in general: it shrinks. If we further consider the non-wandering set of this restricted domain, the set becomes smaller. It is then known that this shrinking step stops at the center [**1**, Proposition 17]. Thus, starting from the sets $\Omega_0(\sigma) = X$ and $\Omega_1(\sigma) = \Omega(\sigma)$,

https://doi.org/10.1017/S014338570200158X Published online by Cambridge University Press

we obtain a decreasing series of closed invariant sets $\{\Omega_\alpha(\sigma)\}$ indexed by the countable ordinal number α $(0 \leq \alpha \leq \gamma)$ such that

$$\Omega_{\alpha+1}(\sigma) = \Omega(\sigma|\Omega_\alpha(\sigma)),$$

and if α is a limit ordinal number,

$$\Omega_\alpha(\sigma) = \bigcap_{\lambda<\alpha} \Omega_\lambda(\sigma).$$

Moreover, the steps end when $\Omega_{\gamma+1}(\sigma) = \Omega_\gamma(\sigma) = \overline{c(\sigma)}$. The minimal such ordinal number γ is called the depth of the center of the homeomorphism σ and is denoted by $d(\sigma)$. It should be noted that, although the fact that the shrinking steps end at the center was known in the time of G. D. Birkhoff, the realization of these steps has only recently been confirmed by Kato [**9**]; i.e. for any given countable ordinal number γ, there exists a homeomorphism σ in some compact metric space X (dendrite) for which $d(\sigma) = \gamma$.

Let $K(\sigma)$ be the largest ideal of type 1 in $A(\Sigma)$. We note that a C^*-algebra A is said to be of type 1 (or post-liminal) if every irreducible representation of A contains the algebra of compact operators. From the point of view of the structure theory of C^*-algebras, it consists of the first tractable classes of C^*-algebras. It is then known that every C^*-algebra A contains the largest ideal K of type 1 such that the quotient algebra A/K contains no non-zero ideal of type 1 (anti-liminal C^*-algebra).

In the present author's joint paper with Aoki [**3**], it is shown that when the space X is metrizable the ideal $K(\sigma)$ is the intersection of all kernels of irreducible representations of $A(\Sigma)$ arising from the points of the set $c(\sigma) \setminus \mathrm{Per}(\sigma)$. Let J be the intersection of all kernels of finite-dimensional irreducible representations of $A(\Sigma)$. We write the intersection of $K(\sigma)$ and J as $J(\sigma)$. This ideal then becomes an ideal of type 1 in $K(\sigma)$ corresponding to the center, $\overline{c(\sigma)}$, in the sense that it is the intersection of all kernels of irreducible representations arising from the points of the center. We must note here that, for a periodic point y, the family of irreducible representations arising from y is parametrized by the torus T written as $\{\tilde{\pi}_{y,\lambda}\}$, while we write the unique infinite-dimensional irreducible representation arising from an aperiodic point x as $\tilde{\pi}_x$. Their unitary equivalences are determined by the orbit $O(y)$ and the parameter λ for a periodic point y and only by its orbit $O(x)$ for an aperiodic point x [**12**, p. 24]. Thus, following [**14**], we denote the intersection of all finite-dimensional irreducible representations associated to the orbit $O(y)$ by $Q(\overline{\bar{y}})$, and the kernel of the infinite-dimensional irreducible representation arising from an aperiodic point x by $P(\bar{x})$. With this, by [**14**, Theorem 2], we have

$$J(\sigma) = \{a \in A(\Sigma) \mid a(n)|\overline{c(\sigma)} = 0\ \forall n\}.$$

Therefore, the ideal $J(\sigma)$ does not have non-trivial finite-dimensional irreducible representations and then every ideal I of $J(\sigma)$, as an ideal of $A(\Sigma)$, becomes a good ideal in the sense of [**14**], which satisfies the condition $E(I) \subset I$. It follows by [**14**, Theorem 2] that there exists a one-to-one correspondence between the family of closed invariant sets $\{S_\lambda\}$ containing the center and the family of ideals $\{I_\lambda\}$ of $J(\sigma)$ given by

$$\begin{aligned} I_\lambda &= \bigcap_{x \in S_\lambda \setminus \overline{c(\sigma)}} P(\bar{x}) \cap J(\sigma) \\ &= \{a \in A(\Sigma) \mid a(n)|S_\lambda = 0\ \forall n\} \end{aligned}$$

https://doi.org/10.1017/S014338570200158X Published online by Cambridge University Press

and

$$S_\lambda = \{x \notin \overline{c(\sigma)} \mid \tilde{\pi}_x(I_\lambda) = 0\} \cup \overline{c(\sigma)}.$$

In this situation, the following lemma is easily verified.

