If the fluid is initially at temperature V_0 and the slab at zero temperature, and no heat is supplied, the temperature in the slab is given by

$$\frac{v}{V_0} = -2H \sum_{s=1}^{\infty} \frac{(\beta_s^2 - kH)\, e^{-\beta_s^2 T}}{P_s \sin \beta_s} \sin \beta_s (1 - x/l), \tag{33}$$

and the temperature of the fluid is

$$\frac{V}{V_0} = 2kH^2 \sum_{s=1}^{\infty} \frac{1}{P_s} e^{-\beta_s^2 T}, \tag{34}$$

where P_s and the β_s are defined in (31) and (32) respectively.

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RELATIVISTIC WAVE EQUATIONS FOR ZERO REST-MASS

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1. In this paper I establish in a certain sense all the relativistically invariant sets of linear partial homogeneous differential equations in which every unknown function u satisfies $\Box u = 0$, $\Box$ being the relativistic Laplace operator. Analogous questions for any orthogonal group are dealt with in a previous paper*, but it was thought that an independent and not too technical treatment for the Lorentz group might be of interest.

Our starting point is a well-known property of Maxwell's equations. Let $x^1, \ldots, x^4$ be real variables and put $g(x, x) = g_{ik} x^i x^k = x^1 x^1 - x^2 x^2 - x^3 x^3 - x^4 x^4$ in the ordinary tensor notation. Put $d_j = \partial/\partial x^j$ and let $F^{jk} = -F^{kj}$ be six functions of $x^1, \ldots, x^4$ transforming like the components of a skew-symmetric tensor. Then

$$d_j F^{jk} = 0, \tag{1·1}$$

$$d^i F^{jk} + d^j F^{ki} + d^k F^{ij} = 0, \tag{1·2}$$

are Maxwell's equations in empty space. They are invariant under the full Lorentz group L leaving $g(x, x)$ invariant, and (1·1) transforms like x^k, and (1·2) like the component x^{ijk} of an entirely antisymmetric tensor. Now, by (1·1) and (1·2),

$$d^j d_i F^{ik} - d^k d_i F^{ij} = 0, \tag{1·1′}$$

$$d_i d^i F^{jk} + d_i d^j F^{ki} + d_i d^k F^{ij} = 0. \tag{1·2′}$$

Adding these equations we get, because of $d_i d_k = d_k d_i$,

$$d_i d^i F^{jk} = (d^1 d^1 - d^2 d^2 - d^3 d^3 - d^4 d^4)\, F^{jk} = \Box F^{jk} = 0.$$

We observe that (1·1′) and (1·2′) both transform like F^{jk}.

* Gårding(4).

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2. The well-known formulae above lead to the following straightforward generalization. Let $u_1, \ldots, u_p$ be p functions* of $x^1, \ldots, x^4$, transforming like the components of some irreducible spinor† when $x^1, \ldots, x^4$ transform under L. Splitting the spinor with the components $d_j u_q$ into irreducible parts, we obtain certain expressions, say $A^k_{\alpha_k} = a^k_{\alpha_k jq} d_j u_q$, which transform irreducibly for k fixed. Here $a^k_{\alpha_k jq}$ are complex constants, $k = 1, \ldots, \bar{n}$; $\alpha_k = 1, \ldots, n_k$; $j = 1, \ldots, 4$; $q = 1, \ldots, p$; and the summation convention is used. Now any linear combinations of the $d_j u_q$ transforming linearly are equivalent to certain $A^k_{\alpha_k}$ (in the sense of mutual linear dependence), k running through a set (k) of values taken from $1, \ldots, \bar{n}$. Hence

$$A^k_{\alpha_k} = a^k_{\alpha_k jq} d_j u_q = 0, \tag{2·1}$$

with (k) arbitrary, are the most general relativistically invariant equations we can get by putting linear combinations of the $d_j u_q$ equal to 0. Now (as will be proved later on) for every q there exist numbers $b^k_{q\alpha_k i}$, not all zero, such that $b^k_{q\alpha_k i} d_i A^k_{\alpha_k} = C^k_q$ (not summed with respect to k) transforms like u_q. By (2·1),

$$C^k_q = b^k_{q\alpha_k i} a^k_{\alpha_k jr} d_i d_j u_r = 0. \tag{2·2}$$

It is obvious that (2·1) is analogous to (1·1) and (1·2), and that (2·2) is analogous to (1·1′) and (1·2′). Now by combining (1·1′) and (1·2′) and using $d_j d_i = d_i d_j$ we get $\Box F^{ik} = 0$. Hence it is natural to look for all sets (k) of values of k giving an equation

$$\sum_{k \subset (k)} c_k C^k_q = \sum_{k \subset (k)} c_k b^k_{q\alpha_k i} a^k_{\alpha_k jr} d_i d_j u_r = \Box u_q, \tag{2·3}$$

in which the c_k are constants and the commutability relation $d_i d_j = d_j d_i$ is to be used. Then by (2·2) we get $\Box u_q = 0$ for every q. Clearly if $(k') \supset (k)$ and (k) gives (2·3), then (k') also gives (2·3). Hence we may ask at once for all 'minimum sets' (k) giving (2·3) and not containing any proper subset with the same property. We proceed to write explicitly all equations (2·1) and (2·2), and it will then be possible to find all minimum sets. Except in certain degenerate cases the result is that (k) *is a minimum set if and only if the number of values of* k *included in* (k) *equals* $\frac{1}{2}\bar{n}$ *if* $\bar{n}$ *is even and* $\frac{1}{2}(\bar{n}+1)$ *if* $\bar{n}$ *is odd*‡. The minimum sets in the degenerate cases will depend on which part of the Lorentz group we are concerned with.

