Ergod. Th. & Dynam. Sys. (2000), **20**, 1231–1251
Printed in the United Kingdom © *2000 Cambridge University Press*

Weak stability of the geodesic flow and Preissmann's theorem

RAFAEL OSWALDO RUGGIERO

Pontificia Universidade Católica do Rio de Janeiro, PUC-Rio,
Departamento de Matemática, Rua Marqués de São Vicente 225, Gávea,
Rio de Janeiro, Brasil
(e-mail: rorr@mat.puc-rio.br)

(*Received* 21 *June* 1998 *and accepted in revised form* 15 *March* 1999)

Abstract. Let (M, g) be a compact, differentiable Riemannian manifold without conjugate points and bounded asymptote. We show that, if the geodesic flow of (M, g) is either topologically stable, or satisfies the ϵ-shadowing property for some appropriate $\epsilon > 0$, then every abelian subgroup of the fundamental group of M is infinite cyclic. The proof is based on the existence of homoclinic geodesics in perturbations of (M, g), whenever there is a subgroup of the fundamental group of M isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

0. *Introduction*

The theorem of Preissmann is one of the first results of the now well-developed geometric group theory of spaces of negative curvature. Preissmann shows [**18**] that every non-trivial abelian subgroup of the fundamental group of a compact manifold of negative curvature is infinite cyclic. The purpose of the present work is to generalize Preissmann's theorem to a wider category of manifolds, where the role of the negative curvature is played by some classical notions of weak stability of the geodesic flow. Recall that a C^∞ Riemannian manifold (M, g), has no conjugate points if the exponential map at every point is non-singular. We shall denote by $\phi_t : T_1M \longrightarrow T_1M$ the geodesic flow of (M, g) acting in the unit tangent bundle T_1M. Let us denote by (T_1M, g) the unit tangent bundle endowed with the metric induced by g. We say that ϕ_t is C^k topologically stable if there exists a C^k neighbourhood V of ϕ_t such that, for every geodesic flow $\psi_t \in V$ corresponding to a metric $(M, \bar{g})$, there is a continuous, surjective map $f : (T_1M, \bar{g}) \longrightarrow (T_1M, g)$ sending the orbits of ψ_t, into orbits of ϕ_t, i.e. for every $\theta \in (T_1(M), \bar{g})$ there exists a continuous surjective map $r : \mathbb{R} \longrightarrow \mathbb{R}$ depending on θ with $r(0) = 0$ such that

$$\phi_{r(t)}(f(\theta)) = f(\psi_t(\theta))$$

for every $t \in \mathbb{R}$. The flow ϕ_t satisfies the ϵ-C^k-shadowing property for some $\epsilon > 0$ if there exists a C^k neighbourhood V of ϕ_t such that for every geodesic flow $\psi_t \in V$

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

corresponding to a metric $(M, \bar{g})$, there is a map $f : (T_1M, \bar{g}) \longrightarrow (T_1M, g)$ such that for every $\theta \in (T_1M, \bar{g})$ there exists a continuous surjective map $r : \mathbb{R} \longrightarrow \mathbb{R}$ with $r(0) = 0$ and

$$d(\phi_t(f(\theta)), \psi_{r(t)}(\theta)) \leq \epsilon$$

for every $t \in \mathbb{R}$. Notice that the map f in the definition of the ϵ-shadowing property is not required to be continuous, or surjective. The semi-equivalence f in the definition of topological stability may not be injective in general, which is the case for structurally-stable systems. In a previous paper [**20**, Theorem 1], we prove that, if the geodesic flow of a compact, analytic Riemannian manifold of non-positive curvature is either C^k topologically stable, or satisfies the ϵ-C^k-shadowing property for $\epsilon = \frac{1}{6}\,\text{inj. rad.}(M)$, then the fundamental group of M is Gromov hyperbolic. Here, inj. rad.(M) is the injectivity radius of (M, g). Hence, the fundamental group of M posseses a rich structure [**12**], and satisfies, in particular, Preissmann's theorem. The main result of the present work is the following.

THEOREM 1. *Let (M, g) be a C^∞, compact, Riemannian manifold without conjugate points and bounded asymptote of constant $B > 0$. If the geodesic flow ϕ_t is, either C^k topologically stable, or satisfies the ϵ-C^k-shadowing property, for some $k \in \mathbb{N}$, where $\epsilon = (1/6B)\,\text{inj. rad.}(M)$, then every abelian subgroup of the fundamental group of M is infinite cyclic.*

A manifold (M, g) without conjugate points is said to have bounded asymptote if there exists a constant $B > 0$ such that every stable Jacobi field $J^s(t)$ satisfies $\|J^s(t)\| \leq B\|J^s(0)\|$ for every $t \geq 0$. For the definition of stable Jacobi fields we refer to the work of Green [**11**]. These manifolds have been studied since the 1970s by many authors (see for instance [**8, 9**]). Manifolds without focal points are examples of such manifolds. However, many important features of the theory of manifolds of non-positive curvature are still open problems in the category of manifolds without conjugate points and bounded asymptote. We shall mention some of them, in the light of [**20**, Theorem 1], to motivate our present study. There are two main ideas in the proof of [**20**, Theorem 1]. On the one hand, if (M, g) is compact, non-positively curved, and $\tilde{M}$ is not Gromov hyperbolic, then it contains a flat, totally geodesic embedding of $\mathbb{R}^m$ [**8**]. Now, if (M, g) is analytic, Bangert and Schroeder [**2**] prove that M contains a flat, totally geodesic, immersed torus. On the other hand, perturbations of a flat metric in the two-torus have homoclinic geodesics, according to the works of Morse [**16**] and Hedlund [**13**]. Homoclinic behaviour of geodesics does not exist in the geometry of non-positive curvature, and we show in [**20**] how to produce homoclinic geodesics in perturbations of metrics of non-positive curvature having totally geodesic, flat, immersed tori. This phenomenon is used to show the topological instability and the absence of shadowing property of the geodesic flow whenever M contains a flat, totally geodesic, immersed torus.

Therefore, three difficult problems arise if we try to extend [**20**, Theorem 1], to manifolds without conjugate points. First of all, the existence of totally geodesic, flat planes in non-Gromov hyperbolic $(\tilde{M}, g)$ is a hard problem if we do not assume the convexity of the metric. Although a non-Gromov hyperbolic, quasiconvex $(\tilde{M}, g)$ contains a quasi-isometric immersion of the flat plane [**20**], such an immersion might be neither

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

isometric nor totally geodesic. Even in the case of bounded asymptotes we do not know any proof of this fact (this problem is related to the flat strip problem, treated by Burns and Knieper [**4**] in certain surfaces without conjugate points. The arguments in [**4**] strongly rely in two-dimensional geometry). Second, even if a flat exists in $(\tilde{M}, g)$, the analyticity is fundamental for the proof of the closing lemma for flats of Bangert and Schroeder. Third, the theory of homoclinic geodesics, or more generally, Mather's theory of connecting orbits [**15**], is well understood in two-dimensional geometry and in the torus [**1, 3, 13**]. We would like to exibit sufficient conditions, in $\pi_1(M)$, for the existence of homoclinic orbits in generic metrics on M, whereas an homological setting is more natural to study the critical points of a Lagrangian action.

Here, we are concerned with the third problem. We show that, if the fundamental group $\pi_1(M)$ of a compact manifold (M, g) with no conjugate points and bounded asymptote contains a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$, then there are perturbations of (M, g) in any C^k topology with a homoclinic geodesic. The limit set of this geodesic is a closed geodesic γ having homotopy class in this subgroup. Moreover, a lifting in $\tilde{M}$ of the homoclinic geodesic is bi-asymptotic to two different, bi-asymptotic liftings of γ, as in the two-dimensional case. Following the ideas in [**20**], we show the instability of the geodesic flow under the hypothesis of Theorem 1.

1. *Preliminaries of the theory of manifolds without conjugate points*

We start with a survey of some results concerning three aspects of the theory of manifolds without conjugate points, namely, the horospheres, the special nature of displacement functions in the universal covering, and the problem of the divergence of geodesic rays. Let us fix some notations. We shall always consider C^∞ Riemannian manifolds. Given $\theta = (p, v) \in T_1M$ we shall often use the notation $\gamma_\theta(t)$ for a geodesic having initial conditions $\gamma'_\theta(0) = v$, $\gamma_\theta(0) = p$. All the geodesics will be parametrized by arc length. A very special property of manifolds with no conjugate points is the existence of the so-called Busemann functions: given $\theta = (p, v) \in T_1\tilde{M}$ the Busemann function $b_\theta : \tilde{M} \longrightarrow \mathbb{R}$ associated to θ is defined by

$$b_\theta(x) = \lim_{t \to +\infty} (d(x, \gamma_\theta(t)) - t).$$

The level sets of b_θ are the horospheres $H_\theta(t)$, where the parameter t means that $\gamma_\theta(t) \in H_\theta(t)$ (notice that $\gamma_\theta(t)$ intersects each level set of b_θ perpendicularly at only one point). The function b_θ is C^1 for every θ, and its gradient ∇b_θ has norm equal to one at every point. Therefore, horospheres are C^1 imbedded submanifolds of $\tilde{M}$ of dimension $n - 1$, where n is the dimension of M, and there is a flow $\psi_t^\theta : \tilde{M} \longrightarrow \tilde{M}$, the Busemann flow, whose orbits are geodesics which are everywhere tangent to the vector field $-\nabla b_\theta$. We shall often refer to an orbit of the Busemann flow as a Busemann asymptote. In particular the geodesic γ_θ is an orbit of this flow and we have that

$$\psi_t^\theta(H_\theta(s)) = H_\theta(s + t)$$

for any real numbers t, s. So this flow acts in $\tilde{M}$ perpendicular to the corresponding horosphere foliation and always preserving it. As a consequence, we have that the distance

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

between two horospheres $H_\theta(s)$, $H_\theta(r)$ is $|s - r|$, and this distance is attained along Busemann asymptotes. It is clear that the map $t \mapsto H_\theta(t)$ is continuous on $t \in \mathbb{R}$ for each $\theta \in T_1(M)$. However, it is not known, in general, whether the map $\theta \mapsto H_\theta(0)$ depends continuously on θ. This is true if the curvature of M is non-positive, due to the convexity of the distance between geodesics. Many authors have dealt with this problem since the 1970s (see for instance [**8, 9, 17**]), replacing curvature restrictions by some assumptions on the asymptotic behaviour of geodesics.

Definition 1.1. Let M be a Riemannian manifold with no conjugate points. Two geodesics $\gamma(t)$, $\beta(t)$ in $\tilde{M}$ are called asymptotic if there exists $C > 0$ such that

$$d(\gamma(t), \beta(t)) \leq C$$

for every $t \geq 0$.