LEMMA 2.1. *Let S be the intersection of the previous kind of closed invariant sets $\{S_\lambda\}$, then the associated ideal I of $J(\sigma)$ is the closure of the union of all associated ideals I_λ.*

Let A be a C^*-algebra of type 1, then the algebra admits an ascending series of ideals $\{I_\alpha\}$ indexed as $0 \leq \alpha \leq \gamma$ by ordinal numbers (called a composition series) such that $I_0 = 0$, $I_\gamma = A$ and if α is a limit ordinal, $I_\alpha = \overline{\bigcup_{\lambda<\alpha} I_\lambda}$. For this composition series we can specify that each quotient algebra $I_{\alpha+1}/I_\alpha$ be a liminal C^*-algebra. Here we call a C^*-algebra liminal if the image of every irreducible representation consists of compact operators. Let $\hat{A}$ be the dual of a C^*-algebra A. In a type 1 C^*-algebra, $\hat{A}$ is identified with the space $\mathrm{Prim}(A)$ of all primitive ideals of A equipped with the hull-kernel topology. We then call A a C^*-algebra with continuous trace if $\hat{A}$ is a Hausdorff space and for each point π_0 of $\hat{A}$ there exist an element a of A and a neighborhood W of π_0 such that $\pi(a)$ is a projection of rank 1 for every irreducible representation π in W [**5**, Proposition 4.5.4]. This algebra necessarily becomes a liminal C^*-algebra and it is known [**5**, Theorem 4.5.5] that the composition series $\{I_\alpha\}$ of the C^*-algebra A can, furthermore, be specified as one in which all quotient algebras are C^*-algebras with continuous trace. This type of composition series of A may well not be unique in general.

Now let S_1 and S_2 be closed invariant sets of X containing the center with $S_1 \supset S_2$ and let I_1 and I_2 be the corresponding ideals of $J(\sigma)$.

The following two facts are the key results of our discussion. Before going into their proofs, however, it would be better to give the detailed structure of the dual of the quotient algebra I_2/I_1 for those readers who are not very familiar with C^*-theory.

Every irreducible representation $\tilde{\pi}$ of I_2/I_1 arises from a point of $S_1 \setminus S_2$, so that the dual of I_2/I_1 is identified with the orbit space $(S_1 \setminus S_2)/Z$. In fact, the representation $\tilde{\pi}$ is regarded as a representation of I_2 vanishing on I_1 and, moreover, it can be considered as the restriction of the irreducible representation of $A(\Sigma)$ (denoted by $\tilde{\pi} = \pi \times u$). Let $\Sigma_\pi = (X_\pi, \sigma_\pi)$ be the dynamical system induced by this representation $\pi \times u$ [**12**, p. 26]. Then it is topologically transitive [**12**, Proposition 4.4] and since X is a compact metric space, there exists a point x in X_π with a dense orbit. Moreover, since $\tilde{\pi}$ does not vanish on I_2, this point x does not belong to the center (nor to S_2); hence, by [**3**, Proposition 1], the image $\tilde{\pi}(A(\Sigma))$ contains the algebra of compact operators. It follows by [**5**, Proposition 4.1.10] that $\tilde{\pi}$ is unitarily equivalent to the irreducible representation $\tilde{\pi}_x$ induced by x because they have the same kernel by [**11**, Proposition 5.2]. Thus, the dual $\widehat{I_2/I_1}$ is identified with the orbit space $(S_1 \setminus S_2)/Z$ as sets and, since the algebra is of type 1, it can also be identified as a topological space. This last assertion holds, however, only because the algebra I_2/I_1 does not have finite-dimensional irreducible representations. In fact, let

$$D = \{\tilde{\pi}_x \mid x \in D' \subset S_1 \setminus S_2\}$$

https://doi.org/10.1017/S014338570200158X Published online by Cambridge University Press

be a subset of $\widehat{I_2/I_1}$, then we have, by [**12**, Proposition 5.2], the following equivalences,

$$\begin{aligned}\tilde{\pi}_z \in \overline{D} \quad \text{in } \widehat{I_2/I_1} &\Longleftrightarrow \bigcap_{x\in D'} P(\bar{x}) \subset P(\bar{z}) \quad \text{and} \quad z \in S_1 \setminus S_2 \\ &\Longleftrightarrow \overline{O(z)} \subset \overline{\bigcup_{x\in D'} O(x)} \\ &\Longleftrightarrow \bar{z} \in \overline{D'} \quad \text{in } (S_1 \setminus S_2)/Z.\end{aligned}$$

With this topological identification, henceforth we denote a point $\tilde{\pi}_x$ of $\widehat{I_2/I_1}$ by $\bar{x}$ taking a point x in $S_1 \setminus S_2$. We also denote the quotient map from $S_1 \setminus S_2$ to its orbit space by q.

