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ARTICLE

Locally Noetherian quiver representations

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Dedicated to the memory of Maurice Auslander (1926–1994)

Abstract

It is shown that a quiver is left Noetherian if and only if the category of quiver representations in any locally Noetherian abelian category is again locally Noetherian. Here, locally Noetherian means that any object is the directed union of its Noetherian subobjects. For a quiver to be left Noetherian means that the left ideals of paths starting at any fixed vertex satisfy the ascending chain condition. The proof generalizes to representations of any small category that admits a Gröbner enrichment.

1. Introduction

Hilbert's basis theorem asserts that a polynomial ring $A[X]$ in one indeterminate is left Noetherian if the coefficient ring A is left Noetherian [7]. The category of $A[X]$-modules identifies with the category of A-linear representations of the *Jordan quiver* consisting of one vertex and a single loop. Thus, an equivalent formulation of Hilbert's theorem is that the category of A-linear representations of the Jordan quiver is locally Noetherian if the category of A-modules is locally Noetherian.

We generalize Hilbert's theorem as follows and write $\mathrm{Rep}(Q, \mathcal{A})$ for the category of representation of a quiver Q in an abelian category $\mathcal{A}$.

Theorem 1.1. *For a quiver Q and a left Noetherian ring A, the following conditions are equivalent:*

1. *The quiver Q is left Noetherian.*
2. *The abelian category $\mathrm{Rep}(Q, \mathcal{A})$ is locally Noetherian for every locally Noetherian abelian category $\mathcal{A}$.*
3. *The abelian category $\mathrm{Rep}(Q, \mathrm{Mod}\, A)$ is locally Noetherian.*

For a quiver to be *left Noetherian* simply means that the left ideals of paths starting at any fixed vertex satisfy the ascending chain condition, and the quiver is *right Noetherian* if the analogous condition for paths terminating at any fixed vertex is satisfied (see Definition 2.2).

When the quiver Q has only finitely many vertices, the category $\mathrm{Rep}(Q, \mathrm{Mod}A)$ identifies with the module category of the *path algebra* $A[Q]$. Thus, we obtain for $A[Q]$ a simple characterization to be left Noetherian (see Corollary 5.1).

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There is an analog of the above theorem for *locally finite categories* of representations. This requires both chain conditions for ideals in a quiver (see Theorem 6.3). Moreover, we provide a generalization of the above theorem which covers representations of any *small category*. This involves the notion of a *Gröbner enrichment* and the proof remains basically unchanged (see Theorem 7.1).

Theorem 1.1 has some predecessors, but in this form, it seems to be new, because we give a simple definition/description of a Noetherian quiver and we consider representations in any locally Noetherian abelian category. In fact, our definition of a locally Noetherian abelian category is more general than the usual one [6, Section II.4]; it simply requires that any object is the directed union of its Noetherian subobjects. For example, the category of finite-dimensional vector spaces over a field is locally Noetherian. We consider representations in abelian categories that are not necessarily module categories; this requires some new techniques since elements cannot be used.

Let us provide the relevant references that contain predecessors of the above theorem. The first is [8] where it is assumed that the quiver has a finite set of vertices, while [9] considers representations of arbitrary quivers and even more general categories. The description of Noetherian quivers in [8], [9] is somewhat complicated. So the results from [9] are reproved in [5], using a more transparent (though incorrect, as pointed out in [1], [10]) description of Noetherian quivers. All mentioned papers restrict to representations in module categories. For a survey on more recent generalizations of Hilbert's basis theorem, including their applications, see [4].

2. Noetherian quivers

Let $Q = (Q_0, Q_1, s, t)$ be a *quiver*, given by a set of vertices Q_0 and a set of arrows Q_1 such that each arrow α starts at $s(\alpha)$ and terminates at $t(\alpha)$. A *path* $\phi\colon x \to y$ of *length* n in Q is a sequence $\alpha_n \dots \alpha_2\alpha_1$ of arrows α_i such that $s(\alpha_{i+1}) = t(\alpha_i)$ for all i, where $s(\phi) := x = s(\alpha_1)$ and $t(\phi) := y = t(\alpha_n)$. For each $x \in Q_0$, there is the *trivial path* $x \to x$ of length zero. For any pair of vertices x, y in Q, let $Q(x, y)$ denote the set of paths $x \to y$. We write $\boldsymbol{Q}$ for the *path category* with set of objects Q_0 and set of morphism $x \to y$ given by $Q(x, y)$ for all $x, y \in Q_0$. The concatenation of paths yields the composition in $\boldsymbol{Q}$.