If γ, β are asymptotic, we say that β is an asymptote of γ. If $\gamma(t), \beta(t)$ are asymptotic and $\bar{\gamma}(t) = \gamma(-t), \bar{\beta}(t) = \beta(-t)$ are also asymptotic then we say that γ, β are bi-asymptotic. One of the nice properties of manifolds with bounded asymptote is that two geodesics are asymptotic in $\tilde{M}$ if and only if they are orbits of the same Busemann flow (see [**7**] for instance). There is no general proof so far of this equivalence for manifolds without conjugate points. The following results are due to Eschenburg [**9**], and express what is known about regularity of horospheres under the bounded asymptote hypothesis.

LEMMA 1.1. *Let M be compact without conjugate points and bounded asymptote. Then:*

(1) *there exists $D > 0$ such that every two orbits of a Busemann flow $\gamma(t)$, $\beta(t)$, satisfy $d(\gamma(t), \beta(t)) \leq Dd(\gamma(0), \beta(0))$ for every $t \geq 0$;*
(2) *horospheres $H_\theta(0)$ are C^2 submanifolds, and depend continuously on $\theta \in T_1\tilde{M}$ uniformly on compact subsets of $\tilde{M}$;*
(3) *A set of the type $H_{(p,v)}(0) \cap H_{(p,-v)}(0)$ is connected, and it consists of the points of intersection of bi-asymptotic orbits of $\psi_t^{(p,v)}$ with $H_{(p,v)}(0)$.*

Assertion (3) in Lemma 1.2, below, is well known for manifolds with non-positive curvature, where intersections between stable and unstable horospheres are convex subsets and determine regions in $\tilde{M}$ foliated by bi-asymptotic geodesics. The problem of the connectedness of intersections $H_{(p,v)}(0) \cap H_{(p,-v)}(0)$ is also related with the nature of the set of critical points of displacement functions, which is the second subject of the section.

Let γ be a closed geodesic of period P. A lift $\bar{\gamma}$ of γ in $\tilde{M}$ defined by $\pi(\bar{\gamma}(t)) = \gamma(t)$ determines a unique isometric $T_{\bar{\gamma}}$ of $\tilde{M}$ that translates $\bar{\gamma}$, i.e. $T_{\bar{\gamma}}(\bar{\gamma}(t)) = \bar{\gamma}(t + P)$, for every $t \in \mathbb{R}$. In particular, the map $T_{\bar{\gamma}}$ preserves $\bar{\gamma}$, and $\bar{\gamma}$ is called an axis of $T_{\bar{\gamma}}$. This gives a well-known representation of $\pi_1(M)$ as a group of isometries acting on $\tilde{M}$. The displacement of $T_{\bar{\gamma}}$ is the function $d_{\bar{\gamma}} : \tilde{M} \longrightarrow \mathbb{R}$ defined by $d_{\bar{\gamma}}(p) = d(p, T_{\bar{\gamma}}(p))$, where $d(p, q)$ is the distance in $\tilde{M}$ induced by the Riemannian metric in M. The special topological nature of the set of critical points of $d_{\bar{\gamma}}$ has been studied by many authors since Busemann. We state next some of the basic properties of $d_{\bar{\gamma}}$ following the work of Croke and Schroeder [**6**].

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

LEMMA 1.2. *Let M be compact without conjugate points.*

(1) *The critical points of $d_{\bar\gamma}$ are the axes of $T_{\bar\gamma}$.*
(2) *All critical points are minimum points, and the minimum value of $d_{\bar\gamma}$ equals the period of γ.*
(3) *If c is a critical value of d_γ, then nc is a critical value for d_{γ^n}.*
(4) *If M is analytic, the set of critical values of d_γ in the quotient $\tilde M / Z(T_{\bar\gamma})$ is connected, where $Z(T_{\bar\gamma})$ is the centralizer of $T_{\bar\gamma}$ in $\pi_1(M)$.*

So, under the bounded asymptote hypothesis, if $\gamma_{(p,v)}$ is an axis of $T_{\bar\gamma}$, the critical points of d_γ are contained in $C(p,v) = \cup_t (H_{(p,v)}(t) \cap H_{(p,-v)}(t))$. The next lemma characterizes the covering translations that commute with $T_{\bar\gamma}$ by the geometry of their action on the axes of $T_{\bar\gamma}$. This simple result will play an important role in the proof of the main theorem.

LEMMA 1.3. *Let M be a compact manifold without conjugate points. Let $\bar\gamma$ be the lift of a closed geodesic γ in M of minimum period P. A covering transformation T in $\pi_1(M)$ commutes with $T_{\bar\gamma}$ if and only if $T(\bar\gamma)$ is asymptotic to $\bar\gamma$.*

Proof. If T commutes with $T_{\bar\gamma}$, it is easy to show that the collection of axes of $T_{\bar\gamma}$ is invariant by T (see [**5**] for instance). In particular, the geodesic $T(\bar\gamma)$ is an axis of $T_{\bar\gamma}$ and hence, $T(\bar\gamma)$ is bi-asymptotic to $\bar\gamma$. The converse of this assertion is the new fact introduced by the lemma. So suppose that there exists $C > 0$ such that $d(T(\bar\gamma(t)), \bar\gamma(t)) \le C$ for every $t \ge 0$. Let $A = d(\bar\gamma(0), T(\bar\gamma(0)))$.

CLAIM 1.1. *$d(\bar\gamma(nP), T(\bar\gamma(nP))) \le A + 2C$, for every $n \in \mathbb{N}$. In particular, the displacement of T restricted to $\bar\gamma[0, +\infty)$ is bounded above.*

For, let $t_n \in \mathbb{R}$ be a point where $d(T(\bar\gamma(nP)), \bar\gamma) = d(T(\bar\gamma(nP)), \bar\gamma(t_n))$, i.e. the distance from $T(\bar\gamma(nP))$ to $\bar\gamma$ is attained. On the one hand, by the triangle inequality and the fact that T is an isometry, we have

$$\begin{aligned} d(\bar\gamma(0), \bar\gamma(t_n)) &\le d(\bar\gamma(0), T(\bar\gamma(0))) + d(T(\bar\gamma(0)), T(\bar\gamma(nP))) + d(T(\bar\gamma(nP)), \bar\gamma(t_n)) \\ &\le A + nP + C, \end{aligned}$$

which implies that

$$d(\bar\gamma(0), \bar\gamma(t_n)) - nP \le A + C.$$

On the other hand, by the same reason we have

$$nP = d(T(\bar\gamma(0)), T(\bar\gamma(nP))) \le A + C + d(\bar\gamma(0), \bar\gamma(t_n)),$$

and hence

$$nP - d(\bar\gamma(0), \bar\gamma(t_n)) \le A + C.$$

Therefore, we conclude that

$$|d(\bar\gamma(0), \bar\gamma(t_n)) - nP| \le A + C.$$

Now, it is not difficult to show that $\bar\gamma(t_n)$ cannot be too far from $\bar\gamma(nP)$. In fact, assume first that $t_n \le nP$. Then

$$d(\bar\gamma(0), \bar\gamma(nP)) = d(\bar\gamma(0), \bar\gamma(t_n)) + d(\bar\gamma(t_n), \bar\gamma(nP)),$$

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

which, by the above inequality, implies that

$$nP \geq nP - (A + C) + d(\bar{\gamma}(t_n), \bar{\gamma}(nP)).$$

Thus, $d(\bar{\gamma}(t_n), \bar{\gamma}(nP)) \leq A + C$. Assuming that $t_n \geq nP$, an analogous reasoning leads to the same conclusion. Hence,

$$\begin{aligned} d(\bar{\gamma}(nP), T(\bar{\gamma}(nP))) &\leq d(\bar{\gamma}(nP), \bar{\gamma}(t_n)) + d(\bar{\gamma}(t_n), T(\bar{\gamma}(nP))) \\ &\leq A + C + C = A + 2C, \end{aligned}$$

as stated in the claim.

CLAIM 1.2. *There exists $k \in \mathbb{Z}$ such that $T(T_{\bar{\gamma}})^k = (T_{\bar{\gamma}})^k T$.*

First of all, by Claim 1.1, we have that

$$\begin{aligned} d(\bar{\gamma}(nP), T(\bar{\gamma}(nP))) &= d((T_{\bar{\gamma}})^n(\bar{\gamma}(0)), T(T_{\bar{\gamma}})^n(\bar{\gamma}(0))) \\ &= d((\bar{\gamma}(0)), (T_{\bar{\gamma}})^{-n} T(T_{\bar{\gamma}})^n(\bar{\gamma}(0))) \\ &\leq A + 2C, \end{aligned}$$

for every $n \in \mathbb{N}$. Then, since the action of the isometries of $\pi_1(M)$ is discrete in $\tilde{M}$, there are only a finite number of different conjugates of $\bar{\gamma}(0)$ of the type $(T_{\bar{\gamma}})^{-n} T(T_{\bar{\gamma}})^n(\bar{\gamma}(0))$. So there exist $0 < n_1 < n_2$ such that

$$(T_{\bar{\gamma}})^{-n_1} T(T_{\bar{\gamma}})^{n_1}(\bar{\gamma}(0)) = (T_{\bar{\gamma}})^{-n_2} T(T_{\bar{\gamma}})^{n_2}(\bar{\gamma}(0)),$$

which implies that

$$(T_{\bar{\gamma}})^{-n_1} T(T_{\bar{\gamma}})^{n_1} = (T_{\bar{\gamma}})^{-n_2} T(T_{\bar{\gamma}})^{n_2},$$

because $\pi_1(M)$ is torsion free. Hence,

$$T = T_{\bar{\gamma}}^{-(n_2 - n_1)} T T_{\bar{\gamma}}^{(n_2 - n_1)},$$

thus proving the claim with $k = n_2 - n_1$.

CLAIM 1.3. $T T_{\bar{\gamma}} = T_{\bar{\gamma}} T$.