3. Before carrying out our programme, it is convenient to give a brief survey of the Lorentz group and the spin representations§. The elements of L are all the real transformations o,

$$x^k \to o^k_i x^i \quad (i, k = 1, \ldots, 4), \tag{3·1}$$

leaving $g(x, x)$ invariant. Put $\sigma_l(o) = \operatorname{sgn} o^1_1 = \pm 1$ and $\sigma_s(o) = \operatorname{sgn} |o^i_k| = \pm 1$ $(i, k = 2, 3, 4)$. Then $\sigma_s(o)$ is a character, i.e. $\sigma_s(o)\sigma_s(o') = \sigma_s(oo')$, and similarly for

* In the following only the transformation properties of d_j and u_q and the commutability $d_j d_k = d_k d_j$ are relevant, so that the actual interpretation of u_q as a function and d_j as a differential operator is not necessary.

† The representations of L given by the spinors are completely reducible, and in a certain sense constitute 'all' representations of L. See Cartan (2, II) and van der Waerden(5). The assumption that $u_1, \ldots, u_p$ transform irreducibly is necessary for the following and cannot be removed without changing the results considerably.

‡ With certain exceptions when the number of variables of the fundamental form is even, this rule is valid for any orthogonal group. See Gårding(4), chapter 5.

§ Cartan(2), Brauer and Weyl(1), van der Waerden(5).

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$\sigma_t(o)$. Further, $\sigma(o) = \sigma_s(o)\sigma_t(o) = |o_k^i| = \pm 1$ $(i, k = 1, \ldots, 4)$. Topologically L consists of four disconnected pieces $L^{\pm\pm}$, o belonging to $L^{\sigma_t(o)\sigma_s(o)}$. The identical transformation belongs to L^{++} and there are four subgroups of L consisting of the pieces $L^{\pm\pm}$, namely L^{++}, $L^{++}+L^{--}$, $L^{++}+L^{+-}$, and $L^{++}+L^{-+}$. The group $L^+ = L^{++}+L^{--}$ contains only proper transformations with $\sigma(o) = 1$, while its co-set contains only improper transformations with $\sigma(o) = -1$. We shall consider equations invariant under some of these groups, and our results will be found to be different according as the actual group does or does not contain improper transformations. Put

$$x^1+x^2 = x^{1\dot{1}}, \quad x^1-x^2 = x^{2\dot{2}}, \quad x^3+ix^4 = x^{1\dot{2}}, \quad x^3-ix^4 = x^{2\dot{1}}, \tag{3·2}$$

with $i^2 = -1$. Then, according as $\sigma(o) = +1$ or $\sigma(o) = -1$, (3·1) may be written as

$$x^{\alpha\dot{\beta}} \to a^{\alpha}_{\alpha'} a^{\dot{\beta}}_{\dot{\beta}'} x^{\alpha'\dot{\beta}'}, \tag{3·3}$$

or

$$x^{\alpha\dot{\beta}} \to a^{\alpha}_{\dot{\alpha}'} a^{\dot{\beta}}_{\beta'} x^{\beta'\dot{\alpha}'}, \tag{3·4}$$

with $\left|a^{\alpha}_{\beta}\right| = \left|a^{\alpha}_{\dot{\beta}}\right| = \left|a^{\dot{\alpha}}_{\beta}\right| = \left|a^{\dot{\alpha}}_{\dot{\beta}}\right| = 1$, the matrices $\left(a^{\alpha}_{\beta}\right)$ and $\left(a^{\dot{\alpha}}_{\dot{\beta}}\right)$ being uniquely determined apart from a simultaneous change of sign. Similarly for $\left(a^{\alpha}_{\dot{\beta}}\right)$ and $\left(a^{\dot{\alpha}}_{\beta}\right)$. A spinor with the components $u^{(\alpha)(\dot{\beta})} = u^{\alpha_1\ldots\alpha_m, \dot{\beta}_1\ldots\dot{\beta}_n}$ symmetrical in each kind of index transforms irreducibly under L^{++}, the transformation (3·1) being represented by

$$u^{\alpha_1\ldots\alpha_m, \dot{\beta}_1\ldots\dot{\beta}_n} \to a^{\alpha_1}_{\alpha_1'} \ldots a^{\alpha_m}_{\alpha_m'} a^{\dot{\beta}_1}_{\dot{\beta}_1'} \ldots a^{\dot{\beta}_n}_{\dot{\beta}_n'} u^{\alpha_1'\ldots\alpha_m', \dot{\beta}_1'\ldots\dot{\beta}_n'}. \tag{3·5}$$

Because of $\left|a^{\alpha}_{\beta}\right| = 1$, the expression $u^{12}-u^{21}$ is invariant under L^{++}, or, more generally, L^+. This gives the usual law of lowering spinor indices, namely $u_2 = u^1$ and $u_1 = -u^2$, so that $u^{12}-u^{21} = u_\alpha^\alpha$. Similarly $u_{\dot{2}} = u^{\dot{1}}$ and $u_{\dot{1}} = -u^{\dot{2}}$. In the following we only consider representations of L^{++} given by (3·5).