For a vertex x in Q, let $Q(x)$ denote the set of paths starting at x. This set is partially ordered via

$$\phi \le \psi \quad :\Longleftrightarrow \quad \phi = \chi\psi \ \text{ for some path } \ \chi\colon t(\psi) \to t(\phi).$$

Let $P = (P, \le)$ be a poset. An *ideal* of P is a non-empty subset $I \subseteq P$ such that $x \le y$ and $y \in I$ implies $x \in I$. The poset P is *Noetherian* if the ascending chain condition holds for the ideals in P. We refer to [2, Section VIII.1] for basic facts about Noetherian posets.[1] For instance, it is easily checked that a poset is Noetherian if and only if every ideal is generated by finitely many elements. For later use, we record another well-known characterization.

Lemma 2.1. *A poset is Noetherian if and only if for every infinite sequence $(x_i)_{i\in\mathbb{N}}$ of elements, there is a map $\nu\colon \mathbb{N} \to \mathbb{N}$ such that $i < j$ implies $\nu(i) < \nu(j)$ and $x_{\nu(j)} \le x_{\nu(i)}$.*

Proof. We prove the direction which is used later on and leave the other one to the interested reader. So assume the poset is Noetherian. First observe that for every infinite sequence $(x_i)_{i\in\mathbb{N}}$ of elements, there exists $i \in \mathbb{N}$ such that $x_j \le x_i$ for infinitely many $j \in \mathbb{N}$. This follows by looking at the ascending chain of ideals generated by the elements $x_0, \dots, x_n$, which must stabilize. Now define $\nu\colon \mathbb{N} \to \mathbb{N}$ recursively by taking for $\nu(0)$ the smallest $i \in \mathbb{N}$ such that $x_j \le x_i$ for infinitely many $j \in \mathbb{N}$. For $n > 0$, set

$$\nu(n) = \min\{i > \nu(n-1) \mid x_j \le x_i \le x_{\nu(n-1)} \text{ for infinitely many } j \in \mathbb{N}\}. \qquad \square$$

[1] Sometimes a poset is defined to be *Noetherian* if the ascending chain condition holds for chains of *elements*. Clearly, one condition implies the other (by passing from an element to the ideal generated by that element), but the conditions are not equivalent.

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The multiplication of paths in Q yields the following notion of an ideal. A non-empty set of paths I in Q is a *left ideal* if for any pair ϕ, ψ of paths with $s(\psi) = t(\phi)$, the condition $\phi \in I$ implies $\psi\phi \in I$. A *right ideal* of paths is defined analogously. Clearly, a subset $I \subseteq Q(x)$ is a poset ideal if and only if it is a left ideal of paths.

Definition 2.2. A quiver Q is called *left Noetherian* if for each vertex x, the poset $Q(x)$ is Noetherian. This means the left ideals of paths starting at any fixed vertex satisfy the ascending chain condition. The quiver is *right Noetherian* if the right ideals of paths terminating at any fixed vertex satisfy the ascending chain condition.

This local condition can be checked by counting the number of *maximal paths* starting at a fixed vertex. By definition, this is either a path $\alpha_n \dots \alpha_2\alpha_1$ such that there is no arrow starting at $t(\alpha_n)$, or it is a path $\dots \alpha_2\alpha_1$ of infinite length.

Proposition 2.3. *Let Q be a quiver. For a vertex x, the following conditions are equivalent:*

1. *The poset $Q(x)$ is Noetherian.*
2. *For all $\phi \in Q(x)$, there are only finitely many arrows starting at $t(\phi)$, and except for finitely many ϕ, this finite number of arrows is at most one.*
3. *There are only finitely many maximal paths starting at x.*

Proof. (1) $\Rightarrow$ (2): Let $\phi \in Q(x)$. The ideal of $Q(x)$ that is generated by all paths $\alpha\phi$ with α an arrow starting at $t(\phi)$ is finitely generated. It follows that there are only finitely many arrows starting at $t(\phi)$. Suppose $\phi_1, \phi_2, \dots$ is an infinite sequence of paths such that there is more than one arrow starting at each $t(\phi_i)$. Because $Q(x)$ is Noetherian, we may assume $\phi_{i+1} \leq \phi_i$ for all i. For each i, we may choose an arrow α_i such that $\phi_{i+1} \not\leq \alpha_i\phi_i$. Then the ideal of $Q(x)$ generated by $\{\alpha_i\phi_i \mid i \in \mathbb{N}\}$ cannot be generated by finitely many elements. This is a contradiction.