Indeed, notice that the geodesic $\bar{\gamma}_1 = T(\bar{\gamma})$ is a lift of γ which has minimum period P. Let $T_{\bar{\gamma}_1}$ be the translation with axis $\bar{\gamma}_1$ and critical displacement $d_{\bar{\gamma}_1}(\bar{\gamma}_1(t)) = P$. Since $(T_{\bar{\gamma}})^k$ commutes with T, the geodesic $\bar{\gamma}_1$ is also an axis of $(T_{\bar{\gamma}})^k$:

$$(T_{\bar{\gamma}})^k(\bar{\gamma}_1) = (T_{\bar{\gamma}})^k T(\bar{\gamma}) = T(T_{\bar{\gamma}})^k(\bar{\gamma}) = T(\bar{\gamma}) = \bar{\gamma}_1.$$

Therefore, $(T_{\bar{\gamma}})^k$ is a power of $T_{\bar{\gamma}_1}$. However, $T_{\bar{\gamma}}$ and $T_{\bar{\gamma}_1}$ have the same critical displacement P along $\bar{\gamma}_1$, and hence, since $\pi_1(M)$ is torsion free, we must have that $T_{\bar{\gamma}} = T_{\bar{\gamma}_1}$. In particular, $\bar{\gamma}_1$ is also an axis of $T_{\bar{\gamma}}$. Finally, recall that $T_{\bar{\gamma}_1}$ is obtained by conjugating $T_{\bar{\gamma}}$:

$$T_{\bar{\gamma}_1} = T T_{\bar{\gamma}} T^{-1}.$$

This follows from the fact that the map $T T_{\bar{\gamma}} T^{-1}$ preserves the geodesic $\bar{\gamma}_1$, where its critical displacement is P (as it is easy to check). Thus, the actions of $T_{\bar{\gamma}}$ and $T T_{\bar{\gamma}} T^{-1}$ coincide along $\bar{\gamma}_1$, and being $\pi_1(M)$ torsion free the two maps must coincide. This finishes the proof of the claim and Lemma 1.3. □

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

Finally, some comments about the divergence of geodesic rays in $\tilde{M}$. Although the norm of non-zero Jacobi fields vanishing somewhere is not bounded, this might not happen uniformly in $\tilde{M}$ (see [**7**]). Therefore, it is not clear that geodesic rays diverge in $\tilde{M}$. Knieper in [**14**] proved that Jacobi fields with a zero diverge at least linearly, more precisely we get the following.

LEMMA 1.4. *Let M be compact without conjugate points and bounded asymptote. Then there exists $D > 0$ such that every Jacobi field $J(t)$ with $J(0) = 0$ satisfies $\|J(t)\| \geq D\|J'(0)\|t$.*

Using this lemma it is not difficult to show that the distance between geodesic rays diverges at least linearly. We shall prove this fact in an appendix for the sake of completeness.

LEMMA 1.5. *Let M be compact without conjugate points and bounded asymptote. Then there exists $A > 0$ such that every pair of geodesics $\alpha(t)$, $\beta(t)$ with $\alpha(0) = \beta(0)$ satisfy $d(\alpha(t), \beta(t)) \geq A\angle(\alpha'(0), \beta'(0))t$.*

COROLLARY 1.6. *Let M be compact without conjugate points and bounded asymptote. If a geodesic $\alpha \subset \tilde{M}$ is not an orbit of the Busemann flow ψ_t^θ, then*

$$\limsup_{t\to+\infty} \angle(\alpha'(t), -\nabla b_\theta(\alpha(t))) > 0.$$

Proof. This is a straightforward consequence of Lemma 1.5. We leave the details to the reader. □

2. *Perturbing (M, g) to create waists*

In this section we shall exhibit a certain type of perturbation of (M, g) that produces a waist in any given homotopy class of M. A waist γ in a free homotopy class $[\gamma] \in \pi_1(M)$ is a closed geodesic whose length is a strict minimum of the length functional restricted to closed loops in $[\gamma]$. Let us denote by $l_g(\gamma)$ the length of γ with respect to the metric g.

LEMMA 2.1. *Let (M, g) be a compact Riemannian manifold and let γ be a closed geodesic in (M, g). Assume that the length of γ minimizes the length of closed loops in its homotopy class. Then, given $k \in \mathbb{N}$, $\delta > 0$, there exists a $\delta - C^k$ conformal perturbation (M, g_δ) of (M, g) such that:*

(1) *γ is a geodesic of (M, g),*
(2) *$l_{g_\delta}(\gamma) < l_g(\gamma)$,*
(3) *Let $\beta : [0, 1] \longrightarrow M$ be a differentiable curve. Then $l_{g_\delta}(\beta) \leq l_g(\beta)$. Moreover, if $l_g(\beta) \geq l_g(\gamma)$ then $l_{g_\delta}(\beta) \geq l_{g_\delta}(\gamma)$, and equality occurs if and only if γ is a subset of β.*

Proof. There are many examples in the literature of perturbations of metrics preserving certain prescribed totally geodesic subsets. The key property of our perturbations is (3). We follow [**20**] for the construction of conformal perturbations making waists. Essentially, γ is a geodesic of every perturbation of g of the type $g_\delta(p) = f_\delta(p)g(p)$, where $f_\delta : M \longrightarrow \mathbb{R}$ is a C^∞ function whose gradient is perpendicular to γ. The condition $f_\delta \leq 1$ implies the

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

shrinking of the arclength of curves. If there exists $p \in \gamma$ such that $f_\delta(p) = m < f(q)$ for every $q \in M - \gamma$, then γ becomes a waist in its homotopy class. Moreover, if $f_\delta(\gamma(t)) = m < f(q)$ for every $t \in \mathbb{R}$ and every $q \in M - \gamma$, then (3) holds. The proof of this statement is very simple: since $f_\delta \leq 1$, $l_{g_\delta}(\beta) \leq l_g(\beta)$ for every curve $\beta : [0, 1] \longrightarrow M$. Let us parametrize β by g-arclength, so $g(\beta'(t), \beta'(t)) = 1$ for every t. If we suppose that $l_g(\beta) \geq l_g(\gamma)$ we have

$$l_{g_\delta}(\beta) = \int_0^{l_g(\beta)} f_\delta(\beta(t))\, dt \geq \int_0^{l_g(\beta)} m\, dt = \int_0^{l_g(\beta)} f_\delta(\gamma(t))\, dt$$
$$\geq \int_0^{l_g(\gamma)} f_\delta(\gamma(t))\, dt = l_{g_\delta}(\gamma).$$

It is clear that, if β has a point outside γ, the above inequality is strict, thus proving (iii). Therefore, it remains to show that there exists f_δ with the required properties. Let V_δ be a tubular neighbourhood of γ of radius δ. Using partitions of unity, we can construct a C^∞ function $h_\delta : M \longrightarrow \mathbb{R}$, with support in V_δ, such that:

- $0 \leq h_\delta \leq 1$,
- $h_\delta(\gamma(t)) = 1$ for every $t \in \mathbb{R}$, and $h_\delta(p) < 1$ for every $p \in M - \gamma$.

Consider the function $H_\delta = (\delta)^{k+1} h_\delta$, and let

$$g_\delta(p) = (1 - H_\delta(p))g(p).$$

It is easy to check that $f_\delta = 1 - H_\delta$ is C^k close to one, and that γ is the set of strict minimum points of f_δ. This concludes the proof of Lemma 2.1. □

3. *The Hedlund effect*

The construction of the perturbations in §2 is motivated by some of the ideas that Hedlund [**13**] applied to investigate minimizing geodesics in certain metrics on the torus $\mathbb{T}^n$. Hedlund changed the flat metric on $\mathbb{T}^n$ in tubular neighbourhoods of closed geodesics generating the homotopy of $\mathbb{T}^n$, by shrinking drastically the metric on these neighbourhoods. This was to show the existence of homoclinic, globally minimizing geodesics in the new metric, generalizing what happened for $\mathbb{T}^2$ (for a recent, in-depth study of this subject we refer to Bangert [**1**]). Our goal in this section is to generalize some of the results by Hedlund, namely, we shall show that the perturbations in §2, when performed in metrics without conjugate points, force the globally minimizing geodesics of the perturbed metric to be most of the time in the support of the perturbation.

First, we give some notations. From now on we assume that (M, g) is a compact manifold without conjugate points having a non-zero homotopy class $a \in \pi_1(M)$ with non-trivial centralizer $Z(a)$, i.e. the quotient $Z(a)/\langle a \rangle$ is not the identity. This is equivalent to the existence of two covering translations T, S that generate a subgroup of $\pi_1(M)$ isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Let γ_0 be an axis of T, i.e. $T = T_{\gamma_0}$, and let $\gamma = \pi(\gamma_0)$ be the closed geodesic of (M, g) covered by γ_0. Let $\theta = (\gamma_0(0), \gamma_0'(0))$, and define $\gamma_1 = S(\gamma_0)$. We shall always parametrize γ_i by g-arclength. The notation $d_g(p, A)$ means the g-distance from a point p to a set A. If σ_1, σ_2 are two geodesics in $\tilde{M}$, the notation $d_g(\sigma_1, \sigma_2)$ means the Hausdorff distance in the metric g. Sometimes we shall use the notation $d(\,,\,)$, meaning d_g. Let (M, g_δ) be a perturbation of the type of §2 that

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

makes γ a strict minimizer in its homotopy class for (M, g_δ). The geodesic γ_1 is bi-asymptotic to γ_0, i.e. there exists $L = L(S) > 0$ such that $d(\gamma_0, \gamma_1) \leq L$. Parametrize $\gamma_i(t)$ such that $\gamma_1(t) = \gamma_1 \cap H_\theta(t)$. Call β_n the geodesic of $(\tilde{M}, g_\delta)$ that minimizes the distance between $\gamma_0(-nl_g(\gamma))$ and $\gamma_1(nl_g(\gamma))$. The forthcoming sections of the paper are devoted to proving that a subsequence of the β_n approaches the lift β_0 of a homoclinic geodesic β of γ, as Morse [**16**] and Hedlund [**13**] proved in the case of surfaces. We shall always parametrize $\beta_n : [0, L_n] \longrightarrow \tilde{M}$ by g_δ-arclength, where $\beta_n(0) = \gamma_0(-nl_g(\gamma))$ and $\beta_n(L_n) = \gamma_0(nl_g(\gamma))$. We shall denote by $d_{T_1\tilde{M}}((p, v), (q, w))$ the g-distance in the unit tangent bundle of $\tilde{M}$ between the tangent vectors (p, v) and (q, w). To shorten notation, we shall call by l the g-length of γ, $l = l_g(\gamma)$, and by $d_{T_1\tilde{M}}(v, w)$ the distance in $T_1\tilde{M}$ between (p, v) and (q, w). Recall that T_{γ_0} acts on γ_0 by translations, i.e. $T_{\gamma_0}(\gamma_0(t)) = \gamma_0(t + l)$.

