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Rarefied gas flow between parallel plates

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Abstract. The flow of a rarefied gas between parallel plates has been studied via the linearized Boltzmann transport equation. Using a general boundary condition, which includes an arbitrary ratio of specular to diffuse reflection from the wall, we have derived an integral equation for the mass flow velocity. The integral equation is solved by using a replication property of the kernel and application of the method of Muskelishvili.

The total volumetric flow rate is obtained and a slip boundary condition is deduced for use with the hydrodynamic equations.

Certain aspects of the eigenvalue spectrum associated with the Boltzmann equation are discussed.

1. *Introduction.* The flow of rarefied gases through long tubes has been studied for a number of years in both its experimental and theoretical aspects (1–9). The curious minimum in the volumetric flow rate as a function of the Knudsen number (i.e. average mean freepath/diameter) has been fully explained. It has been shown, for example, that this arises from a complicated interaction between intermolecular collisions, boundary effects and transport free flight processes. It is essential, therefore, for an accurate prediction of this phenomenon, that the method of solution of the Boltzmann equation treat the boundary condition exactly.

The mathematical problems posed by this requirement are quite severe and, in general, an explicit solution for an arbitrary scattering model is not possible. However, in view of the secondary importance of interparticle interactions, it has been found possible to approximate this interaction by a simple function which preserves the main physical characteristics of the collision process but enables the resulting Boltzmann equation to be solved analytically.

In this paper we shall extend the work of Cercignani (10), who has studied this problem using the Bhatnagar, Gross and Krook, or B.G.K., model (11) with constant collision frequency. Cercignani considers the flow of gas between parallel plates with the assumption of perfectly diffuse scattering from the wall; we shall treat the same problem but with a more realistic description of the boundary conditions which will include an arbitrary amount of specular reflection.

The resulting Boltzmann integro-differential equation is converted to a purely integral one, and this is solved by using a certain replication property possessed by the kernel.

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Finally, we present a slip-flow boundary condition suitable for use with the conventional hydrodynamic equations.

2. *The basic equation of Poiseuille flow.* Consider the flow of gas between parallel plates. Then, if z represents the direction of flow and x a distance perpendicular to this, the appropriate linearized Boltzmann equation may be written

$$\left(c_x\frac{\partial}{\partial x}+c_z\frac{\partial}{\partial z}\right)f(\mathbf{c},x,z)=f_0(\mathbf{c})\,\hat{L}f(\mathbf{c},x,z) \tag{1}$$

where linearization implies that the mean flow speed must be small compared with the speed of sound in the gas.

In equation (1), $f(\mathbf{c},x,z)$ is the distribution function of gas atoms, and f_0 is a Maxwellian function given by

$$f_0(\mathbf{c})=(\rho_0/\pi^{\frac{3}{2}})\,e^{-c^2}, \tag{2}$$

where ρ_0 is a convenient reference density and $c^2=c_x^2+c_y^2+c_z^2$. c is the speed of an atom normalized to the velocity $(2kT/m)^{\frac{1}{2}}$, T being the temperature of the system and m the mass of an atom. $\hat{L}$ is the collision operator and is defined by the nature of the atomic interaction law.

We assume a constant but small density gradient along the length of the tube, and write f as

$$f(\mathbf{c},x,z)=\{n(z)/\pi^{\frac{3}{2}}\}\,e^{-c^2}[1+h(\mathbf{c},x)], \tag{3}$$

with $n(z)=\rho_0(1+kz)$.

Inserting (3) into (1), and neglecting terms of order $\rho_0 k$ and h with respect to unity, we find that h satisfies the following equation:

$$\kappa c_z+c_x\{\partial h(\mathbf{c},x)/\partial x\}=\hat{L}h(\mathbf{c},x), \tag{4}$$

where we have set $\rho_0 k=\kappa$.