(2) $\Rightarrow$ (3): Let n be the number of paths $\phi \in Q(x)$ such that there is more than one arrow starting at $t(\phi)$, and let d be the maximal number of such arrows. Then there are at most d^n maximal paths starting at x.

(3) $\Rightarrow$ (1): Let $I \subseteq Q(x)$ be an ideal, so a left ideal with respect to the composition of paths in Q. For each maximal path starting at x and given by a (finite or infinite) sequence of arrows $\dots \alpha_2\alpha_1$, we check if there exists $n \in \mathbb{N}$ such that $\alpha_n \dots \alpha_2\alpha_1$ belongs to I. In that case, we choose n minimal, and it is easily checked that I is generated by all paths of this form. □

Let us add another criterion for $Q(x)$ to be Noetherian, following [8], [9]. We consider the *path component* Q_x, so the full subquiver of Q with set of vertices $y \in Q_0$ that admit a path $x \to y$. An *oriented cycle* is a path $y \to y$ of length at least one.

Proposition 2.4. *Let Q be a quiver and x a vertex. Then the poset $Q(x)$ is Noetherian if and only if there are arrows $\alpha_1, \dots, \alpha_n$ such that there is a decomposition*

$$Q_x \setminus \{\alpha_1, \dots, \alpha_n\} = Q^0 \amalg Q^1 \amalg \cdots \amalg Q^n \tag{2.1}$$

into full and pairwise disjoint subquivers satisfying the following:

1. *The quiver Q^0 has only finitely many vertices and arrows, and there is precisely one arrow starting at each vertex lying on an oriented cycle.*
2. *For $i \neq 0$, each quiver Q^i is isomorphic to the infinite quiver*

$$A_\infty: \quad \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \cdots .$$

3. *Each arrow α_i starts at a vertex of Q^0 not lying on an oriented cycle and terminates at the source of Q^i.*

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Proof. Suppose the poset $Q(x)$ is Noetherian. Then Proposition 2.3 implies that there are only finitely many maximal paths starting at x. From this finite set of paths, the decomposition (2.1) is constructed as follows. Set $Q^0 = Q_x$ when Q_x is finite. Otherwise, there exists a maximal path $\dots\beta_2\beta_1$ in $Q(x)$ and $r \geq 1$ such that $Q_x \setminus \{\beta_r\} = Q' \amalg Q^1$ with Q^1 of type A_∞ and set of arrows $\{\beta_i \mid i > r\}$. Set $Q^0 = Q'$ if this quiver is finite. Otherwise, there exists another maximal path in Q' and we continue as before by removing finitely many arrows. It is straightforward to check that such a decomposition of Q_x is equivalent to the condition on $Q(x)$ to have only finitely many maximal paths starting at x. So the assertion follows from Proposition 2.3. □

An immediate consequence is the following simple criterion for quivers with finitely many vertices.

Corollary 2.5. *Let Q be quiver having only a finite number of vertices. Then Q is left Noetherian if and only if there are only finitely many arrows and there is precisely one arrow starting at each vertex lying on an oriented cycle.*

Proof. We apply Proposition 2.4 and note that, in the decomposition (2.1), we have $Q_x = Q^0$, since for $i \neq 0$, the quiver Q^i has infinitely many vertices. □

3. Locally Noetherian categories

Let $\mathcal{A}$ be an abelian category. We assume that the subobjects of any object form a complete lattice and that for every directed set of subobjects $(X_i)_{i\in I}$ of an object X and $Y \subseteq X$, one has

$$\left(\sum_{i\in I} X_i\right) \cap Y = \sum_{i\in I}(X_i \cap Y). \tag{AB5}$$

A collection of objects $(G_i)_{i\in I}$ *generates* $\mathcal{A}$ if for each non-zero morphism $\phi\colon X \to Y$ in $\mathcal{A}$, we have $\phi\pi \neq 0$ for some morphism $\pi\colon G_i \to X$. An object of an abelian category is called *Noetherian* if the ascending chain condition holds for its subobjects. We write $\mathrm{Noeth}\mathcal{A}$ for the full subcategory of Noetherian objects in $\mathcal{A}$. An abelian category is *cocomplete* if for any set of objects its coproduct exist.

Definition 3.1. The abelian category $\mathcal{A}$ is *Noetherian* if every object is Noetherian, and $\mathcal{A}$ is *locally Noetherian* if every object is the directed union of its Noetherian subobjects.

The following shows that our definition agrees with the usual one for Grothendieck categories [6, Section II.4].