LEMMA 2.2. *The quotient algebra I_2/I_1 becomes liminal if and only if the boundary set $\partial O(x) = \overline{O(x)} \setminus O(x)$ is contained in S_2 for every point x in $S_1 \setminus S_2$.*

Proof. Take a point x in $S_1 \setminus S_2$ and let $\tilde{\pi} = \pi \times u$ be the irreducible representation on a Hilbert space H induced by x. From the proof of Proposition 1 in [**3**], we see that all points of $O(x)$ are isolated points in the space $\overline{O(x)}$ and the characteristic function of each point becomes a one-dimensional projection in H. Assume the algebra I_2/I_1 is liminal and suppose that there exists a point y_0 in the boundary $\partial O(x)$ which does not belong to S_2. Then there exists a compact neighborhood U of y_0 in S_1 containing infinite points of $O(x)$ which is disjoint from S_2. Take a positive continuous function f on X satisfying the conditions $f|U = 1$ and $f|S_2 = 0$. By definition, f belongs to I_2 and if we consider the induced dynamical system (X_π, σ_π) where $X_\pi = \overline{O(x)}$, the image $\pi(f)$ is regarded as the restriction of f to $\overline{O(x)}$. Furthermore, since $\tilde{\pi}$ is an infinite dimensional irreducible representation, we can apply [**12**, Theorem 5.4] to see that every spectral projection of $\pi(f)$ belongs to $C(\overline{O(x)}) \cap C(H)$. Therefore, $\pi(f)$ cannot be a compact operator on H, a contradiction. Thus, the boundary $\partial O(x)$ is contained in S_2 for every point x in $S_1 \setminus S_2$.

Conversely assume the previous condition and consider the same irreducible representation $\tilde{\pi}$ of $A(\Sigma)$ as before, induced by a point x in $S_1 \setminus S_2$ (also considered as an irreducible representation of I_2/I_1). We assert that the image $\tilde{\pi}(I_2)$ consists of compact operators. Now consider the induced dynamical system $\Sigma_\pi = (X_\pi, \sigma_\pi)$ as before, where $X_\pi = \overline{O(x)}$. Note that, by [**12**, Corollary 5.1 A], the image $\tilde{\pi}(A(\Sigma))$ has the crossed product structure as $A(\Sigma_\pi)$ and if we take the canonical projection E_π in the algebra, we have the relation

$$\pi \circ E(a) = E_\pi \circ \tilde{\pi}(a) \quad \text{for every } a \text{ in } A(\Sigma).$$

Hence, we can see that the ideal $\tilde{\pi}(I_2)$ is also a well-behaved ideal of the homeomorphism algebra $A(\Sigma_\pi)$, i.e. $E_\pi(\tilde{\pi}(I_2)) \subset \tilde{\pi}(I_2)$. Thus in order to show that $\tilde{\pi}(I_2)$ consists of compact operators, it is enough, by [**14**, Theorem 2], to show that the image $E_\pi(\tilde{\pi}(I_2))$ consists of compact operators. Put $f = \pi(E(a))$ for an element $a \in I_2$. The element $E(a)$ belongs to I_2; hence, f vanishes on $\partial O(x)$ because we may regard f as the restriction of $E(a)$ to $\overline{O(x)}$. Therefore, for any positive number ε, we can find a $\delta > 0$ such that $d(\partial O(x), y) < \delta$ implies $|f(y)| < \varepsilon$, where $d(\partial O(x), y)$ means the distance between y and the boundary set. Hence, the set $\{y \mid |f(y)| \geq \varepsilon\}$ is a closed subset of $O(x)$, and it becomes a finite set, say $\{y_1, y_2, \ldots, y_n\}$. Let p_i be the characteristic function of $\{y_i\}$ and

https://doi.org/10.1017/S014338570200158X Published online by Cambridge University Press

let the function g be given by

$$g(y) = \sum_{i=1}^{n} f(y_i)p_i(y).$$

Obviously, the element g is a compact operator in H and, by definition, $\|f - g\| < \varepsilon$. Hence, the operator f is a compact operator.