Definition 3.1. Given $\epsilon > 0$, we say that a g_δ-geodesic segment $\alpha : (a, b) \longrightarrow \tilde{M}$ is ϵ, θ-critical if it satisfies the two following conditions:

(1) $\angle(\alpha'(t), -\nabla b_\theta(\alpha(t))) \leq \epsilon$ for every $t \in (a, b)$, where the angle $\angle(\,,\,)$ is measured with respect to the metric g;

(2) there is a lift $\tilde{\gamma}$ of γ which can be parametrized so that $d_{T_1\tilde{M}}(\alpha'(t), \tilde{\gamma}'(t)) \leq \epsilon$ for every $t \in (a, b)$.

A $0, \theta$-critical geodesic segment is a subset of a lift of γ_0 that is an orbit of the Buseman flow ψ^θ (recall that every lift of γ_0 is also a g_δ-geodesic by the construction of g_δ). In particular, this lift is asymptotic to γ_0 and therefore, by Lemma 1.3, it is an axis of T_{γ_0}.

LEMMA 3.1. *A g_δ-geodesic segment $\eta : (a, b) \longrightarrow \tilde{M}$ that is a limit of ϵ, θ-critical segments is an ϵ, θ-critical geodesic segment. Moreover, if $\alpha(a, b)$ is ϵ, θ-critical then $(T_{\gamma_0})^k(\alpha(a, b))$ is also ϵ, θ-critical for every $k \in \mathbb{Z}$.*

Proof. Since the proof is easy, we just give an outline for the sake of completeness. The first assertion of the lemma follows from the fact that items (1) and (2) in Definition 3.1 are given by closed conditions. The second assertion follows from two facts. On the one hand, the map T_{γ_0} is an isometry of $(\tilde{M}, g)$ that preserves the Buseman flow ψ^θ, i.e. $(DT_{\gamma_0})^k(-\nabla_p b_\theta) = -\nabla_{T_{\gamma_0}(p)} b_\theta$, so T_{γ_0} preserves the g-angles of curves with respect to $-\nabla b_\theta$. On the other hand, (M, g_δ) is a conformal perturbation of (M, g). Thus, $(\tilde{M}, g_\delta)$ is equivariant by the action of the covering isometries of $(\tilde{M}, g)$ and, in particular, T_{γ_0} is an isometry of $(\tilde{M}, g_\delta)$. Now, if $\alpha(a, b)$ is a g_δ-geodesic segment, then $T^k_{\gamma_0}(\alpha(a, b))$ is a g_δ-geodesic segment for every $k \in \mathbb{Z}$. Moreover, if $\alpha'(t)$ is close to the vector field $-\nabla b_\theta$ for every $t \in (a, b)$, then $DT^k_{\gamma_0}(\alpha'(t))$ remains close to the vector field $DT^k_{\gamma_0}(-\nabla b_\theta) = -\nabla b_\theta$ for every $t \in (a, b)$. Furthermore, if $\alpha(a, b)$ is close to a lift β of γ, then $(T_{\gamma_0})^k(\alpha(a, b))$ is close to $T_{\gamma_0}(\beta)$ that is a lift of γ. This concludes the proof of the lemma. □

LEMMA 3.2. *There exists ϵ_0 such that, if $\epsilon \leq \epsilon_0$ and two consecutive g_δ-geodesic segments $\alpha(a, b), \alpha(b, c)$ are ϵ, θ-critical, then $\alpha(a, c)$ is also ϵ, θ-critical.*

Proof. If $\alpha(a, b)$, $\alpha(b, c)$ are ϵ, θ-critical, then $\angle(\alpha'(t), -\nabla_{\alpha(t)} b_\theta) \leq \epsilon$ for every $t \in (a, b) \cup (b, c)$, and there exist two lifts β, σ of γ such that

$$d_{T_1\tilde{M}}(\alpha'(t), \beta'(t)) \leq \epsilon$$

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

for every $t \in (a, b)$, and

$$d_{T_1\tilde{M}}(\alpha'(t), \sigma'(t)) \leq \epsilon$$

for every $t \in (b, c)$. The point is that if ϵ is small enough, then $\beta = \sigma$. Indeed, since the action of $\pi_1(M)$ is discrete in $\tilde{M}$, there exists $\epsilon_1 > 0$ such that if two lifts σ_1, σ_2 of γ have points at distance less than ϵ_1, then either $\sigma_1 = \sigma_2$, or σ_1 intersects σ_2 transversally at some point making an angle of at least ϵ_1. This clearly implies that there exists $\epsilon_2(\epsilon_1) > 0$ such that, given two lifts σ_1, σ_2 of γ, then either $\sigma_1 = \sigma_2$, or $d_{T_1\tilde{M}}(\sigma_1'(t), \sigma_2'(t)) > \epsilon_2$ for every $t \in \mathbb{R}$. Now, let $\epsilon_0 = \epsilon_2/2$. If $d_{T_1\tilde{M}}(\alpha'(t), \beta'(t)) \leq \epsilon_0$ for every $t \in (a, b)$, and $d_{T_1\tilde{M}}(\alpha'(t), \sigma'(t)) \leq \epsilon_0$ for every $t \in (b, c)$, then

$$d_{T_1\tilde{M}}(\beta'(b), \sigma'(b)) \leq 2\epsilon_0 \leq \epsilon_2,$$

and hence $\beta = \sigma$ according to the definition of ϵ_2. This implies that the geodesic segment $\alpha(a, c)$ is within a distance ϵ_0 from a single lift of γ, thus concluding the proof of the lemma. □

From now on, we assume that the number δ, that is the radius of the support V_δ of g_δ, is smaller than ϵ_0. To simplify the notation during this section, let us call by H_k the horosphere of γ_0 in $(\tilde{M}, g)$ containing the point $\gamma_0(kl)$. Take a partition of β_n by connected segments $\beta_n[t_i, t_{i+1}]$, for $-n \leq i \leq n$, such that $\beta_n(t_i) \in H_i$ and $t_i < t_{i+1}$.

LEMMA 3.3. *For each $0 < \epsilon \leq \epsilon_0$ there is a number $N(\epsilon)$ such that, for any $n \in \mathbb{N}$, all except at most $N(\epsilon)$ of the segments $\beta_n([t_i, t_{i+1}])$ are ϵ, θ-critical. Furthermore, the complement of the union of the ϵ, θ-critical segments $\beta_n([t_i, t_{i+1}])$ has g_δ-length bounded above by $N(\epsilon)(l_{g_\delta}(\gamma) + L)$.*

Proof. By horosphere geometry (§1) we have that the g-distance between H_i and H_{i+m} is equal to $ml_g(\gamma) = ml$, so

$$l_g(\beta_n[t_i, t_{i+1}]) \geq l_g(\gamma)$$

for every $i \in [-n, n]$. This means by Lemma 2.1(3) that

$$l_{g_\delta}(\beta_n[t_i, t_{i+1}]) \geq l_{g_\delta}(\gamma)$$

for every i. From the fact that β_n is g_δ-globally minimizing and the triangle inequality we obtain

$$l_{g_\delta}(\beta_n) = \sum_{i=-n}^{n-1} l_{g_\delta}(\beta_n[t_i, t_{i+1}]) \leq 2nl_{g_\delta}(\gamma) + L,$$

which implies that

$$0 \leq \sum_{i=-n}^{n-1} [l_{g_\delta}(\beta_n[t_i, t_{i+1}]) - l_{g_\delta}(\gamma)] \leq L. \tag{1}$$

This is perhaps the most important remark of the paper, from now we shall analyze some of its strong consequences. Since each term $[l_{g_\delta}(\beta_n[t_i, t_{i+1}]) - l_{g_\delta}(\gamma)]$ is positive, the fact that their sum is bounded by L for all n implies that given $\lambda > 0$ there exist $N = N(\lambda, \delta, L)$ and a subset $A_{N,n}$ of indexes, with $\text{card}(A_{N,n}) \leq N$, such that

$$\sum_{i \in [-n,n] - A_{N,n}} [l_{g_\delta}(\beta_n[t_i, t_{i+1}]) - l_{g_\delta}(\gamma)] \leq \lambda.$$

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

Since g_δ and L are fixed along the proof, we can assume without loss of generality that N depends only on λ. Notice that N does not depend on n. In particular, $|l_{g_\delta}(\beta_n[t_i, t_{i+1}]) - l_{g_\delta}(\gamma)| \leq \lambda$ for every $i \in [-n, n] - A_{N,n}$.

CLAIM 3.1. *Given $\lambda > 0$ there exists $\mu = \mu(\lambda) > 0$, and covering isometries $a_{i,n} \in \pi_1(M)$, such that, if $|l_{g_\delta}(\beta_n[t_i, t_{i+1}]) - l_{g_\delta}(\gamma)| \leq \lambda$, then*

$$d_{T_1\tilde{M}}(\beta_n'(t), a_{i,n}(\gamma_0'(x(t)))) \leq \mu,$$

and

$$\angle(\beta_n'(t), -\nabla b_\theta(\beta_n(t))) \leq \mu,$$

for every $t \in [t_i, t_{i+1}]$, where $x(t) \in \mathbb{R}$.

Otherwise, the contradiction of the hypothesis of Claim 3.1 implies that there exists $t \in [t_i, t_{i+1}]$ such that, either $d(\beta_n(t), a(\gamma_0)) > \mu$ for every $a \in \pi_1(M)$, or the angle between $\beta_n'(t)$ and $-\nabla b_\theta(\beta_n(t))$ is at least μ. In the first case, there exists $\alpha(\mu, \delta) > 0$ such that

$$[l_{g_\delta}(\beta_n[t_i, t_{i+1}]) - l_{g_\delta}(\gamma)] > \alpha(\mu, \delta), \tag{2}$$

by Lemma 2.1(3). In the latter case, there exists $\omega(\mu) > 0$ such that $[l_g(\beta_n[t_i, t_{i+1}]) - l_g(\gamma)] > \omega(\mu)$, since the g-distance between H_i and H_{i+1} is attained at the asymptotes of γ_0 and equals $l = l_g(\gamma)$ (Corollary 1.6). Thus, again by Lemma 2.1, there exists $\bar{\alpha}(\mu, \delta) > 0$ such that

$$[l_{g_\delta}(\beta_n[t_i, t_{i+1}]) - l_{g_\delta}(\gamma)] > \bar{\alpha}. \tag{3}$$

However, the number of indexes $i \in [-n, n-1]$ such that $[t_i, t_{i+1}]$ satisfies either (2) or (3) is bounded above by $N(\inf\{\alpha, \bar{\alpha}\})$. It is clear that $\lim_{\lambda \to 0} \mu(\lambda) = 0$. This finishes the proof of Claim 3.1.