Lemma 3.2. *The category $\mathcal{A}$ is locally Noetherian if and only if there is a collection of Noetherian generators.*

Proof. Suppose that $\mathcal{A}$ is locally Noetherian. Then we may take the Noetherian objects as the collection of generators, because for a morphism $\phi\colon X \to Y$ with $X = \sum_i X_i$ written as the directed union of its Noetherian subobjects, we have

$$\mathrm{Hom}(X, Y) \cong \lim_i \mathrm{Hom}(X_i, Y).$$

If $\phi \neq 0$, then one of the composites $X_i \hookrightarrow X \to Y$ needs to be non-zero.

Now suppose there is a collection of Noetherian generators $(G_i)_{i\in I}$. Let X be an object and $X' \subseteq X$ the directed union of its Noetherian subobjects. It is clear that every composite $G_i \to X \twoheadrightarrow X/X'$ is zero. Thus, $X' = X$. □

Let $Y \in \mathcal{A}$ be an object, written as the directed union $Y = \sum_j Y_j$ of a set of subobjects. Then it is easily checked that, for each Noetherian object $X \in \mathcal{A}$, the canonical map

$$\operatorname{colim}_j \mathrm{Hom}(X, Y_j) \longrightarrow \mathrm{Hom}(X, \sum_j Y_j) \tag{3.1}$$

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is a bijection. Clearly, the map is injective, and because X is Noetherian any morphism $X \to Y$ factors through the inclusion $Y_{j_0} \to Y$ for some index j_0. Here, one uses the condition (AB5). Thus, the map is also surjective.

Lemma 3.3. *Let $X = \sum_i X_i$ and $Y = \sum_j Y_j$ be objects in a locally Noetherian category, written as directed union of their Noetherian subobjects. Then there is a natural isomorphism of abelian groups*

$$\lim_i \operatorname*{colim}_j \operatorname{Hom}(X_i, Y_j) \xrightarrow{\sim} \operatorname{Hom}(X, Y). \tag{3.2}$$

Proof. Combine (3.1) with the fact that $\operatorname{Hom}(-,-)$ preserves colimits in the first argument. □

Now assume that $\operatorname{Noeth}\mathcal{A}$ is essentially small. We write $\operatorname{Lex}((\operatorname{Noeth}\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$ for the category of additive contravariant functors $\operatorname{Noeth}\mathcal{A} \to \mathrm{Ab}$ into the category of abelian groups that are left exact and consider the *restricted Yoneda functor*

$$\mathcal{A} \longrightarrow \bar{\mathcal{A}} := \operatorname{Lex}((\operatorname{Noeth}\mathcal{A})^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto h_X := \operatorname{Hom}(-, X)|_{\operatorname{Noeth}\mathcal{A}}.$$

Then the following is a slight generalization of a theorem of Gabriel [6].

Proposition 3.4. *Let $\mathcal{A}$ be a locally Noetherian abelian category. Then the category* $\operatorname{Lex}((\operatorname{Noeth}\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$ *is a locally Noetherian and cocomplete abelian category. The restricted Yoneda functor $\mathcal{A} \to \bar{\mathcal{A}}$ is fully faithful and exact. It restricts to an equivalence* $\operatorname{Noeth}\mathcal{A} \xrightarrow{\sim} \operatorname{Noeth}\bar{\mathcal{A}}$*, and its essential image is closed under subobjects.*

Proof. Let $X = \sum_i X_i$ and $Y = \sum_j Y_j$ be objects in $\mathcal{A}$, written as directed union of their Noetherian subobjects. Using (3.1), we have

$$h_X \cong \operatorname*{colim}_i \operatorname{Hom}(-, X_i)$$

and compute

$$\begin{aligned} \operatorname{Hom}(h_X, h_Y) &\cong \operatorname{Hom}(\operatorname*{colim}_i \operatorname{Hom}(-, X_i), \operatorname*{colim}_j \operatorname{Hom}(-, Y_j)) \\ &\cong \lim_i \operatorname{Hom}(\operatorname{Hom}(-, X_i), \operatorname*{colim}_j \operatorname{Hom}(-, Y_j)) \\ &\cong \lim_i \operatorname*{colim}_j \operatorname{Hom}(X_i, Y_j) \\ &\cong \operatorname{Hom}(X, Y). \end{aligned}$$

The second isomorphism uses that $\operatorname{Hom}(-,-)$ preserves colimits in the first argument, the third follows from Yoneda's lemma, and the last is (3.2). Thus, $\mathcal{A} \to \bar{\mathcal{A}}$ is fully faithful. For the rest of the assertion, see [6, Section II.4]. □

Example 3.5. Let A be a ring. Then the regular module A is a finitely generated generator for the category $\operatorname{Mod}A$ of left A-modules. Thus, $\operatorname{Mod}A$ is locally Noetherian if and only if A is left Noetherian [12, V.4].