This completes the proof. □

PROPOSITION 2.3. *With the above notation, the algebra I_2/I_1 becomes a C^*-algebra with continuous trace if and only if the following conditions hold for the space $S_1 \setminus S_2$:*

(1) *all points of $S_1 \setminus S_2$ are wandering points with respect to the homeomorphism $\sigma|S_1$; and*

(2) *for any neighborhood U of x in $S_1 \setminus S_2$, the set $\overline{\bigcup_{n\in Z} \sigma^n(U)} \setminus \bigcup \sigma^n(U)$ is contained in S_2.*

Proof. Suppose that the algebra I_2/I_1 is a C^*-algebra with continuous trace and take a point x_0 of $S_1 \setminus S_2$. Then there exist a neighborhood W of $\bar{x}_0$ in the dual $\widehat{I_2/I_1}$ and an element a of I_2 such that for every $\bar{x}$ in W the image $\tilde{\pi}_x(a)$ is a one-dimensional projection in the representation space H_x. Put $U = q^{-1}(W)$, then as we have mentioned in the proof of Lemma 2.2, we can find a real-valued continuous function f on X such that the restriction $f|\overline{O(x_0)}$ becomes the characteristic function of x_0 and, hence, a one-dimensional projection in H_{x_0}. In this situation, we obtain a unitary element u in $A(\Sigma)$ such that $\tilde{\pi}_{x_0}(u)$ intertwines the previous two projections ([**7**, Theorem 1] or by elementary spectral calculus). Put $b = uau*$, then $\tilde{\pi}_x(b)$ becomes a projection of rank 1 for every point x in U with $\tilde{\pi}_{x_0}(b) = \pi_{x_0}(f)$, whence $\tilde{\pi}_{x_0}(b - f) = 0$. Now, since the function $\bar{x} \to \|\tilde{\pi}_x(b - f)\|$ is continuous on $\widehat{I_1/I_2}$ because of its Hausdorff property [**5**, Corollary 3.3.9], for any positive ε we find a neighborhood W' of $\bar{x}_0$ in W such that, putting $U' = q^{-1}(W')$, we have

$$\|\tilde{\pi}_x(b - f)\| < \varepsilon \quad \text{for every point } x \in U'.$$

It follows that the spectrum of $\pi_x(f)$ is quite near to that of $\tilde{\pi}_x(b)$. Hence, if we choose ε small enough, we can find a suitable continuous function g on the interval $[-1, 2]$ so that the composed function $h = g \circ f$ satisfies the condition that $\pi_x(h)$ is a projection for every x in U'. Moreover, as we may assume here that

$$\|\pi_x(h) - \tilde{\pi}_x(b)\| < 1,$$

$\pi_x(h)$ is equivalent to $\tilde{\pi}_x(b)$ and becomes a projection of rank 1 for every x in U'. Thus, we see that $\pi_x(h) = h|\overline{O(x)}$ is the characteristic function of a point of $O(x)$. Now let

$$V = \{x \in U' \mid |h(x) - h(x_0)| < 1\}.$$

Then the above arguments lead us to the fact that V is a wandering neighborhood of x_0 with respect to the map $\sigma|S_1$.

For the second condition, let V be a neighborhood of x_0. If there exists a point y_0 in $S_1 \setminus S_2$ that belongs to the set $\partial\overline{\bigcup_{n\in Z} \sigma^n(V)}$, two orbits $O(x_0)$ and $O(y_0)$ could not be separated in the orbit space $(S_1 \setminus S_2)/Z$, contradicting the Hausdorff property.

https://doi.org/10.1017/S014338570200158X Published online by Cambridge University Press

Next, suppose that we have the latter conditions for the space $(S_1 \setminus S_2)$. The second condition then shows that the orbit space $S_1 \setminus S_2/Z$ becomes a Hausdorff space. Now consider a wandering neighborhood U of a point x_0. We can find a continuous function f in I_2 vanishing outside U and taking the value 1 on a neighborhood V contained in U. It is then easily verified that for any point x in V, the image $\pi_x(f)$ on $\overline{O(x)}$ becomes the characteristic function of the set $\{x\}$. Therefore it is a minimal projection of $\tilde{\pi}_x(A(\Sigma))$. Thus, putting $W = q(V)$ in $\widehat{I_1/I_2}$, we see that the algebra I_1/I_2 is a C^*-algebra with continuous trace. This completes the proof. □

It should be noted here that, in general, the second condition need not hold for gaps of non-wandering sets; hence, the notion of C^*-algebras with continuous trace seems to be too strong to describe these gaps. For a similar reason, neither can we use another class of generalized C^*-algebras with continuous trace introduced by Dixmier [**4**]. Thus we reach the following theorem.