If we take the plates to be symmetric about $x=0$ and situated at $x=\pm d$, the boundary conditions associated with equation (4) may be written as follows:

$$h(-c_x,c_y,c_z,d)=\frac{2}{\pi}\alpha\int_0^\infty dc_x'\int_{-\infty}^\infty dc_y'\int_{-\infty}^\infty dc_z'\,c_x'\,e^{-c'^2}h(c_x',c_y',c_z',d)+(1-\alpha)\,h(c_x,c_y,c_z,d) \tag{5}$$

where $c_x>0$.

A similar equation at $x=-d$ exists but with c_x replaced by $-c_x$.

The first term on the right-hand side of equation (5) corresponds physically to elastic diffuse scattering with the scattered particles being emitted according to the cosine law. The second term represents the amount of specular reflection. α is an empirical constant which determines the relative amounts of specular and diffuse scattering; it is called the accommodation coefficient.

Further progress in the solution of equation (4) can only be made if some assumptions are made about $\hat{L}$. For the purpose of this paper we assume that the linearized B.G.K. model applies with constant collision frequency. Thus (12)

$$\hat{L}h\equiv-\lambda h(\mathbf{c})+\pi^{-\frac{3}{2}}\lambda\left\{\int d\mathbf{c}'\,e^{-c'^2}h(\mathbf{c}')+2\mathbf{c}\cdot\int\mathbf{c}'\,e^{-c'^2}h(\mathbf{c}')\,d\mathbf{c}'+\tfrac{2}{3}(c^2-\tfrac{3}{2})\int d\mathbf{c}'(c'^2-\tfrac{3}{2})\,e^{-c'^2}h(\mathbf{c}')\right\}. \tag{6}$$

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Insertion of (6) into (4) and use of symmetry leads to

$$\kappa c_z + c_x \frac{\partial h}{\partial x} + \lambda h = \pi^{-\frac{3}{2}} \lambda \left\{ \int d\mathbf{c}' \, e^{-c'^2} h(\mathbf{c}') + 2c_z \int c_z' \, e^{-c'^2} h(\mathbf{c}') \, d\mathbf{c}' \right.$$
$$\left. + \tfrac{2}{3}(c^2 - \tfrac{3}{2}) \int d\mathbf{c}' (c'^2 - \tfrac{3}{2}) \, e^{-c'^2} h(\mathbf{c}') \right\}. \quad (7)$$

Before reducing equation (7) further, consider the mass flow velocity in the z-direction, viz.

$$q(x) = \int d\mathbf{c} \, c_z f(\mathbf{c}, x, z). \quad (8)$$

Using (3), we find

$$q(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dc_x \exp(-c_x^2) \, Z(x, c_x) \quad (9)$$

where

$$Z(x, c_x) = \frac{1}{\pi} \int_{-\infty}^{\infty} dc_y \int_{-\infty}^{\infty} dc_z c_z \exp(-c_y^2 - c_z^2) \, h(c_x, c_y, c_z, x). \quad (10)$$

But, if we multiply equation (7) by $c_z \exp(-c_y^2 - c_z^2)$ and integrate over c_y and c_z, we find

$$\tfrac{1}{2}\kappa + c_x \frac{\partial Z}{\partial x} + \lambda Z = \frac{\lambda}{\sqrt{\pi}} \int_{-\infty}^{\infty} dc_x' \exp(-c_x'^2) \, Z(x, c_x'). \quad (11)$$

Using the same procedure with the boundary conditions leads to

$$Z(d, -c_x) = (1-\alpha) Z(d, c_x) \quad (c_x > 0), \quad (12)$$

$$Z(-d, c_x) = (1-\alpha) Z(-d, -c_x) \quad (c_x > 0). \quad (13)$$

We also have the symmetry condition

$$Z(x, c_x) = Z(-x, -c_x). \quad (14)$$

Equation (11), with its boundary conditions, is the basic equation for study.