4. Representations of Noetherian quivers

Let Q be a quiver and Q its path category. A *representation* of Q in a category $\mathcal{A}$ is by definition a functor $Q \to \mathcal{A}$. Morphisms between representations are the natural transformations, and we write $\operatorname{Rep}(Q, \mathcal{A})$ for the category of representations.

Let $\mathcal{A}$ be a cocomplete abelian category. We assume that the subobjects of any object form a complete lattice and that the condition (AB5) holds. Given an object $M \in \mathcal{A}$, the functor

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$\mathrm{Hom}(M,-)\colon \mathcal{A} \to \mathrm{Set}$ admits a left adjoint, which takes a set X to the coproduct $M[X]$ of copies of M indexed by the elements of X. Thus, for each object $N \in \mathcal{A}$, there is a natural isomorphism

$$\mathrm{Hom}(M[X], N) \cong \mathrm{Hom}(X, \mathrm{Hom}(M, N)).$$

Given a vertex $x \in Q_0$, we write $\mathbb{1}_x$ for the subquiver of Q consisting of the single vertex x and no arrows. Restriction along the inclusion $\mathbb{1}_x \hookrightarrow Q$ yields the evaluation functor

$$\mathrm{Rep}(Q,\mathcal{A}) \longrightarrow \mathrm{Rep}(\mathbb{1}_x,\mathcal{A}) = \mathcal{A}, \quad M \mapsto M(x).$$

The left Kan extension provides a left adjoint, which is given by

$$\mathcal{A} \longrightarrow \mathrm{Rep}(Q,\mathcal{A}), \quad N \mapsto N[Q(x,-)].$$

Thus, we have for objects $N \in \mathcal{A}$ and $M \in \mathrm{Rep}(Q,\mathcal{A})$ a natural isomorphism

$$\mathrm{Hom}(N[Q(x,-)], M) \cong \mathrm{Hom}(N, M(x)). \tag{4.1}$$

Lemma 4.1. *Let $(M_i)_{i\in I}$ be a collection of generators of $\mathcal{A}$. Then the representations $M_i[Q(x,-)]$ with $i \in I$ and $x \in Q_0$ generate* $\mathrm{Rep}(Q,\mathcal{A})$.

Proof. This follows from the adjunction isomorphism (4.1). □

An object X is *finitely generated* if, for any directed set of subobjects $(X_i)_{i\in I}$ with $X = \sum_{i\in I} X_i$, there is $i_0 \in I$ such that $X = X_{i_0}$.

Lemma 4.2. *If $M \in \mathcal{A}$ is finitely generated, then $M[Q(x,-)]$ is finitely generated in* $\mathrm{Rep}(Q,\mathcal{A})$ *for each $x \in Q_0$.*

Proof. Let $M[Q(x,-)] = \sum_{i\in I} N_i$ be a directed union of subobjects N_i. Then the adjunction isomorphism (4.1) yields a morphism $M \to \sum_{i\in I} N_i(x)$, which factors through $N_{i_0}(x) \to \sum_{i\in I} N_i(x)$ for some $i_0 \in I$ since M is finitely generated. Thus, the identity $M[Q(x,-)] \to \sum_{i\in I} N_i$ factors through $N_{i_0} \to \sum_{i\in I} N_i$, so $M[Q(x,-)] = N_{i_0}$. □

Let Q be a left Noetherian quiver and fix $x \in Q_0$. We choose a total order $\preceq$ on the countable set of arrows in Q that lie on a path in $Q(x)$. This order extends lexicographically to a total order $\preceq$ on $Q(x)$ such that $\phi \prec \psi$ implies $\phi \prec \psi$. Note that $(Q(x),\preceq)$ is Noetherian, because any ideal with respect to $\preceq$ is also one with respect to $\leq$.

Lemma 4.3. *Let Q be a left Noetherian quiver and x a vertex. If $M \in \mathcal{A}$ is a Noetherian object, then the representation $M[Q(x,-)]$ is Noetherian.*

Proof. Let $U \subseteq M[Q(x,-)]$ be a subrepresentation and $\phi \in Q(x,y)$ a path. We set

$$Q(x,y)_\phi := \{\psi \in Q(x,y) \mid \phi \preceq \psi\}$$

and write U_ϕ for the image of the composite

$$U(y) \cap M[Q(x,y)_\phi] \hookrightarrow M[Q(x,y)_\phi] \twoheadrightarrow M,$$

where the second map denotes the projection corresponding to ϕ. For $\psi \in Q(x,y)$, we have

$$\phi \preceq \psi \quad \Longrightarrow \quad U_\psi \subseteq U_\phi. \tag{4.2}$$

More precisely, if there is a path χ such that $\phi = \chi\psi$, then composition with χ induces a map

$$Q(x,y)_\phi \longrightarrow Q(x,y)_\psi$$

such that ϕ is the only element that is mapped to ψ. Thus, $U(\chi)$ maps U_ψ into U_ϕ.