THEOREM 2.4. *With the above notation, the set S_2 becomes a non-wandering set with respect to the homeomorphism σ on S_1 if and only if I_2 is the largest ideal of $J(\sigma)$ containing the ideal I_1 which satisfies the following conditions:*

(1) *the quotient algebra I_2/I_1 is liminal; and*

(2) *for each point $\bar{x}_0$ of the dual $\widehat{I_2/I_1}$ there exist a neighborhood W of $\bar{x}_0$ and a continuous function f belonging to I_2 such that $\pi_x(f)$ is a projection of rank 1 for every irreducible representation $\tilde{\pi}_x$ with $\bar{x} \in W$.*

Note that in the theorem we lose the assumption of the Hausdorff property but have instead the stronger condition that an element f is required to be in $C(X)$.

Proof. Suppose that S_2 is the non-wandering set for σ on S_1. Since for each point x in S_1, every point of the set $\overline{O(x)} \setminus O(x)$ becomes a non-wandering point for σ, S_1 is absorbed into S_2, and by Lemma 2.1, I_2/I_1 is a liminal C^*-algebra. Take a point x_0 in $S_1 \setminus S_2$ and let U be a wandering neighborhood of x_0. We can then find a positive continuous function f supported in U with $f(x) = 1$ on a smaller neighborhood V. Write $W = q(V)$, which becomes a neighborhood of $\bar{x}_0$ in $\widehat{I_2/I_1}$. Since U is a wandering neighborhood of x_0, for any point x in V the restriction of f to the set $\overline{O(x)}$ becomes the characteristic function of the set $\{x\}$. It follows from the proof of Proposition 1 in [**3**] that the image $\pi_x(f)$ (identified with $f|\overline{O(x)}$) becomes a projection of rank 1.

We assert next that I_2 is the largest ideal of $J(\sigma)$ with the required properties. In fact, let I be an ideal of $J(\sigma)$ with those properties and let S be the corresponding closed invariant set of X. For the conclusion it is enough to show that the difference $S_1 \setminus S$ consists of wandering points for $\sigma|S_1$. Note that this also proves the converse implication in the theorem.

Now, by assumption, each point x_0 in $S_1 \setminus S$ has a neighborhood U whose image W in the dual space $\widehat{I/I_1}$ satisfies the assigned condition for a continuous function f vanishing on S. Then the condition means that for every point x of U the restriction of f to $\overline{O(x)}$ becomes the characteristic function of some point $\{\sigma^k x\}$. Thus, we may assume here that $f|\overline{O(x_0)}$ is the characteristic function of the set $\{x_0\}$. Write the neighborhood V of x_0 as

$$V = \{x \in U \mid f(x) > \tfrac{1}{2}\}.$$

https://doi.org/10.1017/S014338570200158X Published online by Cambridge University Press

It follows that $f(x) = 1$ for every x in V, whereas $f(\sigma^k x) = 0$ for any integer $k \neq 0$. Therefore, V is a wandering neighborhood and x_0 is a wandering point. This completes the proof. □

In Theorem 2.4, we are not sure whether f may be replaced by a general element a in I_2. If this holds, we could say that the shrinking steps of non-wandering sets are also algebraically invariant as in the case of other elementary sets. Here we call a property of topological dynamical systems an algebraic invariant when two dynamical systems have the same property if their homeomorphism C^*-algebras are isomorphic (cf. [**14**, §4]).

We finally discuss C^*-versions of the cases where the depth of the center is 0 or 1, i.e. $X = \Omega(\sigma)$ and $\Omega(\sigma|\Omega(\sigma)) = \Omega(\sigma)$. As we mentioned before, these amount to the cases where $X = \overline{c(\sigma)}$ and $\Omega(\sigma) = \overline{c(\sigma)}$.

PROPOSITION 2.5.

(1) *The depth of the center is 0 if and only if the largest ideal of type 1, $K(\sigma)$, in $A(\Sigma)$ becomes a residually finite-dimensional C^*-algebra or zero.*

(2) *The depth is 1 if and only if the ideal $J(\sigma)$ is a liminal ideal of $A(\Sigma)$ and satisfies the second condition of Theorem 2.4.*

Here a C^*-algebra is said to be residually finite dimensional if there exist sufficiently many finite-dimensional irreducible representations.