As a consequence of the claim we get that, given $\epsilon > 0$ there exists $\lambda = \lambda(\epsilon)$ such that $\mu(\lambda) \leq \epsilon$, since $\lim_{\lambda \to 0} \mu(\lambda) = 0$. Let

$$I_n^1 = I_n^1(\epsilon) = \{\beta_n(t) \mid d_{T_1M}(\beta_n'(t), T(\gamma_0')) > \epsilon \ \forall T \in \pi_1(M)\}.$$

and

$$I_n^2 = I_n^2(\epsilon) = \{\beta_n(t) \mid \angle(\beta_n'(t), -\nabla b_\theta(\beta_n(t))) > \epsilon\}.$$

Let $I_n = I_n^1 \cup I_n^2$. By inequality (1), and Claim 3.1, the cardinality of the collection of the $\beta_n[t_i, t_{i+1}]$ that contain points of I_n is bounded above by $A_{N,n} \leq N = N(\lambda(\epsilon)) = N(\epsilon)$. Moreover, by inequality (1), $l_{g_\delta}(\beta_n[t_i, t_{i+1}]) \leq l_{g_\delta}(\gamma) + L$, and hence, the length of the union of the $\beta_n[t_i, t_{i+1}]$ having points in I_n is bounded by $N(l_{g_\delta}(\gamma) + L)$ (that is a number independent of n). Finally, observe that if $\beta_n[t_i, t_{i+1}]$ does not contain points of I_n then $\beta_n[t_i, t_{i+1}]$ is an ϵ, θ-critical geodesic segment. This finishes the proof of the lemma. □

Lemma 3.3 motivates the following definition.

Definition 3.2. Given $\epsilon > 0$, we say that a g_δ-geodesic segment $\alpha : (a, b) \longrightarrow \tilde{M}$ is mostly ϵ, θ-critical if there exist an integer $r \leq N(\epsilon)$, and a collection of intervals $[c_1, d_1], \ldots, [c_r, d_r]$ such that

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

(1) $a \leq c_1 < d_1 < c_2 < \cdots < c_r < d_r \leq b$;
(2) the g_δ-length of the set $\cup_{i=1}^{r} \alpha([c_i, d_i])$ is at most $N(\epsilon)(l_{g_\delta}(\gamma) + L) = Q(\epsilon)$.

Clearly, an ϵ, θ-critical geodesic segment is mostly ϵ, θ-critical.

COROLLARY 3.4. *Given $0 < \epsilon \leq \epsilon_0$, the geodesic segment β_n is mostly ϵ, θ-critical for every $n \in \mathbb{N}$.*

Proof. According to Lemma 3.3, each β_n contains at most $r_n \leq N(\epsilon)$ intervals $\beta_n[t_{i_j}, t_{i_j+1}]$, $0 \leq j \leq r_n$, which are not ϵ, θ-critical. The total g_δ-length of these geodesic segments is at most $Q(\epsilon)$. Each connected component in the complement of this collection is a union of ϵ, θ-critical g_δ-geodesic segments. Since $\epsilon \leq \epsilon_0$ by hypothesis, Lemma 3.2 implies that such a component is ϵ, θ-critical, thus proving the corollary. □

Corollary 3.4 improves Lemma 3.3: the geodesic segments β_n are most of the time (up to a subset of length $\leq Q(\epsilon)$) close to the vector field $-\nabla b_\theta$, and simultaneously close to a certain number of lifts of γ. This number of lifts is bounded above by $N(\epsilon)$, which is a number not depending on n. We shall use these facts in the next section to show that, actually, some of the above lifts of γ must be bi-asymptotic to γ_0.

4. *β_n must approach an axis of T_{γ_0}*

We continue with the terminology of §3. The purpose of this section is to apply the results in §3 on manifolds with bounded asymptote (observe that the bounded asymptote condition has not been used so far). The main result of the section is the following.

PROPOSITION 4.1. *Let (M, g) be a compact manifold without conjugate points and bounded asymptote. There exists a subsequence β_{n_k} that converges to a g_δ-geodesic $\beta : (-\infty, \infty) \longrightarrow \tilde{M}$ which is bi-asymptotic to γ_0. Moreover, there exists an axis α of T_{γ_0} that is different from γ_0, with*

$$\lim_{t \to -\infty} d(\gamma_0, \beta(t)) = 0 = \lim_{t \to +\infty} d(\alpha, \beta(t)).$$

We shall subdivide the proof into several lemmas. Our first result is a generalization of Lemma 3.1. The proof is straightforward, we leave it to the reader.

LEMMA 4.2. *Any limit of mostly ϵ, θ-critical geodesic segments is mostly ϵ, θ-critical. Also, the image of a mostly ϵ, θ-critical geodesic segment by any power of T_{γ_0} is a mostly ϵ, θ-critical geodesic segment.*

LEMMA 4.3. *Assume that (M, g) has bounded asymptote. Let $\epsilon_0 > 0$ be the constant defined in Lemma 3.2. Suppose that a g_δ-ray $\beta : (0, +\infty) \longrightarrow \tilde{M}$ is mostly ϵ, θ-critical for every $\epsilon \leq \epsilon_0$. Then there exists an axis α of T_{γ_0} that is asymptotic to β. Furthermore, the geodesic α is a lift of γ.*

Proof. Let us first prove that, under the hypothesis of Lemma 4.3, we have the following.

CLAIM 4.1. $\lim_{t \to \infty} \angle(\beta'(t), -\nabla b_\theta(\beta(t))) = 0$.

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

By contradiction, assume that this is not the case. Then there exist some $0 < \epsilon < \epsilon_0$, and a sequence $s_n \to +\infty$, such that $\angle(\beta'(s_n), -\nabla b_\theta(\beta(s_n))) > \epsilon$ for every $n \in \mathbb{N}$. By the continuity of geodesics with respect to initial conditions, and the compactness of (M, g), there exists a constant $\eta > 0$ such that

$$\angle(\beta'(s_n + t), -\nabla b_\theta(\beta(s_n + t))) > \frac{\epsilon}{2}$$

for every $|t| \leq \eta$. This clearly contradicts the condition of mostly $\epsilon/2, \theta$-critical assumed for β. This finishes the proof of the claim.

From the fact that β is mostly ϵ, θ-critical for every $\epsilon \leq \epsilon_0$, we deduce that there exists $r_0 > 0$ such that every point $\beta(t)$, $t \geq r_0$, is within a distance $\epsilon_0/2$ from some lift α_t of γ_0. By Lemma 3.2, all these lifts in fact coincide on a single lift that we shall call α.

CLAIM 4.2. $\lim_{t\to\infty} d(\beta(t), \alpha) = 0$.

Indeed, since β is mostly ϵ, θ-critical for every $\epsilon \leq \epsilon_0$, given $\epsilon < \epsilon_0$, there exists $r(\epsilon) > 0$ such that $\beta(t)$ is at distance at most ϵ from a lift of γ_0. However, we observed that there is only one lift of γ_0 very close to $\beta(t)$ if t is large, and this lift is α. This proves the claim.

Now, let us show that $\lim_{t\to\infty} \angle(\alpha'(t), -\nabla b_\theta(\alpha(t))) = 0$. We argue as in Claim 4.1: otherwise, there would exist $\epsilon > 0$ and a sequence $x_n \to +\infty$ such that $\angle(\alpha'(x_n), -\nabla b_\theta(\alpha(x_n))) \geq \epsilon$ for every $n \in \mathbb{N}$. By Claim 4.2, this implies the existence of $R > 0$ large such that

$$\angle(\beta'(x_n), -\nabla b_\theta(\beta(x_n))) \geq \frac{\epsilon}{2},$$

for every $x_n \geq R$. However, by Claim 4.1, this contradicts the hypothesis on β. To finish the proof of Lemma 4.3, we apply Corollary 1.6 to conclude that α is asymptotic to γ_0, and hence, by Lemma 1.3, α must be an axis of T_{γ_0}. □

LEMMA 4.4. *Let $\gamma_0, \gamma_1, \beta_n$ be as in §3. Assume that, given $\epsilon > 0$, there exist n_0 such that, for every $n \geq n_0$ there is an interval $[s_n, r_n]$—with $|r_n - s_n| \geq n$—and an axis $\alpha = \alpha_n$ of T_{γ_0} such that*

$$d_{g_\delta}(\alpha, \beta_n[s_n, r_n]) \leq \epsilon.$$

Then the angle between γ_0 and β_n at $\beta_n(0)$ must tend to zero as $n \to +\infty$. Analogously, the angle between γ_1 and β_n tends to zero.

Proof. Lemma 4.4 is, in fact, a consequence of Morse's proof of [**16**, Theorem 6], for surfaces. Notice that we do not need to assume the bounded asymptote condition. The idea extends to any dimension, and we shall sketch it here for the sake of completeness (see Figure 1). According to the hypothesis, for every $n \geq n_0$ there exists a subsegment $\beta_n[s_n, r_n]$ of large length which is at distance at most ϵ from some axis α_n of T_{γ_0}. By Lemma 3.1, inequality (1), given $m > 0$, there exists $n_m > 0$ such that for every $n \geq n_m$, in a subset $[t_n, t_n + l_g(\gamma)] \subset [s_n, r_n]$ we have $d(\beta_n(t), \alpha_n) \leq 1/m$ for every $t \in [t_n, t_n + l_g(\gamma)]$. If there exists $a > 0$ such that the angle between $\beta_n(0)$ and $\gamma_0(-nl_g(\gamma))$ is bounded below by a, the same will happen to any translate $(T_{\gamma_0})^k(\beta_n)$. Namely, the angle between γ_0 and $(T_{\gamma_0})^k(\beta_n(t))$ at $t = 0$ is at least a. This allows us to

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

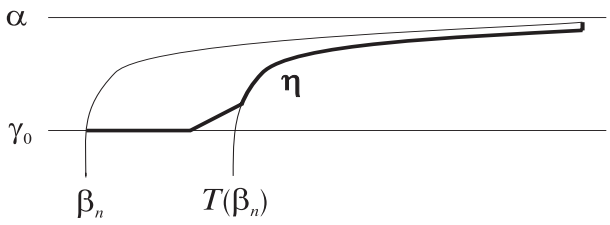


FIGURE 1.