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Given subrepresentations $U \subseteq V \subseteq M[Q(x,-)]$, we have $U_\phi \subseteq V_\phi$ for all $\phi \in Q(x)$. If $U \neq V$, we claim that there exists $\phi \in Q(x)$ such that $U_\phi \neq V_\phi$. To see this, choose $y \in Q_0$ such that $U(y) \neq V(y)$. Observe that the union

$$Q(x,y) = \bigcup_{\phi \in Q(x,y)} Q(x,y)_\phi \tag{4.3}$$

is directed, and therefore condition (AB5) implies

$$U(y) = \sum_{\phi \in Q(x,y)} U(y) \cap M[Q(x,y)_\phi].$$

Thus, there exists $\phi \in Q(x,y)$ with

$$U(y) \cap M[Q(x,y)_\phi] \neq V(y) \cap M[Q(x,y)_\phi].$$

Using that $Q(x)$ is Noetherian, we may choose ϕ maximal with respect to $\preceq$. Next observe that

$$Q(x,y)_\phi \setminus \{\phi\} = \bigcup_{\phi \prec \psi} Q(x,y)_\psi \tag{4.4}$$

is a directed union. This yields the following commutative diagram with exact rows, using again the condition (AB5):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \sum_{\phi \prec \psi} U(y) \cap M[Q(x,y)_\psi] & \longrightarrow & U(y) \cap M[Q(x,y)_\phi] & \longrightarrow & U_\phi & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sum_{\phi \prec \psi} M[Q(x,y)_\psi] & \longrightarrow & M[Q(x,y)_\phi] & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

For $\phi \prec \psi$, the maximality of ϕ implies

$$U(y) \cap M[Q(x,y)_\psi] = V(y) \cap M[Q(x,y)_\psi]$$

and therefore $U_\phi \neq V_\phi$.

Now suppose there is a proper ascending chain

$$U^1 \subseteq U^2 \subseteq U^3 \subseteq \cdots$$

of subrepresentations of $M[Q(x,-)]$. Pick for each $i \in \mathbb{N}$ a path ϕ_i such that $U^i_{\phi_i} \subsetneq U^{i+1}_{\phi_i}$. The set $\{\phi_i \mid i \in \mathbb{N}\}$ is infinite since M is Noetherian. So we may assume $\phi_i \neq \phi_j$ for all $i \neq j$ in $\mathbb{N}$. Because $Q(x)$ is Noetherian, there exists a map $\nu\colon \mathbb{N} \to \mathbb{N}$ such that $i < j$ implies $\nu(i) < \nu(j)$ and $\phi_{\nu(j)} \leq \phi_{\nu(i)}$ (see Lemma 2.1). Thus, for all $i \in \mathbb{N}$, we have

$$U^{\nu(i)}_{\phi_{\nu(i)}} \subsetneq U^{\nu(i)+1}_{\phi_{\nu(i)}} \subseteq U^{\nu(i+1)}_{\phi_{\nu(i)}} \subseteq U^{\nu(i+1)}_{\phi_{\nu(i+1)}},$$

where the last inclusion follows from (4.2). This infinite chain is impossible since M is Noetherian, and therefore the representation $M[Q(x,-)]$ is Noetherian. □

Let Q be a quiver and A a ring. By default, we assume $A \neq 0$ and fix $x \in Q_0$. For any ideal $I \subseteq Q(x)$ and $y \in Q_0$, set $I(x,y) = I \cap Q(x,y)$. Then $A[I(x,-)]$ yields a subrepresentation of $A[Q(x,-)]$ in $\mathrm{Rep}(Q, \mathrm{Mod}A)$.

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Lemma 4.4. *For $x \in Q_0$ and a ring A, the following holds:*

1. *The assignment $I \mapsto A[I(x,-)]$ induces an embedding of posets from the ideals of $Q(x)$ into the subrepresentations of $A[Q(x,-)]$.*
2. *The assignment $J \mapsto J[Q(x,-)]$ induces an embedding of posets from the left ideals of A into the subrepresentations of $A[Q(x,-)]$.*

Proof. (1) Any inclusion of sets $X \subseteq Y$ induces an inclusion $A[X] \subseteq A[Y]$ of A-modules, with equality $A[X] = A[Y]$ only if $X = Y$.