4. We shall have to consider representations of L and its various subgroups reducing to the representations (3·5) of L^{++}. Put $f(\tau, o) = 1$, $\sigma_t(o)$, $\sigma_s(o)$, $\sigma(o)$ according as $\tau = 1, t, s, ts$. Then $f(\tau, o)\, f(\tau, o') = f(\tau, oo')$. A spinor with components

$$u_\tau^{(\alpha)(\dot{\beta})} = u_\tau^{\alpha_1\ldots\alpha_m, \dot{\beta}_1\ldots\dot{\beta}_n},$$

symmetrical in each kind of index, transforms irreducibly under $L^+ = L^{++}+L^{--}$, the proper transformation (3·3) being represented by

$$u_\tau^{\alpha_1\ldots\alpha_m, \dot{\beta}_1\ldots\dot{\beta}_n} \to f(\tau, o)\, a^{\alpha_1}_{\alpha_1'} \ldots a^{\alpha_m}_{\alpha_m'} a^{\dot{\beta}_1}_{\dot{\beta}_1'} \ldots a^{\dot{\beta}_n}_{\dot{\beta}_n'} u_\tau^{\alpha_1'\ldots\alpha_m', \dot{\beta}_1'\ldots\dot{\beta}_n'}. \tag{4·1}$$

Now for L^+, $\sigma_t(o)\sigma_s(o) = 1$, so that there are only two inequivalent transformations (4·1) corresponding to, for instance, $\tau = 1$ and $\tau = t$. They constitute all the irreducible representations of L^+ reducing to the spin representation of L^{++}. If $m \neq n$, then $u_\tau^{(\alpha)(\dot{\beta})} = u_\tau^{\alpha_1\ldots\alpha_m, \dot{\beta}_1\ldots\dot{\beta}_n}$ and $u_\tau^{(\beta)(\dot{\alpha})} = u_\tau^{\beta_1\ldots\beta_n, \dot{\alpha}_1\ldots\dot{\alpha}_m}$ transform irreducibly under L, (4·1) being valid for both quantities for the proper transformation (3·3), and

$$u_\tau^{\alpha_1\ldots\alpha_m, \dot{\beta}_1\ldots\dot{\beta}_n} \to f(\tau, o)\, a^{\alpha_1}_{\dot{\alpha}_1'} \ldots a^{\alpha_m}_{\dot{\alpha}_m'} a^{\dot{\beta}_1}_{\beta_1'} \ldots a^{\dot{\beta}_n}_{\beta_n'} u_\tau^{\beta_1'\ldots\beta_n', \dot{\alpha}_1'\ldots\dot{\alpha}_m'}, \tag{4·2}$$

$$u_\tau^{\beta_1\ldots\beta_n, \dot{\alpha}_1\ldots\dot{\alpha}_m} \to f(\tau, o)\, a^{\beta_1}_{\beta_1'} \ldots a^{\beta_n}_{\beta_n'} a^{\dot{\alpha}_1}_{\dot{\alpha}_1'} \ldots a^{\dot{\alpha}_m}_{\dot{\alpha}_m'} u_\tau^{\alpha_1'\ldots\alpha_m', \dot{\beta}_1'\ldots\dot{\beta}_n'} \tag{4·3}$$

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corresponding to the improper transformation (3·4). Here again the possibilities for τ reduce to two inequivalent ones, $\tau = 1$ and $\tau = t$, because $u_\tau^{(\alpha)(\beta)}$ and $u_\tau^{(\beta)(\dot\alpha)}$ transform like $u_{\tau ts}^{(\alpha)(\beta)}$ and $-u_{\tau ts}^{(\beta)(\dot\alpha)}$ respectively, if $t^2 = s^2 = 1$ and $ts = st$. When $m = n$, $u_\tau^{(\alpha)(\beta)}$ alone transforms irreducibly according to (4·1) and (4·2), and now the four values of τ give four inequivalent transformations. Under $L^{++} + L^{+-}$, $u_\tau^{(\alpha)(\beta)}$ transforms like $u_{\tau t}^{(\alpha)(\beta)}$. This means that there is one irreducible representation of $L^{++} + L^{+-}$ given by $u_\tau^{(\alpha)(\beta)}$, $u_\tau^{(\beta)(\dot\alpha)}$ if $m \neq n$ and two irreducible inequivalent representations given by $u_1^{(\alpha)(\beta)}$ and $u_s^{(\alpha)(\beta)}$ respectively when $m = n$. Similarly for the group $L^{++} + L^{-+}$.