take a g_δ-shortcut from $\beta_n(0)$ to $\beta_n(t_n + l_g(\gamma))$ that is different from β_n. The point is that the broken g_δ-geodesic

$$\gamma_0[-nl_g(\gamma), (-n+1)l_g(\gamma)] \cup T_{\gamma_0}(\beta_n[0, t_n])$$

can be shortened at $\gamma_0((-n+1)l_g(\gamma))$, due to the fact that the angle

$$\angle(\gamma_0'((-n+1)l_g(\gamma)), T_{\gamma_0}(\beta_n'(0)))$$

is greater than $a > 0$. Denote by $[p, q]$, a g_δ-shortest path joining two points p, q in $\pi_1(M)$. The argument in [**16**, Theorem 6], shows that there exists $x \in (-nl_g(\gamma), (-n+1)$ $l_g(\gamma))$, $y > 0$, and $w = w(a) > 0$, such that the broken geodesic (see Figure 1)

$$\eta = \gamma_0[-nl_g(\gamma), x] \cup [\gamma_0(x), T_{\gamma_0}(\beta_n(y))] \cup T_{\gamma_0}(\beta_n[y, t_n]) \cup [T_{\gamma_0}(\beta_n(t_n)), \beta_n(t_n + l_g(\gamma))],$$

which contains the points $\beta_n(0)$ and $\beta_n(t_n + l_g(\gamma))$, satisfies

$$l_{g_\delta}(\eta) - l_{g_\delta}(\beta_n[0, t_n + l_g(\gamma)]) \leq -w + \frac{2}{m}.$$

Thus, β_n would be no longer g_δ-minimizing for large m, n. □

Proof of Proposition 4.1. Let β_n be the g_δ-geodesic segments defined in §3. Recall that by Corollary 3.4, the geodesics $\beta_n : [0, L_n] \longrightarrow \tilde{M}$ are mostly ϵ, θ-critical for every $\epsilon \leq \epsilon_0$, where ϵ_0 is the constant defined in Lemma 3.2. Let $\beta_{n,1}, \beta_{n,2}, \ldots, \beta_{n,k(n)}$, be the ϵ_0, θ-critical subsegments of β_n, where $k(n) \leq N(\epsilon_0) + 1$ by Lemma 3.3. The ordering of the second subscript corresponds to the order in which the subsegments occur along β_n. Recall also that the total length of the gaps between the $\beta_{n,i}$'s is bounded above by $Q(\epsilon_0)$. Since the length of β_n tends to ∞ as $n \to \infty$, there exists a sequence $\beta_{n,i(n)}$ whose length tends to ∞. Define a sequence $j(n)$ by the following properties:

(1) $\lim_{n\to\infty} l_{g_\delta}(\beta_{n,j(n)}) = \infty$,
(2) the sequence $j(n)$ is minimal with respect to item (1): if any sequence $\beta_{n,i(n)}$ satisfies $\lim_{n\to\infty} l_{g_\delta}(\beta_{n,i(n)}) = \infty$, then $j(n) \leq i(n)$ for every $n \in \mathbb{N}$.

It is clear, by the choice of the sequence $j(n)$, that there exists a constant $A > 0$ such that for every $n \in \mathbb{N}$, $l_{g_\delta}(\beta_{n,i}) \leq A$ for every $i < j(n)$. Hence, the initial points of the subsegments $\beta_{n,j(n)}$ lie within a distance $(N(\epsilon)+1)A + Q(\epsilon)$ from γ_0. This allows us to choose a sequence of iterates $(T_{\gamma_0})^{m_n}$ such that the initial points of the segments $(T_{\gamma_0})^{m_n}(\beta_{n,j(n)})$ all lie within a bounded distance from $\gamma_0(0)$. Let $\bar{\beta}_n = (T_{\gamma_0})^{m_n}(\beta_n)$, and reparametrize β_n in order that $\bar{\beta}_n(0)$ is the initial point of $(T_{\gamma_0})^{m_n}(\beta_{n,j(n)})$. Then, the

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

sequence $(\bar{\beta}_n(0), \bar{\beta}_n'(0))$ has a convergent subsequence, whose limit will be called (q, w). To shorten notation, we shall assume that the whole sequence $\bar{\beta}_n$ is convergent. Let β be the g_δ limit geodesic of the subsequence, i.e. $\beta(0) = q$, $\beta'(0) = w$. By the choice of the indices $j(n)$, $\beta[0, +\infty)$ is a g_δ-ray which is the limit of a sequence of ϵ, θ-critical geodesic segments for every $\epsilon \leq \epsilon_0$. By Lemma 3.1, $\beta[0, +\infty)$ is ϵ, θ-critical for every $\epsilon \leq \epsilon_0$. Furthermore, by Lemma 4.3, there exists an axis α of T_{γ_0} that is asymptotic to β, and

$$\lim_{t \to +\infty} d(\beta(t), \alpha) = 0.$$

Hence, the sequence $\bar{\beta}_n$ of mostly ϵ, θ-geodesic segments approaches an axis of T_{γ_0}. According to Lemma 4.4, this implies that

$$\lim_{n \to +\infty} \angle(\bar{\beta}_n'(0), \gamma_0) = 0, \quad \lim_{n \to +\infty} \angle(\bar{\beta}_n'(L_n), \gamma_1) = 0.$$

In particular, every g_δ-geodesic segment $\beta_{n,j(n)}$ starts at $\gamma_0(-nl)$, and hence $j(n) = 1$ for every n; and every geodesic $\beta_{n,k(n)}$ ends at $\gamma_1(nl)$. If the gap between $\beta_{n,1} = \beta_{n,j(n)}$ and $\beta_{n,k(n)}$ had uniformly bounded g_δ-length, then the limit geodesic β would satisfy the statement of Proposition 4.1, with $\alpha = \gamma_1$. However, since this might not be the case, we have to choose more carefully some good pieces of β_n to find the limit geodesic β required by Proposition 4.1. We have already proved that the collection of ϵ, θ-geodesic segments $\beta_{n,i}$ whose associated axis of T_{γ_0} is precisely γ_0 is non-empty. Define the sequence $y(n) \leq k(n)$ by:

(1) the associated axis of $\beta_{n,y(n)}$ is γ_0, and $\lim_{t \to +\infty} l_{g_\delta}(\beta_{n,y(n)}) = \infty$,
(2) $y(n)$ is maximal with respect to the property of item (1).

Also, the collection of the $\beta_{n,i}$'s whose associated axis is γ_1 is non-empty. So define the sequence $y(n) < z(n) \leq k(n)$ by:

(1) the associated axis of $\beta_{n,z(n)}$ is not γ_0, and $\lim_{t \to +\infty} l_{g_\delta}(\beta_{n,y(n)}) = \infty$,
(2) the sequence $z(n)$ is minimal with respect to the property of item (1).

Parametrize $\beta_{n,y(n)} : [0, 1] \longrightarrow \tilde{M}$, $\beta_{n,z(n)} : [0, 1] \longrightarrow \tilde{M}$. Consider the subset of β_n from $\beta_{n,y(n)}(0)$ to $\beta_{n,z(n)}(1)$, let us denote it by $[\beta_{n,y(n)}(0), \beta_{n,z(n)}(1)]$. This g_δ-geodesic segment is mostly ϵ, θ-critical, and by the construction, its length goes to ∞ as $n \to +\infty$. Also, by the choice of $y(n)$, $z(n)$, we have the following.

CLAIM 4.3. *There exists a constant C such that the length of the subset of β_n from $\beta_{n,y(n)}(1)$ to $\beta_{n,z(n)}(0)$ is bounded above by C for every $n \in \mathbb{N}$.*

Indeed, otherwise, there would exist a sequence $n_p \to +\infty$ and a sequence of ϵ, θ-critical segments $\beta_{n_p,i(n_p)}$ contained in $[\beta_{n_p,y(n_p)}(1), \beta_{n_p,z(n_p)}(0)]$, i.e. $y(n_p) < i(n_p) < z(n_p)$, such that $\lim_{p \to +\infty} l_{g_\delta}(\beta_{n_p,i(n_p)}) = \infty$. This is because the length of the complement of the union of the ϵ_0, θ-critical components of β_n is bounded above by $Q(\epsilon_0)$ for every n; but now, the associated axis of $\beta_{n_p,i(n_p)}$ cannot be γ_0, since $y(n_p) < i(n_p)$, nor any other axis different from γ_0, since $i(n_p) < z(n_p)$. This contradicts the fact that $\beta_{n_p,i(n_p)}$ is an ϵ, θ-critical geodesic segment, thus proving the claim.

Finally, we repeat the limit argument made in the beginning of the proof of Proposition 4.1: consider iterates $(T_{\gamma_0})^{q_n}$ such that the points $(T_{\gamma_0})^{q_n}(\beta_{n,y(n)}(1))$ lie within a distance $\epsilon_0 + l$ from $\gamma_0(0)$. Thus, the sequence $(T_{\gamma_0})^{q_n}([\beta_{n,y(n)}(0), \beta_{n,z(n)}(1)])$ has a

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

convergent subsequence, whose limit β is a g_δ-geodesic that is mostly ϵ, θ-critical for every $\epsilon \leq \epsilon_0$. Furthermore, by the choice of $y(n)$, $z(n)$ we get the following:

(1) $\lim_{t\to-\infty} d(\beta(t), \gamma_0) = 0$,

(2) there exists an axis $\alpha \neq \gamma_0$ of T_{γ_0} such that $\lim_{t\to+\infty} d(\beta(t), \alpha) = 0$.

This concludes the proof of Proposition 4.1. □

5. *The existence of homoclinic geodesics and applications*

Let us summarize in a proposition the main results obtained until now.

PROPOSITION 5.1. *Let (M, g) be a compact manifold without conjugate points and bounded asymptote. Assume that there exists $a \in \pi_1(M)$ such that $Z(a)/\langle a\rangle$ is non-trivial. Then, for every $\delta > 0$, $k \in N$ there exists δ-C^k perturbations (M, g_δ) of (M, g) such that:*

(1) *there is a unique 'waist' in the homotopy class a, i.e. a unique, closed geodesic γ in (M, g_δ) that is a strict minimum of the length of closed loops freely homotopic to a. The geodesic γ is also a geodesic of (M, g);*

(2) *there exists a homoclinic g_δ-geodesic β in $\tilde{M}$, i.e. there exist two different, bi-asymptotic lifts γ_0, γ_1 of γ in $\tilde{M}$ such that*

$$\lim_{t\to+\infty} d_{g_\delta}(\gamma_1(t), \beta(t)) = 0 = \lim_{t\to-\infty} d_{g_\delta}(\gamma_0(t), \beta(t));$$

(3) *the geodesic γ_0 is an axis of a covering transformation T_{γ_0}, and there exists a covering translation S commuting with T_{γ_0} such that $\gamma_1 = S(\gamma_0)$. The subgroup generated by $\langle T_{\gamma_0}, S\rangle$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.*

Items (1), (2) are essentially Lemma 2.1 and Proposition 4.1. Item (3) follows from Proposition 4.1 and Lemma 1.3. Now, we proceed to prove the main theorem. We begin by pointing out a particular fact of the geometry of manifolds with bounded asymptote, namely the quasi-conformality of the action of the geodesic flow restricted to central Jacobi vector fields. The proof is straightforward from the definition of bounded asymptote.