Proof. (1) Suppose that $\Omega(\sigma) = X$, then, as previously mentioned,

$$\Omega(\sigma) = \overline{c(\sigma)} = \overline{c(\sigma) \setminus \mathrm{Per}(\sigma)} \cup \overline{\mathrm{Per}(\sigma)}.$$

Let $\tilde{\rho}$ be the natural homomorphism from $A(\Sigma)$ to $A(\overline{c(\sigma) \setminus \mathrm{Per}(\sigma)})$ derived from the restriction map of the functions to the previous invariant set. Here this means the homeomorphism C^*-algebra on this set. We then have, by [**3**, Theorem 2],

$$\tilde{\rho}^{-1}(0) = K(\sigma) = \{a \in A(\Sigma) \mid a(n)|c(\sigma) \setminus \mathrm{Per}(\sigma) = 0 \text{ for every } n\}$$

and

$$X \setminus (\overline{c(\sigma) \setminus \mathrm{Per}(\sigma)}) \subset \overline{\mathrm{Per}(\sigma)}.$$

Therefore, since for every non-zero element a in $K(\sigma)$ we have the non-zero image $E(a * a)$, we see that this function does not vanish on the set $\mathrm{Per}(\sigma)$. It follows that there exists a finite-dimensional irreducible representation $\tilde{\pi}_{x,\lambda}$ which does not kill the element a (cf. [**14**, Theorem 2]). Thus, $K(\sigma)$ is residually finite-dimensional.

Conversely, let $K(\sigma)$ be such an algebra and suppose that $\overline{c(\sigma)} \neq X$. There then exists a non-zero continuous function f vanishing on $c(\sigma)$. By [**14**, Theorem 2] this function f belongs to $K(\sigma)$ and, moreover, f is killed by every finite-dimensional irreducible representation $\tilde{\pi}_{x,\lambda}$. This contradicts the assumption; hence, $\overline{c(\sigma)} = X = \Omega(\sigma)$.

For assertion (2), it is enough simply to apply Theorem 2.4 to $J(\sigma)$.

We meet the first case, for instance, when the dynamical system is topologically transitive (although, in this case, $K(\sigma) = 0$ if X is infinite). For the case of depth one, perhaps a good class of examples is a dynamical system satisfying the pseudo-orbit tracing property (POTP).

https://doi.org/10.1017/S014338570200158X Published online by Cambridge University Press

Recall that, for positive δ, a sequence $\{x_i\}$ is called a δ-pseudo orbit for σ if

$$d(\sigma(x_i), x_{i+1}) < \delta \quad \text{for every } i.$$

For a positive ε, it is then said to be ε-traced if there exists a point $x \in X$ such that

$$d(\sigma^i(x), x_i) < \varepsilon \quad \text{for every } i.$$

We say that a dynamical system $\Sigma = (X, \sigma)$ satisfies the POTP if, for every ε, there exists $\delta = \delta(\varepsilon) > 0$ such that every δ-pseudo orbit is ε-traced. It is then known [**2**, Theorem 7.20] that the restriction of σ to $\Omega(\sigma)$ also has the POTP. Since, in this case, the set of chain recurrent points coincides with the non-wandering set, we have

$$\Omega(\sigma|\Omega(\sigma)) = X(\sigma|X(\sigma)) = X(\sigma) = \Omega(\sigma).$$

Hence, by [**1**, Proposition 17], $\Omega(\sigma) = \overline{c(\sigma)}$, i.e. the depth of the center is 1. □

3. *Related results*

In this section, we treat problems related to our preceding main results. One may first recognize that the C^*-algebra $A(\Sigma)$ may rarely become a C^*-algebra with continuous trace. That this is actually the case is assured in the next result. In the following, we write

$$\mathrm{Per}^n(\sigma) = \{x \in X \mid \sigma^n x = x\}$$

and denote the set of n-periodic points by $\mathrm{Per}_n(\sigma)$. We recall, moreover, that the primitive ideals $P(\bar{x})$ for an aperiodic point x and $P(\bar{y}, \lambda)$ for a periodic point y are kernels of those irreducible representations $\tilde{\pi}_x$ and $\tilde{\pi}_{y,\lambda}$ respectively.

PROPOSITION 3.1. *The algebra $A(\Sigma)$ becomes a C^*-algebra with continuous trace if and only if the dynamical system Σ consists of periodic points with finite numbers periodicities, $\{n_1, n_2, \ldots, n_k\}$, and each set $Per_{n_j}(\sigma)$ is open and closed.*