5. We are now able to write the expressions (2·1) explicitly. Suppose that $p^{1\dot 1} = d^1 + d^2$, etc., analogously to (3·2), $p^{\alpha\beta}$ transforming as $u_1^{\alpha\beta}$. We require our equations to be invariant under L^+. The results will then be valid for L^{++} as well. The components of the wave function must be $u_\tau^{(\alpha)(\beta)} = u_\tau^{\alpha_2 \dots \alpha_m, \beta_2 \dots \beta_n}$, $m = 1$ and $n = 1$ denoting the absence of undotted and dotted indices respectively. Let s be a symmetry operator which operates on the undotted indices of a spinor making it symmetrical where it is not symmetrical before, so that, for example,

$$sp^{\alpha_1\dot\beta_1} u_\tau^{\alpha_2 \dots \alpha_m, \dot\beta_2 \dots \dot\beta_n} = p^{\alpha_1\dot\beta_1} u_\tau^{\alpha_2 \dots \alpha_m, \dot\beta_2 \dots \dot\beta_n} + p^{\alpha_2\dot\beta_1} u_\tau^{\alpha_1\alpha_3 \dots \alpha_m, \dot\beta_2 \dots \dot\beta_n} + \dots + p^{\alpha_m\dot\beta_1} u_\tau^{\alpha_2 \dots \alpha_{m-1}\alpha_1, \dot\beta_2 \dots \dot\beta_n}.$$

Similarly for $\dot s$. Then the following sets of equations certainly transform irreducibly:

$$s\dot s p^{\alpha_1\dot\beta_1} u_\tau^{\alpha_2 \dots \alpha_m, \dot\beta_2 \dots \dot\beta_n} = 0, \tag{5·1}$$

$$sp^{\alpha_1}{}_{\dot\beta_1} u_\tau^{\alpha_2 \dots \alpha_m, \dot\beta_1\dot\beta_3 \dots \dot\beta_n} = 0, \tag{5·2}$$

$$\dot s p_{\alpha_1}{}^{\dot\beta_1} u_\tau^{\alpha_1\alpha_3 \dots \alpha_m, \dot\beta_2 \dots \dot\beta_n} = 0, \tag{5·3}$$

$$p_{\alpha_1\dot\beta_1} u_\tau^{\alpha_1\alpha_3 \dots \alpha_m, \dot\beta_1\dot\beta_3 \dots \dot\beta_n} = 0. \tag{5·4}$$

There are no other equations because the number of linearly independent left-hand sides equals $4mn$, which is the number of linearly independent products $p^{\alpha_1\dot\beta_1} u_\tau^{\alpha_2 \dots \alpha_m, \dot\beta_2 \dots \dot\beta_n}$. Clearly (5·2) and (5·4) cease to exist if $n = 1$ and similarly for (5·3) and (5·4) if $m = 1$. It should be noticed that (5·1) and (5·2) together may be written as

$$sp^{\alpha_1\dot\beta_1} u_\tau^{\alpha_2 \dots \alpha_m, \dot\beta_2 \dots \dot\beta_n} = 0,$$

and similarly for (5·1) and (5·3), (5·2) and (5·4), (5·3) and (5·4).

Now let us find the equations (2·2). Starting from (5·1) we get

$$p_{\alpha_1\dot\beta_1} s\dot s p^{\alpha_1\dot\beta_1} u_\tau^{\alpha_2 \dots \alpha_m, \dot\beta_2 \dots \dot\beta_n}$$

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transforming as $u_{\tau}^{\alpha_2\dots\alpha_m,\dot\beta_2\dots\dot\beta_n}$. Here, carrying out the operations s and $\dot s$, in virtue of $p_{\alpha_1\dot\beta}p^{\alpha_2\dot\beta} = \delta_{\alpha_1}^{\alpha_2}\Box$ and $p_{\alpha\dot\beta_1}p^{\alpha\dot\beta_2} = \delta_{\dot\beta_1}^{\dot\beta_2}\Box$, we find

$$\sum_{j=1}^{m}\sum_{k=1}^{n} p_{\alpha_1\dot\beta_1}p^{\alpha_j\dot\beta_k}u_{\tau}^{\dots\alpha_1\dots,\dots\dot\beta_1\dots} = \left(\sum_{j=2}^{m}\sum_{k=2}^{n}+\sum_{j=2}^{m}\sum_{k=1}^{1}+\sum_{j=1}^{1}\sum_{k=2}^{n}+\sum_{j=1}^{1}\sum_{k=1}^{1}\right)p_{\alpha_1\dot\beta_1}p^{\alpha_j\dot\beta_k}u_{\tau}^{\dots\alpha_1\dots,\dots\dot\beta_1\dots}$$