LEMMA 5.2. *Let (M, g) be a compact manifold without conjugate points and bounded asymptote, i.e. every stable Jacobi field $J(t)$ satisfies $\|J(t)\| \leq B\|J(0)\|$ for every $t \geq 0$. Let σ_1, σ_2 be two bi-asymptotic geodesics in $\tilde{M}$. Then*

$$B^{-1}d(\sigma_2(0), \sigma_1) \leq d(\sigma_2(t), \sigma_1) \leq Bd(\sigma_2(0), \sigma_1)$$

for every $t \in \mathbb{R}$.

LEMMA 5.3. *Let (M, g) be compact without conjugate points. Assume that there exist two commuting covering translations T, S generating a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Let γ_0 be an axis of T, and suppose that $d(\gamma_0, S(\gamma_0)) \leq D$. Then, there exists a covering translation Γ commuting with T whose critical displacement is at most $D + (l(\gamma)/2)$.*

Proof. Since T and S commute, the geodesic $\gamma_1(t) = S(\gamma_0(t))$ is an axis of T, and thus, preserved by the iterates of T. Let $\gamma_1(x)$ be a point where the distance $d(\gamma_0(0), \gamma_1)$ is realized. There exists $k \in \mathbb{Z}$ such that $d(T^k(\gamma_1(0)), \gamma_1(x)) \leq l(\gamma)/2$, since T restricted to

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

γ_1 is a translation by $l(\gamma)$. Let $\Gamma = T^k S$, let $\bar{\eta}$ be an axis of Γ, and let $\eta = \pi(\bar{\eta})$. Since $l(\eta) \leq d(\gamma_0(0), \Gamma(\gamma_0(0))) = d(\gamma_0(0), T^k(\gamma_1(0)))$, by the triangular inequality we get

$$l(\eta) \leq d(\gamma_0(0), \gamma_1(x)) + d(\gamma_1(x), T^k(\gamma_1(0))) \leq D + \frac{l(\gamma)}{2},$$

which proves the lemma. □

LEMMA 5.4. *Let (M, g) be a compact manifold without conjugate points and bounded asymptote. Suppose that there exists a subgroup of $\pi_1(M)$ isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Then the geodesic flow of (M, g) is not C^k-topologically stable for any $k \in \mathbb{N}$.*

Proof. The proof is a generalization of the proof of [**20**, Theorem 1], that solved the problem in the case of manifolds of non-positive curvature. Assume that the geodesic flow is topologically stable in some C^k topology. Let T_{γ_0}, S be two commuting covering translations generating a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$, and let $\gamma = \pi(\gamma_0)$. Let (M, g_δ) be a C^k perturbation of (M, g) of the type constructed in §2, that turns γ into a waist in its homotopy class. According to Theorem 1, (M, g_δ) contains a homoclinic geodesic β_0 in $\tilde{M}$ approaching two different lifts γ_0 and $P(\gamma_0)$ of γ (respectively in past and future), where P is a covering translation commuting with T_{γ_0} that is not an element of the group generated by T_{γ_0}. Let $\beta = \pi(\beta_0)$ and let $f : (T_1 M, g_\delta) \longrightarrow (T_1 M, g)$ be the semi-equivalence provided by the topological stability of the geodesic flow of (M, g). The map f lifts to a semi-equivalence $\bar{f} : (T_1\tilde{M}, g_\delta) \longrightarrow (T_1\tilde{M}, g)$ in the unit tangent bundle of $\tilde{M}$ that commutes with the covering map of M. To simplify the notation, we denote also by β, γ, β_0 and γ_0 the orbits of the geodesic flows in $(T_1 M, g_\delta)$, $(T_1\tilde{M}, g_\delta)$ corresponding to these geodesics.

CLAIM 5.1. $f(\beta) = f(\gamma)$.

Indeed, otherwise, by the continuity of f, we would have that $\bar{f}(\beta_0)$ would be bi-asymptotic to $\bar{f}(\gamma_0)$. Moreover, $\lim_{t \to -\infty} d(\bar{f}(\beta_0(t), \bar{f}(\gamma_0))) = 0$. However, recall that (M, g) has bounded asymptote, and that $\bar{f}(\beta_0)$, $\bar{f}(\gamma_0)$ are geodesics of $(\tilde{M}, g)$. Hence, Lemma 5.2 implies that these two geodesics must be equal.

Now, we proceed as in the proof of Theorem 1 in [**20**]. Consider the closed loops c_n in M formed in the following way: join the point $\gamma_0(-nl)$ to a close point in β_0, say $\beta_0(t_n)$, with a geodesic segment $[\gamma_0(-nl), \beta_0(t_n)]$; join the point $\gamma_1(nl)$ to a close point in β_0, say $\beta_0(s_n)$, with a geodesic segment $[\gamma_0(nl), \beta_0(s_n)]$; consider the curve

$$C_n = [\gamma_0(-nl), \beta_0(t_n)] \cup \beta_0([t_n, s_n]) \cup [\gamma_1(nl), \beta_0(s_n)];$$

and let $c_n = \pi(C_n)$. Observe that, for n large, the length of the geodesic segments $[\gamma_0(-nl), \beta_0(t_n)]$ and $[\gamma_1(nl), \beta_0(s_n)]$ is very small. Moreover, since f is continuous, and $f(\beta) = f(\gamma)$, the images $f(\beta(t_n))$ and $f(\beta_0(s_n))$ are close to $f(\gamma(0))$. So we can suppose without loss of generality that $f(\beta(t_n)) = f(\beta_0(s_n))$. Since f is a surjective semi-equivalence, f is surjective on the closed orbits of the geodesic flow and then, the induced linear map $f_* : H_1(T_1 M, Z) \longrightarrow H_1(T_1 M, Z)$ is surjective. A linear, surjective transformation of $\mathbb{R}^m$ onto itself must be an isomorphism. Denote by $[c]$ the homology class of a closed curve in $T_1 M$. Notice that the homology class of c_n is

$$[c_n] = 2n[\gamma] + x,$$

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

where $x \in H_1(T_1M, \mathbb{Z})$ is a non-trivial homology class depending on P (recall that T_{γ_0} and P represent two commuting homotopy classes generating a non-cyclic abelian subgroup of $\pi_1(M)$). Hence, we have that $f_*([c_n]) = 2nf_*([\gamma]) + f_*(x)$. However, taking n large, $f(c_n)$ is a closed curve composed by $f(\gamma)$ together with two small closed curves homological to zero. Therefore, $f_*([c_n]) = 2nf_*([\gamma]) + f_*(x) = 2nf_*([\gamma])$, which implies that $f_*(x) = 0$ contradicting the fact that f_* is an isomorphism. □

LEMMA 5.5. *Let (M, g) be a compact manifold without conjugate points and bounded asymptote. Let B be the constant in the definition of bounded asymptote, and suppose that there exists a subgroup of $\pi_1(M)$ isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Then, if $\epsilon \leq$ inj. rad.$(M)/6B$, the geodesic flow does not satisfy the ϵ-C^k-shadowing property for any k.*

Proof. Assume, by contradiction, that the geodesic flow ϕ_t of (M, g) satisfies the ϵ-C^k-shadowing property for some $\epsilon \leq$ inj. rad.$(M)/6B$. Let T_{γ_0}, S be two covering translations generating a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$. We can assume without loss of generality that the critical displacement of T_{γ_0} is minimal among the isometries in $Z(T_{\gamma_0})$. Let $\gamma = \pi(\gamma_0)$, and let us consider a C^k perturbation (M, g_δ) of (M, g) satisfying Lemma 2.1, where γ now is strictly minimizing in its homotopy class with respect to the g_δ-length. Let β be the homoclinic geodesic of $(\tilde{M}, g_\delta)$ given by Proposition 5.1. The geodesic $\pi(\beta)$ has a geodesic shadow β_0 in (M, g), according to the contradiction hypotheses. By the choice of ϵ, there exists a lift $\bar{\beta}$ of the geodesic β_0 to $(\tilde{M}, g)$ within distance ϵ from β. Since there exists $\Gamma \in Z(T_{\gamma_0}) - \langle T_{\gamma_0} \rangle$, with $\Gamma(\gamma_0) = \gamma_1$, such that

$$\lim_{t \to -\infty} d(\beta(t), \gamma_0) = 0, \quad \lim_{t \to +\infty} d(\beta(t), \gamma_1) = 0,$$

then we have,

$$\limsup_{t \to -\infty} d(\bar{\beta}(t), \gamma_0) \leq \epsilon, \quad \limsup_{t \to +\infty} d(\bar{\beta}(t), \gamma_1) \leq \epsilon.$$

By the bounded asymptote condition, Lemma 5.2, the above inequalities imply

$$\sup_{t \in \mathbb{R}} d(\bar{\beta}(t), \gamma_i) \leq B\epsilon,$$

for $i = 0, 1$. Therefore, $d(\gamma_0, \gamma_1) \leq 2B\epsilon$. By Lemma 5.3, there exists an element in $Z(T_{\gamma_0})$ whose critical displacement is at most $2B\epsilon + (l_g(\gamma)/2)$. However, by the choice of ϵ, this number must be less than

$$\frac{l_g(\gamma)}{3} + \frac{l_g(\gamma)}{2} < l_g(\gamma).$$

This contradicts the fact that the critical displacement of T_{γ_0}, $l_g(\gamma)$, minimizes the critical displacements in $Z(T_{\gamma_0})$. □

Theorem 1 follows from Lemmas 5.4 and 5.5.

Acknowledgements. The author was partially supported by CNPq of the Brazilian Government.

The author specially thanks Professor K. Burns for fruitful discusions about many subjects related to this paper. The author also thanks Professor H. Weiss, the Department

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

of Mathematics of Northwestern University and the Department of Mathematics of Penn. State University at State College, for their kind hospitality while part of this work was in progress. We would like to thank the referee for the detailed and careful report about the paper. The remarks and suggestions in the report improved substantially the exposition in §§3 and 4.

Appendix

The purpose of the section is to show Lemma 1.5. Throughout the section, (M, g) will be a compact Riemannian manifold without conjugate points and bounded asymptote, i.e. there exists $B > 0$ such that every stable Jacobi field $J(t)$ satisfies $\|J(t)\| \leq B\|J(0)\|$ for every $t \geq 0$. We shall use the notation $[p, q]$ for a geodesic segment joining two points p, q in $\tilde{M}$.