Proof. Suppose that $A(\Sigma)$ is a C^*-algebra with continuous trace. It is then a liminal C^*-algebra and since $A(\Sigma)$ is unital, every irreducible representation becomes finite-dimensional. Hence, by [**12**, Theorem 4.6], X consists of periodic points and, as we have already mentioned, the dual of $A(\Sigma)$ is regarded as the product set (not as the product topological space) $X/Z \times T$ with the correspondence $[\tilde{\pi}_{x,\lambda}] \leftrightarrow (\bar{x}, \lambda)$. Now take an arbitrary point $(\bar{x}_0, \lambda_0)$ in $\widehat{A(\Sigma)}$ where $x_0 \in \mathrm{Per}_p(\sigma)$. By the assumption, there exist an element a of $A(\Sigma)$ and a neighborhood U of $(\bar{x}_0, \lambda_0)$ such that $\tilde{\pi}_{x,\lambda}(a)$ is a projection of rank 1 for every point $(\bar{x}, \lambda)$ in U. Let $\{e_{ij}\}$ be a matrix units of $\tilde{\pi}_{x_0,\lambda_0}(A(\Sigma))$ $(=M_p)$ with $e_{11} = \tilde{\pi}_{x_0,\lambda_0}(a)$. We can then find another neighborhood V in U for which there exists a set $\{a_{ij}\}$ in $A(\Sigma)$ such that $a_{11} = a$ and $\{\tilde{\pi}_{x,\lambda}(a_{ij})\}$ keep all relations of the matrix units for every $(\bar{x}, \lambda)$ [**8**, Lemmas 10 and 11], inducing the system $\{e_{ij}\}$ at $(\bar{x}_0, \lambda_0)$. Here the element $1_{x,\lambda} - \sum_{i=1}^{p} \tilde{\pi}_{x,\lambda}(a_{ij})$ is a projection at each point $(\bar{x}, \lambda)$ in V. Since the norm functions are continuous by the Hausdorff property of the dual [**5**, Corollary 3.3.4], there exists a neighborhood W in V on which this projection becomes zero. Thus, the phenomenon of so-called 'dimension drops' does not occur in our case and the p-dimensional component $\widehat{A(\Sigma)}_p$ becomes open in $\widehat{A(\Sigma)}$.

https://doi.org/10.1017/S014338570200158X Published online by Cambridge University Press

We assert that the set $\mathrm{Per}_p(\sigma)$ is open in X. Let G be the image of W in the space $\mathrm{Per}_p(\sigma)/Z$ by the projection map on $\mathrm{Per}_p(\sigma)/Z \times T$. We show that G is an open neighborhood of $\bar{x}_0$ in $\mathrm{Per}(\sigma)/Z$ so that its inverse image becomes an open set of X contained in $\mathrm{Per}_p(\sigma)$. In fact, let G^c be the complement of G. It follows then that the product space $G^c \times T$ is included in the complement W^c, which is a closed subset in the dual space. Take a point $\bar{y}_0$ in G, then we find a number μ_0 in T such that $(\bar{y}_0, \mu_0)$ belongs to W. Since the topology of $\widehat{A(\Sigma)}$ is regarded as the hull–kernel topology, the ideal $P(\bar{y}_0, \mu_0)$ does not include the ideal of the intersection:

$$\bigcap_{G^c \times T} P(\bar{x}, \lambda) = \bigcap_{G^c} Q(\bar{x}).$$

Now suppose that there exists a sequence $\{\bar{x}_n\}$ in G^c converging to $\bar{y}_0$ in X/Z. We then have

$$O(y_0) \subseteq \overline{\bigcup_n O(x_n)}.$$

It follows that

$$\bigcap_{G^c} Q(\bar{x}) \subseteq \bigcap_n Q(\bar{x}_n) \subseteq Q(\bar{y}_0) \subseteq P(\bar{y}_0.\mu_0),$$

a contradiction. After all, the inverse of G in $\mathrm{Per}(\sigma)_p$ becomes an open subset of X/Z; hence, $\mathrm{Per}_p(\sigma)$ is an open set of X. Since X consists of periodic points, it must also be a closed set. Thus, only finite numbers of periodicities in Σ appear.

Conversely, suppose that we have such a dynamical system Σ and let $\{q_1, q_2, \ldots, q_k\}$ be characteristic functions of the sets $\{\mathrm{Per}_{p_i}(\sigma)\}$. Since these sets are invariant, one may easily see that these functions (regarded as projections) are invariant central projections of $A(\Sigma)$ and $A(\Sigma)$ can be written as a direct sum of $A(\Sigma)q_i$. Here, each direct summand becomes a p_i-dimensional homogeneous C^*-algebra, by which we mean its irreducible representation is always p_i-dimensional. It is then well known that a homogeneous C^*-algebra is a C^*-algebra with continuous trace. Therefore, as a direct sum of a finite number of such algebras, $A(\Sigma)$ becomes a C^*-algebra with continuous trace. □

We next consider the case where $\Omega(\sigma)$ is a finite set. We often meet this type of dynamical systems, e.g. in the case of Morse–Smale systems. In this case, the set $\Omega(\sigma)$ naturally consists of only periodic points and $A(\Sigma)$ is of type 1.