$$= \sum_{j=2}^{m}\sum_{k=2}^{n} p_{\alpha_1\dot\beta_1}p^{\alpha_j\dot\beta_k}u_{\tau}^{\dots\alpha_1\dots,\dots\dot\beta_1\dots} + \sum_{j=2}^{m}\delta_{\alpha_1}^{\alpha_j}\Box u_{\tau}^{\dots\alpha_1\dots,\dot\beta_2\dots\dot\beta_n} + \sum_{k=2}^{n}\delta_{\dot\beta_1}^{\dot\beta_k}\Box u_{\tau}^{\alpha_2\dots\alpha_m,\dots\dot\beta_1\dots}$$

$$+2\Box u_{\tau}^{\alpha_2\dots\alpha_m,\dot\beta_2\dots\dot\beta_n} = (m+n)\Box u_{\tau}^{\alpha_2\dots\alpha_m,\dot\beta_2\dots\dot\beta_n} + s\dot s p_{\alpha\dot\beta}p^{\alpha_2\dot\beta_2}u_{\tau}^{\alpha\alpha_3\dots\alpha_m,\dot\beta\dot\beta_3\dots\dot\beta_n}.$$

A similar computation gives us from (5·2)

$$\dot s p_{\alpha_1}{}^{\dot\beta_2} s p^{\alpha_1}{}_{\dot\beta_1} u_{\tau}^{\alpha_2\dots\alpha_m,\dot\beta_1\dot\beta_3\dots\dot\beta_n} = (1-n)\Box u_{\tau}^{\alpha_2\dots\alpha_m,\dot\beta_2\dots\dot\beta_n} + s\dot s p_{\alpha}{}^{\dot\beta_2}p^{\alpha_2}{}_{\dot\beta}u_{\tau}^{\alpha\alpha_3\dots\alpha_m,\dot\beta\dot\beta_3\dots\dot\beta_n}.$$

Now if the $p^{\alpha\dot\beta}$ are supposed to commute, it is easily verified that

$$p_{\alpha}{}^{\dot\beta_2}p^{\alpha_2}{}_{\dot\beta} - p_{\alpha\dot\beta}p^{\alpha_2\dot\beta_2} = -\delta_{\alpha}^{\alpha_2}\delta_{\dot\beta}^{\dot\beta_2}\Box.$$

This means that we may write the last expression as

$$-m(n-1)\Box u_{\tau}^{\alpha_2\dots\alpha_m,\dot\beta_2\dots\dot\beta_n} + s\dot s p_{\alpha\dot\beta}p^{\alpha_2\dot\beta_2}u_{\tau}^{\alpha\alpha_3\dots\alpha_m,\dot\beta\dot\beta_3\dots\dot\beta_n}.$$

The equations (5·3) and (5·4) are treated in the same way. Putting

$$v_{\tau}^{(\alpha)(\dot\beta)} = s\dot s p_{\alpha\dot\beta}p^{\alpha_2\dot\beta_2}u_{\tau}^{\alpha\alpha_3\dots\alpha_m,\dot\beta\dot\beta_3\dots\dot\beta_n} \quad\text{and}\quad u_{\tau}^{(\alpha)(\dot\beta)} = u_{\tau}^{\alpha_2\dots\alpha_m,\dot\beta_2\dots\dot\beta_n},$$

we get finally,

$$(m+n)\Box u_{\tau}^{(\alpha)(\dot\beta)} + v_{\tau}^{(\alpha)(\dot\beta)} = 0, \tag{5·1′}$$

$$-m(n-1)\Box u_{\tau}^{(\alpha)(\dot\beta)} + v_{\tau}^{(\alpha)(\dot\beta)} = 0, \tag{5·2′}$$

$$-n(m-1)\Box u_{\tau}^{(\alpha)(\dot\beta)} + v_{\tau}^{(\alpha)(\dot\beta)} = 0, \tag{5·3′}$$

$$v_{\tau}^{(\alpha)(\dot\beta)} = 0. \tag{5·4′}$$

Now if $m>1$ and $n>1$, then $\Box u_{\tau}^{(\alpha)(\dot\beta)}$ and $v_{\tau}^{(\alpha)(\dot\beta)}$ are linearly independent. Also $m+n>0$, $-m(n-1)<0$, $-n(m-1)<0$, and $m(n-1)\neq n(m-1)$ unless $m=n$. Hence, by taking the difference between any two of the equations (5·1′)–(5·4′), except (5·2′) and (5·3′) when $m=n$, we conclude that $\Box u_{\tau}^{(\alpha)(\dot\beta)} = 0$. If $m=1$, $n>1$ or $m>1$, $n=1$, then $v_{\tau}^{(\alpha)(\dot\beta)} = 0$, and any one of (5·1), (5·2) and (5·1), (5·3) respectively constitutes a minimum set. Finally, if $n=1$, $m=1$, only (5·1) remains and constitutes a minimum set.

We sum up in a table.