(2) Any inclusion of A-modules $M \subseteq N$ induces an inclusion $M[Q(x,-)] \subseteq N[Q(x,-)]$, with equality only if $M = N$. □

We are ready to prove our main result.

Proof of Theorem 1.1. (1) $\Rightarrow$ (2): Let $M \in \mathrm{Rep}(Q,\mathcal{A})$. We need to show that M is the directed union of its Noetherian subobjects. We may assume that $\mathcal{A}$ is essentially small, because it suffices to consider the full subcategory of $\mathcal{A}$ that is obtained by closing the set of objects $M(x)$ with $x \in Q_0$ under finite coproducts, subobjects, and quotients. Let $\bar{\mathcal{A}}$ denote the completion of $\mathcal{A}$, as in Proposition 3.4. The representations $N[Q(x,-)]$ with $N \in \mathrm{Noeth}\bar{\mathcal{A}}$ and $x \in Q_0$ are Noetherian and generate $\mathrm{Rep}(Q,\bar{\mathcal{A}})$ by Lemmas 4.1 and 4.3. Thus, $\mathrm{Rep}(Q,\bar{\mathcal{A}})$ is locally Noetherian (see Lemma 3.2). It follows that M is the directed union of its Noetherian subobjects, because we can view $\mathrm{Rep}(Q,\mathcal{A})$ as a full subcategory which is closed under subobjects.

(2) $\Rightarrow$ (3): This is clear since $\mathrm{Mod}A$ is locally Noetherian for every left Noetherian ring A (see Example 3.5).

(3) $\Rightarrow$ (1): Fix $x \in Q_0$. The regular module A is a finitely generated object in $\mathrm{Mod}A$, and therefore the representation $A[Q(x,-)]$ is finitely generated in $\mathrm{Rep}(Q,\mathrm{Mod}A)$ by Lemma 4.2. It follows that $A[Q(x,-)]$ is Noetherian since $\mathrm{Rep}(Q,\mathrm{Mod}A)$ is locally Noetherian, and we conclude from Lemma 4.4 that the poset $Q(x)$ is Noetherian. □

Xiao-Wu Chen kindly pointed out the following consequence of Theorem 1.1 and its proof.

Corollary 4.5. *Let Q be a quiver having at least one vertex and A any ring. Then $\mathrm{Rep}(Q,\mathrm{Mod}A)$ is locally Noetherian if and only if the quiver Q and the ring A are both left Noetherian.*

Proof. We give the argument for the implication that is not covered by Theorem 1.1. So suppose that $\mathrm{Rep}(Q,\mathrm{Mod}A)$ is locally Noetherian. As in the above proof of (3) $\Rightarrow$ (1), we deduce from the first part of Lemma 4.4 that Q is left Noetherian. Using the second part of that lemma, it follows that the ring A is left Noetherian. □

5. Modules over path algebras

Let Q be a quiver with a finite set of vertices and A a ring. The *path algebra* is by definition the A-module

$$A[Q] := \coprod_{x,y \in Q_0} A[Q(x,y)]$$

with multiplication induced by the composition of paths in Q. From the isomorphism (4.1), it follows that $A[Q]$ identifies with the endomorphism ring of the representation $\coprod_{x \in Q_0} A[Q(x,-)]$, and that the assignment $M \mapsto \coprod_{x \in Q_0} M(x)$ induces an equivalence

$$\mathrm{Rep}(Q,\mathrm{Mod}A) \xrightarrow{\sim} \mathrm{Mod}A[Q]. \tag{5.1}$$

The following is well known, at least for path algebras over a field [3, §1].

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Corollary 5.1. *Let Q be a quiver with a finite set of vertices and A a left Noetherian ring. Then the path algebra $A[Q]$ is left Noetherian if and only if Q has only finitely many arrows and there is precisely one arrow starting at each vertex lying on an oriented cycle.*

Proof. The assertion follows from Theorem 1.1 and Corollary 2.5, because of the equivalence (5.1). □

6. Locally finite representations

Let $\mathcal{A}$ be an abelian category. We assume that the subobjects of any object form a complete lattice and that the condition (AB5) holds. Recall that an object is of *finite length* if it admits a finite composition series. An equivalent condition is that both chain conditions hold for subobjects.

Definition 6.1. The abelian category $\mathcal{A}$ is a *length category* if every object is of finite length, and $\mathcal{A}$ is *locally finite* if every object is the directed union of its finite length subobjects.