LEMMA A.1. *Let α, β be two geodesic rays in $\tilde{M}$, with $\alpha(0) = \beta(0) = p$. Let $S_t(p)$ be the sphere of radius t centered at p. There exists $D > 0$ such that the spherical distance $d_{S_t(p)}$—the restriction of the distance in $\tilde{M}$ to $S_t(p)$—satisfies $d_{S_t(p)}(\alpha(t), \beta(t)) \geq D\angle(\alpha'(0), \beta'(0))t$, for every $t > 0$.*

This lemma is a straightforward consequence of Lemma 1.4, we simply state it without proof. We would like to show Lemma A.1 taking the distance d, induced by g in $\tilde{M}$, instead of $d_{S_t(p)}$.

LEMMA A.2. *Let α, β be two geodesic rays in $\tilde{M}$, with $\alpha(0) = \beta(0) = p$. If $d(p, [\alpha(t), \beta(t)]) < t/n$ then $d(\alpha(t), \beta(t)) \geq (2 - 2/n)t$. In particular, if $n = 2$ we get $d(\alpha(t), \beta(t)) \geq t$.*

Proof. Parametrize $[\alpha(t), \beta(t)] : [0, 1] \longrightarrow \tilde{M}$, and let $[\alpha(t), \beta(t)](t_0)$ be the point where the distance $d(p, [\alpha(t), \beta(t)](s)$ is minimal. Then

$$d(\alpha(t), \beta(t)) = d(\alpha(t), [\alpha(t), \beta(t)](t_0)) + d([\alpha(t), \beta(t)](t_0), \beta(t)),$$

since $[\alpha(t), \beta(t)]$ minimizes the distance between any two points in it. On the other hand, by the triangle inequality,

$$d(\alpha(t), [\alpha(t), \beta(t)](t_0)) \geq l(\alpha[0, t]) - d(p, [\alpha(t), \beta(t)](t_0)) \geq t - \frac{t}{n},$$
$$d(\beta(t), [\alpha(t), \beta(t)](t_0)) \geq l(\beta[0, t]) - d(p, [\alpha(t), \beta(t)](t_0)) \geq t - \frac{t}{n}.$$

This implies that $d(\alpha(t), \beta(t)) \geq 2t - (2/n)t$, thus proving the lemma. □

LEMMA A.3. *Let D be as in Lemma 1.3, and let α, β be two geodesic rays in $\tilde{M}$ with $\alpha(0) = \beta(0) = p$. Let $L > 0$ and suppose that $d(\alpha(t), \beta(t)) = L$ for some $t > 0$. Then there exists $T = T(D, L) > 0$ such that:*

(1) *$d(\alpha(s), \beta(s)) \geq T\angle(\alpha'(0), \beta'(0))s$ for every $0 \leq s \leq t$,*

(2) *$d(\alpha(t + s), \beta(t + r)) \geq Td(\alpha(t), \beta(t))$ for every $r, s > 0$.*

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

Proof. Assume, without loss of generality, that $t \leq 2$. The main point of the proof is that, since $d(\alpha(t), \beta(t)) \leq L$ then $d_{S_t(p)}(\alpha(s), \beta(s))$ is comparable with $d(\alpha(t), \beta(t))$, i.e. there exists $G > 0$ depending on L and the bounded geometry (curvature bounds) of spheres of radius greater than one, such that

$$\frac{1}{G} d_{S_t(p)}(\alpha(t), \beta(t)) \leq d(\alpha(t), \beta(t)) \leq G d_{S_t(p)}(\alpha(t), \beta(t)).$$

The spheres of radius at least one have bounded geometry because $\tilde{M}$ has bounded geometry and M has no conjugate points (see for instance [**17**]). Thus, item (1) follows from this comparison between the distances $d_{S_t(p)}$, d, and Lemma A.1. Let us prove item (2) by contradiction. Assume that there is no constant $C > 0$ such that $d(\alpha(t+s), \beta(t+r)) \geq C d(\alpha(t), \beta(t))$ for every $s, t > 0$. Then there exist sequences s_n, r_n of positive numbers such that $d(\alpha(t+s_n), \beta(t+r_n)) \leq n^{-1} d(\alpha(t), \beta(t)) \leq L/n$. By the comparison inequality we deduce that, given $\delta > 0$, there exists n_δ such that $d_{S_t(p)}(\alpha(t+s_n), \beta(t+s_n)) \leq \delta$. However, this contradicts Lemma A.1, since δ is arbitrary and the angle $\angle(\alpha'(0), \beta'(0))$ is fixed. Hence, there must exist some $C > 0$ such that $d(\alpha(t+s), \beta(t+r)) \geq C d(\alpha(t), \beta(t))$ for every $t, s > 0$. Finally, to get the constant T take $T = \min\{1/G, C\}$. □

Proof of Lemma 1.5. We can assume, without loss of generality, that $t \geq 2$ and $t^{-1} d(p, [\alpha(t), \beta(t)]) \geq \frac{1}{2}$ (Lemma A.2). Let $[\alpha(t), \beta(t)](t_0)$ be, as in the proof of Lemma A.2, a point where $d(p, [\alpha(t), \beta(t)](s))$ is minimal. Let $P : [\alpha(t), \beta(t)] \longrightarrow S_{t_0}(p)$ be the projection along geodesic rays starting at $p = \alpha(0) = \beta(0)$. Denote by $b(s)$ the curve $P([\alpha(t), \beta(t)])$, where s is the arclength parameter. Let $l(b)$ be the length of b, and assume that $b(0) = P(\alpha(t))$, $b(l(b)) = P(\beta(t))$. Take a partition of $b(s)$ in subcurves $b[s_i, s_{i+1}]$, for $i = 0, 1, \ldots, m$, such that $s_0 = 0$, $s_m = l(b)$, $l(\beta[s_i, s_{i+1}]) = 1$ for every $0 \leq i \leq m-2$, and $l(\beta[s_{m-1}, s_m]) \leq 1$. Let γ_i be the geodesic ray joining $p = \alpha(0)$ to $b(s_i)$, and let $\gamma_i(x_i) = \gamma_i \cap [\alpha(t), \beta(t)]$. It is clear that $x_i \geq t_0$ for every $0 \leq i \leq m$. Let T be the constant given by Lemma A.3, for $L = 1$. Therefore, we have

$$d(\alpha(t), \beta(t)) = \sum_{i=0}^{m-1} l[\gamma_i(x_i), \gamma_{i+1}(x_{i+1})] \geq \sum_{i=0}^{m-1} T d(\gamma_i(t_0), \gamma_{i+1}(t_0)),$$

by Lemma A.3(1). Applying Lemma A.3(2) we get

$$d(\alpha(t), \beta(t)) \geq \sum_{i=0}^{m-1} T^2 \angle(\gamma_i'(0), \gamma_{i+1}'(0)) t_0,$$

and hence, by the assumption $t_0/t = t^{-1} d(p, [\alpha(t), \beta(t)]) \geq \frac{1}{2}$ we have

$$d(\alpha(t), \beta(t)) \geq T^2 \frac{1}{2} t \sum_{i=0}^{m-1} \angle(\gamma_i'(0), \gamma_{i+1}'(0)).$$

However, observe that the sum of the angles $\angle(\gamma_i'(0), \gamma_{i+1}'(0))$ is larger than $\angle(\alpha'(0), \beta'(0))$, with equality holding if and only if the curve $b(s)$ is a geodesic in $S_1(p)$ with respect to the metric $d_{S_1}(p)$. This finishes the proof of Lemma 1.5. □

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press

REFERENCES

[1] V. Bangert. Minimal geodesics. *Ergod. Th. & Dynam. Sys.* **10** (1989), 263–286.

[2] V. Bangert and V. Schroeder. Existence of flat tori in analytic manifolds of non-positive curvature. *Ann. Sci. Ec. Norm. Sup.* **24** (1991), 605–634.

[3] S. Bolotin. Homoclinic orbits in invariant tori of Hamiltonian systems. *Dynamic Systems and Classical Mechanics (Amer. Math. Soc. Transl. Ser. 2, 168)*. American Mathematical Society, Providence, RI, 1995, pp. 21–90.

[4] K. Burns and G. Knieper. Rigidity of surfaces with no conjugate points. *J. Diff. Geom.* **34** (1991), 623–650.

[5] H. Busemann. *The Geometry of Geodesics*. Academic Press, New York, 1955.

[6] C. Croke and V. Schroeder. The fundamental group of compact manifolds wihout conjugate points. *Comm. Math. Helv.* **61** (1986), 161–175.

[7] P. Eberlein. When is a geodesic flow of Anosov type I. *J. Diff. Geom.* **8** (1973), 437–463.

[8] P. Eberlein. Geodesic flows in certain manifolds without conjugate points. *Trans. Amer. Math. Soc.* **167** (1972), 151–170.

[9] J.-H. Eschenburg. Horospheres and the stable part of the geodesic flow. *Math. Z.* **153** (1977), 237–251.

[10] A. Freire and R. Mañé. On the entropy of the geodesic flow in manifolds without conjugate points. *Inv. Math.* **69** (1982), 375–392.

[11] L. Green. A theorem of E. Hopf. *Michigan Math. J.* **5** (1958), 31–34.

[12] M. Gromov. Hyperbolic groups. *Essays in Group Theory*. Ed. S. M. Gersten. Springer, New York, pp. 75–263.

[13] G. Hedlund. Geodesics on a two-dimensional Riemannian manifolds with periodic coefficients. *Ann. Math.* **33** (1932), 719–739.

[14] G. Knieper. Mannifaltigkeiten ohne konjugierte Punkte. *Bonner Mathematische Schriften* **168** (1986).

[15] J. Mather. Variational construction of connecting orbits. *Ann. Inst. Fourier* **43** (1993), 1349–1386.

[16] M. Morse. A fundamental class of geodesics on any closed surface of genus greater than one. *Trans. Amer. Math. Soc.* **26** (1924), 25–60.

[17] Ya. B. Pesin. Geodesic flows on closed Riemannian manifolds without focal points. *Math. USSR Izvestija* **11** (1977), 1195–1228.

[18] A. Preissmann. Quelques propriétés globales des espaces de Riemann. *Comm. Math. Helv.* **15** (1943), 175–216.

[19] R. Ruggiero. On nonhyperbolic quasiconvex spaces. *Trans. Amer. Math. Soc.* **350**(2) (1998), 665–687.

[20] R. Ruggiero. Topological stability and Gromov hyperbolicity. *Ergod. Th. & Dynam. Sys.* **19** (1999), 143–154.

https://doi.org/10.1017/S0143385700000663 Published online by Cambridge University Press