PROPOSITION 3.2. *The set $\Omega(\sigma)$ is a finite set if and only if $J(\sigma)$ is the ideal of $A(\Sigma)$ having properties (1) and (2) in Theorem 2.4 (as $I_1 = 0$) such that its quotient algebra is a direct sum of finite numbers of algebras of all matrix valued continuous functions on the torus.*

Proof. Suppose that $\Omega(\sigma)$ is finite. Then the ideal $J(\sigma)$ coincides with the ideal J and it has the properties in Theorem 2.4. Since the quotient algebra is naturally considered as the homeomorphism C^*-algebra on the set $\Omega(\sigma)$, it is a direct sum of finite numbers of matrix valued continuous functions on the torus T by [**13**, Proposition 3.5]. Here, the central projection for each direct summand appears as the quotient image of a continuous function with the value 1 on an assigned periodic orbit and vanishing on other periodic orbits.

Conversely if $A(\Sigma)$ has that structure it is of type 1. Moreover, it implies that the non-wandering set consists of finite numbers of periodic points by Theorem 2.4. □

https://doi.org/10.1017/S014338570200158X Published online by Cambridge University Press

From this observation we see the following result.

PROPOSITION 3.3. *The property $\Omega(\sigma)$ is finite is an algebraic invariant, i.e. if $\Sigma' = (Y, \tau)$ is a dynamical system for which $A(\Sigma')$ is isomorphic to the algebra $A(\Sigma)$ of such a dynamical system, it has the same property.*

Proof. Let I be the ideal of $A(\Sigma')$ corresponding to $J(\sigma)$. From the definition, every irreducible representation of $J(\sigma)$ is infinite-dimensional and arises from a non-recurrent point of Σ. Since such an irreducible representation is characterized as the one containing the algebra of compact operators [**3**, Proposition 1], every irreducible representation of I comes from a non-recurrent point in Y. It follows by the structure of $A(\Sigma')$ that the space Y consists of non-recurrent points together with finite numbers of periodic points. Thus, Σ' has the same property. □

Acknowledgements. The author is deeply indebted to H. Kato and K. Hiraide for a clear understanding of non-wandering sets. Thanks are also due to F. Hiai, M. Nagisa and A. Sannami for stimulating conversations during this investigation. A substantial part of this paper was written while the author was visiting the H. C. Oersted Institute in Copenhagen in the summer of 1998. The author expresses his deep gratitude to G. K. Pedersen and his colleagues for their kind hospitality.

REFERENCES

[1] E. Akin. *The General Topology of Dynamical Systems (Graduate Studies in Mathematics, 1)*. American Mathematical Society, Providence, RI, 1993.
[2] N. Aoki. Topological dynamics. *Topics in General Topology*. North-Holland, Amsterdam, 1989, pp. 625–740.
[3] N. Aoki and J. Tomiyama. Characterizations of topological dynamical systems whose transformation group C^*-algebras are antiliminal and of type 1. *Ergod. Th. & Dynam. Sys.* **13** (1993), 1–5.
[4] J. Dixmier. Traces sur les C^*-algébres. *Ann. Inst. Fourier* **13** (1963), 219–262.
[5] J. Dixmier. *C^*-algebras*. North-Holland, Amsterdam, 1977.
[6] J. M. G. Fell. The structure of algebras of operator fields. *Acta Math.* **106** (1961), 233–280.
[7] J. Glimm and R. V. Kadison. Unitary operators in C^*-algebras. *Pacific J. Math.* **10** (1960), 547–556.
[8] J. Glimm. Type 1 C^*-algebras. *Ann. Math.* **73** (1961), 572–612.
[9] H. Kato. The depth of centers of maps on dendrites. *J. Austral. Math.* **63** (1997), 1–10.
[10] M. Pimsner. Embedding of some transformation group C^*-algebras into AF algebras. *Ergod. Th. & Dynam. Sys.* **3** (1983), 613–626.
[11] J. Tomiyama. *Invitation to C^*-algebras and Topological Dynamical Systems*. World Scientific, Singapore, 1987.
[12] J. Tomiyama. The interplay between topological dynamics and theory of C^*-algebras. *Lecture Note No. 2*, Research Institute of Mathematics, Seoul, 1992.
[13] J. Tomiyama. C^*-algebras and topological dynamical systems. *Rev. Math. Phys.* **8** (1996), 741–760.
[14] J. Tomiyama. Structure of ideals and isomorphism problems of C^*-crossed products by single homeomorphisms. To appear in *Tokyo J. Math*.

https://doi.org/10.1017/S014338570200158X Published online by Cambridge University Press