Definition 6.2. A quiver is called *left finite* if, for each vertex x, the set of paths starting at x is finite. A quiver is *right finite* if, for each vertex x, the set of paths terminating at x is finite.

Using Proposition 2.3, it is easily checked that $Q(x)$ is left finite if and only if both chain conditions hold for ideals of $Q(x)$. This observation suggests the following analog of Theorem 1.1.

Theorem 6.3. *For a quiver Q and a left artinian ring A, the following conditions are equivalent:*

1. *The quiver Q is left finite.*
2. *The abelian category* $\mathrm{Rep}(Q, \mathcal{A})$ *is locally finite for every locally finite abelian category $\mathcal{A}$.*
3. *The abelian category* $\mathrm{Rep}(Q, \mathrm{Mod}A)$ *is locally finite.*

Proof. Recall that a ring is left artinian if and only if its category of left modules is locally finite [12, Section VIII.1]. On the other hand, the quiver Q is left finite if and only if both chain conditions hold for the ideals of $Q(x)$ for all vertices x. Thus, we can adapt the proof of Theorem 1.1. □

7. Gröbner enrichments

Let C be a *small category*. Thus, the objects in C form a set. For a pair of objects x, y, let $C(x, y)$ denote the set of morphisms $x \to y$ and let $C(x)$ denote the set of all morphisms with domain x. A partial order $\preceq$ on $C(x)$ is called a *Gröbner enrichment* if the following holds for each $y \in C$:

(G1) $\phi \prec \psi$ implies $\omega\phi \prec \omega\psi$ for all $\phi, \psi \in C(x, y)$ and $\omega \in C(y)$.
(G2) The poset $C(x, y)$ is totally ordered and every non-empty subset has a maximal element.

For any abelian category $\mathcal{A}$, let $\mathrm{Rep}(C, \mathcal{A})$ denote the category of functors $C \to \mathcal{A}$. The following result generalizes Theorem 1.1.

Theorem 7.1. *Let C be a small category and A a left Noetherian ring. Suppose there exists a Gröbner enrichment for each object in C. Then the following conditions are equivalent:*

1. *The left ideals of morphisms in C starting at any fixed object satisfy the ascending chain condition.*
2. *The abelian category* $\mathrm{Rep}(C, \mathcal{A})$ *is locally Noetherian for every locally Noetherian abelian category $\mathcal{A}$.*
3. *The abelian category* $\mathrm{Rep}(C, \mathrm{Mod}A)$ *is locally Noetherian.*

Proof. The special case that C is the path category of a quiver is Theorem 1.1. In fact, the Gröbner enrichment for a path category is specified right before Lemma 4.3. The proof of the general case is essentially the same. The crucial ingredient is Lemma 4.3 where the Gröbner enrichment is used; its definition is extracted from the proof of that lemma. For instance, condition (G1) is needed for (4.2), and condition (G2) ensures that in (4.3) and (4.4) the union is directed. □

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Remark 7.2. The above theorem is a variation of the main result of [9]. This was rediscovered by Sam and Snowden in [11] and they provide many interesting examples. Their notion of a *Gröbner category* and our notion of a Gröbner enrichment basically agree, except that they require each poset $C(x, y)$ to be *well-ordered*. Passing from $\preceq$ to its opposite shows that both concepts are equivalent. The work in [9], [11] restricts to representations in module categories; so these proofs are different from the one given here.

Remark 7.3. The above definition of a Gröbner enrichment for every object in C means that the category C is enriched in the category of posets satisfying condition (G2), with morphisms the strictly increasing maps.

Remark 7.4. We write $\phi \preceq \psi$ for $\phi, \psi \in C(x)$ if $\phi = \chi\psi$ for some morphism χ. In many examples, the partial order $\preceq$ is a *refinement* of $\leq$, which means

$$\phi \preceq \psi \quad \Longrightarrow \quad \phi \leq \psi.$$

In that case, the condition in (G2) that every non-empty subset has a maximal element is automatic when (1) in Theorem 7.1 holds. So it seems natural to impose this condition in (G2).

Remark 7.5. The above condition (G2) can be relaxed. We need that, for every collection $\mathcal{X}$ of subsets of $C(x, y)$ that is closed under taking subsets and directed unions, the following holds: if $C(x, y) \notin \mathcal{X}$, there exists $\phi \in C(x, y)$ such that

$$\{\psi \in C(x, y) \mid \phi \prec \psi\} \in \mathcal{X} \quad \text{and} \quad \{\psi \in C(x, y) \mid \phi \preceq \psi\} \notin \mathcal{X}.$$

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