	$\bar n$	Minimum sets of equations (5·1)–(5·4)
$m>1,\ n>1,\ m\neq n$	4	Any two
$m>1,\ n>1,\ m=n$	4	Any two except (5·2) and (5·3)
$m=1,\ n>1$	2	Any one
$m>1,\ n=1$	2	Any one
$m=1,\ n=1$	1	Any one

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Our general rule of § 2 is thus seen to be valid except in the degenerate case $m = n$, when (5·2) and (5·3) together transform irreducibly under L. If we require our equations to be invariant under L^{++} instead of L^{+}, the only effect will be a reduction of the relevant possibilities for τ from two to one, not affecting our final conclusions. The wave equation for zero rest-mass proposed by Dirac* reads $p_{\alpha}^{\ \dot\beta} u_{\tau}^{\alpha\alpha_3\dots\alpha_m} = 0$, and fits into the above scheme under the heading $m > 1$, $n = 1$.

6. We now require our equations to be invariant under L. The results will be seen to be valid for $L^{++} + L^{+-}$ and $L^{++} + L^{-+}$ as well. The components of the wave function have to be $u_{\tau}^{(\alpha)(\dot\beta)} = u_{\tau}^{\alpha_2\dots\alpha_m,\dot\beta_2\dots\dot\beta_n}$ and $u_{\tau}^{(\beta)(\dot\alpha)} = u_{\tau}^{\beta_2\dots\beta_n,\dot\alpha_2\dots\dot\alpha_m}$ provided that $m \neq n$. Then the equations analogous to (5·1)–(5·4) read

$$ss\dot{s}p^{\alpha_1\dot\beta_1} u_{\tau}^{\alpha_2\dots\alpha_m,\dot\beta_2\dots\dot\beta_n} = 0, \qquad ss\dot{s}p^{\beta_1\dot\alpha_1} u_{\tau}^{\beta_2\dots\beta_n,\dot\alpha_2\dots\dot\alpha_m} = 0, \tag{6·1}$$

$$sp^{\alpha_1}_{\ \dot\beta_1} u_{\tau}^{\alpha_2\dots\alpha_m,\dot\beta_1\dot\beta_3\dots\dot\beta_n} = 0, \qquad \dot{s}p_{\beta_1}^{\ \dot\alpha_1} u_{\tau}^{\beta_1\beta_3\dots\beta_n,\dot\alpha_2\dots\dot\alpha_m} = 0, \tag{6·2}$$

$$\dot{s}p_{\alpha_1}^{\ \dot\beta_1} u_{\tau}^{\alpha_1\alpha_3\dots\alpha_m,\dot\beta_2\dots\dot\beta_n} = 0, \qquad sp^{\beta_1}_{\ \dot\alpha_1} u_{\tau}^{\beta_2\dots\beta_n,\dot\alpha_1\dot\alpha_3\dots\dot\alpha_m} = 0, \tag{6·3}$$

$$p_{\alpha_1\dot\beta_1} u_{\tau}^{\alpha_1\alpha_3\dots\alpha_m,\dot\beta_1\dot\beta_3\dots\dot\beta_n} = 0, \qquad p_{\beta_1\dot\alpha_1} u_{\tau}^{\beta_1\beta_3\dots\beta_n,\dot\alpha_1\dot\alpha_3\dots\dot\alpha_m} = 0. \tag{6·4}$$

These are the only possibilities if $|m-n| \neq 2$. If, e.g., $n-m = 2$, then (6·2) breaks up into two irreducible sets of equations:

$$sp^{\alpha_1}_{\ \dot\beta_1} u_{\tau}^{\alpha_2\dots\alpha_m,\dot\beta_1\dot\beta_3\dots\dot\beta_n} + \dot{s}p_{\alpha}^{\ \dot\beta_3} u_{\tau}^{\alpha\alpha_1\dots\alpha_m,\dot\beta_4\dots\dot\beta_n} = 0, \tag{6·2′}$$

$$sp^{\alpha_1}_{\ \dot\beta_1} u_{\tau}^{\alpha_2\dots\alpha_m,\dot\beta_1\dot\beta_3\dots\dot\beta_n} - \dot{s}p_{\alpha}^{\ \dot\beta_3} u_{\tau}^{\alpha\alpha_1\dots\alpha_m,\dot\beta_4\dots\dot\beta_n} = 0. \tag{6·2″}$$

The left-hand side of (6·2′) transforms like $u_{\tau}^{\alpha_1\dots\alpha_m,\dot\beta_3\dots\dot\beta_n}$, while the left-hand side of (6·2″) transforms like $u_{\tau st}^{\alpha_1\dots\alpha_m,\dot\beta_3\dots\dot\beta_n}$. There is an analogous splitting of (6·3) if $m-n = 2$. If $n = 1$ or $m = 1$, then (6·2), (6·4) and (6·3), (6·4) respectively cease to exist. Writing the equations corresponding to (5·1′)–(5·4′) is to the greater part done in § 5, and, because of the many cases occurring, it is also rather tedious. Therefore we only give the result of the investigation in a table similar to that of § 5.

$m \neq n$	$\bar n$	Minimum sets of equations (6·1)–(6·4)
$m > 1$, $n > 1$, $\lvert m-n\rvert \neq 2$	4	Any two
$m > 1$, $n > 1$, $n-m = 2$	5	Any two, (6·2′) and (6·2″) counting as one-half each
$m > 1$, $n > 1$, $m-n = 2$	5	Any two, (6·3′) and (6·3″) counting as one-half each
$m = 1$, $n-m \neq 2$	2	Any one
$m = 1$, $n = 3$	3	Any one, (6·2′) and (6·2″) counting as one-half each
$m-n \neq 2$, $n = 1$	2	Any one
$m = 3$, $n = 1$	3	Any one, (6·3′) and (6·3″) counting as one-half each

There are deviations from the general rule of § 2 only in the degenerate cases $|n-m| = 2$, when two sets of equations transform in the same way under L^{+}.

* Dirac(3), equation (35), p. 453, written in the notation of § 5, pp. 458f.

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If $m = n$, the components of the wave function are $u_{\tau}^{(\alpha)(\dot{\beta})} = u_{\tau}^{\alpha_2 \dots \alpha_m, \dot{\beta}_2 \dots \dot{\beta}_m}$, and the possible irreducible sets of equations read

$$s\dot{s} p^{\alpha_1 \dot{\beta}_1} u_{\tau}^{\alpha_2 \dots \alpha_m, \dot{\beta}_2 \dots \dot{\beta}_m} = 0, \tag{6·5}$$

$$s p^{\alpha_1}{}_{\dot{\beta}_1} u_{\tau}^{\alpha_2 \dots \alpha_m, \dot{\beta}_1 \dot{\beta}_3 \dots \dot{\beta}_m} = 0, \qquad \dot{s} p_{\beta_1}{}^{\dot{\alpha}_1} u_{\tau}^{\beta_1 \beta_3 \dots \beta_m, \dot{\alpha}_2 \dots \dot{\alpha}_m} = 0, \tag{6·6}$$

$$p_{\alpha_2 \dot{\beta}_2} u_{\tau}^{\alpha_2 \dots \alpha_m, \dot{\beta}_2 \dots \dot{\beta}_m} = 0. \tag{6·7}$$

If $m = n = 1$, only (6·5) remains. The table reads

	$\bar{n}$	Minimum sets of equations (6·5)–(6·7)
$m = n > 1$	3	Any two
$m = n = 1$	1	Any one

in accordance with the general rule. If we restrict ourselves to $L^{++} + L^{+-}$ or $L^{++} + L^{-+}$ instead of L, the only result will be a reduction of the relevant values of τ, not affecting our final conclusions.

7. Examining the position of Maxwell's equations in our present scheme, we have to face the degenerate case $|m - n| = 2$. The components of the wave function are $u^{\alpha\beta} = u_1^{\alpha\beta}$ and $u^{\dot{\alpha}\dot{\beta}} = u_1^{\dot{\alpha}\dot{\beta}}$ and the irreducible sets of equations are

$$s p^{\alpha_1 \dot{\beta}_1} u^{\alpha_2 \alpha_3} = 0, \qquad \dot{s} p^{\alpha_1 \dot{\beta}_1} u^{\dot{\beta}_2 \dot{\beta}_3} = 0, \tag{7·1}$$

$$p^{\alpha_1}{}_{\beta} u^{\beta_1 \beta} + p_{\alpha}{}^{\beta_1} u^{\alpha_1 \alpha} = 0, \tag{7·2}$$

$$p^{\alpha_1}{}_{\beta} u^{\beta_1 \beta} - p_{\alpha}{}^{\beta_1} u^{\alpha_1 \alpha} = 0. \tag{7·3}$$

Maxwell's equations are (7·2), transforming like the vector component $u_1^{\alpha_1 \dot{\beta}_1}$, and (7·3), transforming like the trivector component $u_{st}^{\alpha_1 \dot{\beta}_1}$. Owing to the above there are two minimum sets, one consists of Maxwell's equations, the other of (7·1) alone.

It might be worth noticing the connexion between the number p of components of the wave function and the number q of the equations in a minimum set. By trying all cases one finds that $q \geqslant p$, and that $q = p$ only in the following cases. For L^+

$$s p^{\alpha_1}{}_{\beta} u_{\tau}^{\alpha_2 \dots \alpha_m, \dot{\beta}} = 0, \quad p_{\alpha\dot{\beta}} u_{\tau}^{\alpha\alpha_3 \dots \alpha_m, \dot{\beta}} = 0, \tag{7·4}$$

and similar equations when $m = 2$ and n is arbitrary. For L we have (7·4) together with

$$\dot{s} p_{\beta}{}^{\dot{\alpha}_1} u_{\tau}^{\beta, \dot{\alpha}_2 \dots \dot{\alpha}_m} = 0, \quad p_{\beta\dot{\alpha}} u_{\tau}^{\beta, \dot{\alpha}\dot{\alpha}_3 \dots \dot{\alpha}_m} = 0. \tag{7·5}$$

These include Dirac's equations for the electron when its mass is zero, namely

$$p^{\alpha}{}_{\beta} u^{\beta} = 0 \quad \text{and} \quad p_{\alpha}{}^{\dot{\beta}} u^{\alpha} = 0.$$

As to the analytical properties of the systems of linear homogeneous partial differential equations established as 'minimum sets', they will not be the object of the present paper. Some of the equations, e.g. Maxwell's equations, are thoroughly known, but others have not been studied at all. It is an open question whether they have in common

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other interesting mathematical properties than the purely algebraical property of being minimum sets in the sense used here.

8. *Note.* From the equations given above we may easily get similar inhomogeneous equations in which every component u of the wave function satisfies $\Box u = au$ with some constant a, zero or not zero. We give one example. Starting from a minimum set, e.g. (5·1) and (5·2), we write with four constants $a_1, \ldots, a_4$, supposing that $m > 1, n > 1$,

$$s\dot{s}p^{\alpha_1\dot{\beta}_1} u_\tau^{\alpha_2\ldots\alpha_m,\dot{\beta}_2\ldots\dot{\beta}_n} = a_1 w_\tau^{\alpha_1\ldots\alpha_m,\dot{\beta}_1\ldots\dot{\beta}_n}, \tag{8·1}$$

$$p_{\alpha_1\dot{\beta}_1} w_\tau^{\alpha_1\ldots\alpha_m,\dot{\beta}_1\ldots\dot{\beta}_n} = a_2 u_\tau^{\alpha_2\ldots\alpha_m,\dot{\beta}_2\ldots\dot{\beta}_n}, \tag{8·2}$$

$$sp_{\beta}^{\alpha_1} u_\tau^{\alpha_2\ldots\alpha_m,\beta\dot{\beta}_3\ldots\dot{\beta}_n} = a_3 w_\tau^{\alpha_1\ldots\alpha_m,\dot{\beta}_3\ldots\dot{\beta}_n}, \tag{8·3}$$

$$\dot{s}p_{\alpha}^{\dot{\beta}_2} w_\tau^{\alpha\alpha_2\ldots\alpha_m,\dot{\beta}_3\ldots\dot{\beta}_n} = a_4 u_\tau^{\alpha_2\ldots\alpha_m,\dot{\beta}_2\ldots\dot{\beta}_n}. \tag{8·4}$$

Then, by (8·1) and (8·2),

$$p_{\alpha_1\dot{\beta}_1} s\dot{s}p^{\alpha_1\dot{\beta}_1} u_\tau^{\alpha_2\ldots\alpha_m,\dot{\beta}_2\ldots\dot{\beta}_n} = a_1 a_2 u_\tau^{\alpha_2\ldots\alpha_m,\dot{\beta}_2\ldots\dot{\beta}_n},$$

and a similar equation from (8·3) and (8·4). Hence, by (5·1′) and (5·2′) with

$$u_\tau^{(\alpha)(\dot{\beta})} = u_\tau^{\alpha_2\ldots\alpha_m,\dot{\beta}_2\ldots\dot{\beta}_n}, \quad \text{etc.},$$

$$(m+n)\Box u_\tau^{(\alpha)(\dot{\beta})} + v_\tau^{(\alpha)(\dot{\beta})} = a_1 a_2 u_\tau^{(\alpha)(\dot{\beta})}, \quad -m(n-1)\Box u_\tau^{(\alpha)(\dot{\beta})} + v_\tau^{(\alpha)(\dot{\beta})} = a_3 a_4 u_\tau^{(\alpha)(\dot{\beta})}.$$

Hence, with $a_1 a_2 - a_3 a_4 = n(m+1)a$ we get $\Box u_\tau^{(\alpha)(\dot{\beta})} = a u_\tau^{(\alpha)(\dot{\beta})}$. Now, by (8·1),

$$\begin{aligned} a_1 \Box w_\tau^{\alpha_1\ldots\alpha_m,\dot{\beta}_1\ldots\dot{\beta}_n} &= s\dot{s}p^{\alpha_1\dot{\beta}_1} \Box u_\tau^{\alpha_2\ldots\alpha_m,\dot{\beta}_2\ldots\dot{\beta}_n} \\ &= as\dot{s}p^{\alpha_1\dot{\beta}_1} u_\tau^{\alpha_2\ldots\alpha_m,\dot{\beta}_2\ldots\dot{\beta}_n} = a_1 a w_\tau^{\alpha_1\ldots\alpha_m,\dot{\beta}_1\ldots\dot{\beta}_n}. \end{aligned}$$

Hence, if $a_1 \neq 0$, $\quad \Box w_\tau^{\alpha_1\ldots\alpha_m,\dot{\beta}_1\ldots\dot{\beta}_n} = a w_\tau^{\alpha_1\ldots\alpha_m,\dot{\beta}_1\ldots\dot{\beta}_n}$.

In the same way $\quad \Box w_\tau^{\alpha_1\ldots\alpha_m,\dot{\beta}_3\ldots\dot{\beta}_n} = a w_\tau^{\alpha_1\ldots\alpha_m,\dot{\beta}_3\ldots\dot{\beta}_n}$,

provided $a_3 \neq 0$. It is clear from this example that (5·1) and (5·2) may be replaced by any minimum set if an appropriate change is made in (8·1)–(8·4). Thus the equations for non-zero rest-mass proposed by Dirac* are seen to be similar to (8·1)–(8·4).

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* Dirac(3), p. 459, the last two equations.

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