3. *Conversion to an integral equation.* From the definition (9) we may obtain an integral equation for $q(x)$. Thus, integrating (11) as a first-order differential equation, assuming the right-hand side to be known, we find:

$$Z(x, c_x) = \exp\{\lambda(d-x)/c_x\} \, Z(d, c_x) - \frac{1}{c_x} \int_x^d \exp\{\lambda(x'-x)/c_x\}$$
$$\times \{\lambda q(x') - \tfrac{1}{2}\kappa\} \, dx' \quad (c_x < 0), \quad (15a)$$

$$Z(x, c_x) = \exp\{-\lambda(d+x)/c_x\} \, Z(-d, c_x) + \frac{1}{c_x} \int_{-d}^x \exp\{\lambda(x'-x)/c_x\}$$
$$\times \{\lambda q(x') - \tfrac{1}{2}\kappa\} \, dx' \quad (c_x > 0). \quad (15b)$$

Setting $x = -d$ in (15a), and $+d$ (15b), using the boundary conditions and symmetry, we obtain

$$Z(x, c_x) = \frac{(1-\alpha)\exp\{-\lambda(d+x)/c_x\}}{1-(1-\alpha)\exp(-2\lambda d/c_x)} \frac{1}{c_x} \int_{-d}^d \exp\{\lambda(x'-d)/c_x\} \{\lambda q(x') - \tfrac{1}{2}\kappa\} \, dx'$$
$$+ \frac{1}{c_x} \int_{-d}^x \exp\{\lambda(x'-x)/c_x\} \{\lambda q(x') - \tfrac{1}{2}\kappa\} \, dx' \quad (c_x > 0) \quad (16a)$$

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and $$Z(x,c_x) = -\frac{(1-\alpha)\exp\{\lambda(d-x)/c_x\}}{1-(1-\alpha)\exp(2d\lambda/c_x)}\frac{1}{c_x}\int_{-d}^{d}\exp\{\lambda(x'+d)/c_x\}\{\lambda q(x')-\tfrac{1}{2}\kappa\}\,dx'$$
$$-\frac{1}{c_x}\int_{x}^{d}\exp\{\lambda(x'-x)/c_x\}\{\lambda q(x')-\tfrac{1}{2}\kappa\}\,dx' \quad (c_x<0). \tag{16b}$$

Inserting these expressions into equation (9) leads to the following integral equation for $q(x)$:

$$q(x) = \frac{1}{\sqrt{\pi}}\int_{-d}^{d} dx' K(|x-x'|)\{\lambda q(x')-\tfrac{1}{2}\kappa\} \tag{17}$$

where $$K(|x-x'|) = 2(1-\alpha)\int_0^\infty \frac{dc_x}{c_x}\frac{\exp(-c_x^2-2\delta/c_x)\cosh\{\lambda(x'-x)/c_x\}}{1-(1-\alpha)\exp(-2\delta/c_x)}$$
$$+\int_0^\infty \frac{dc_x}{c_x}\exp(-c_x^2-\lambda|x-x'|/c_x), \tag{18}$$

with $\lambda d = \delta$.

4. *Solution of integral equation.* To solve equation (17) we shall employ a replication property possessed by the kernel $K(|x-x'|)$. Such a technique was suggested by Case (13) and has been used by Ferziger (14) in rarefied gas flow, and Williams (15) in neutron transport theory.

We seek a solution to equation (17) in the form

$$q(x) = A + Bx^2 + C\cosh\left(\frac{x}{\bar{\nu}}\right) + \int_0^\infty \psi(\nu)\cosh\left(\frac{x\lambda}{\nu}\right)d\nu. \tag{19}$$

The first two terms on the right-hand side of (19) are suggested by the solution of the hydrodynamic equations. The remaining terms are conjectures based on the structure of $K(|x-x'|)$ and the physical requirements of the problem.

Algebraic manipulation shows that the following relations hold:

$$\int_{-d}^{d} dx' K(|x-x'|) = \frac{\sqrt{\pi}}{\lambda} - \frac{2\alpha}{\lambda}\int_0^\infty \frac{\exp(-c_x^2-\delta/c_x)\cosh(\lambda x/c_x)\,dc_x}{1-(1-\alpha)\exp(-2\delta/c_x)}, \tag{20}$$

$$\int_{-d}^{d} dx' x'^2 K(|x-x'|) = \frac{\sqrt{\pi}}{\lambda}x^2 + \frac{\sqrt{\pi}}{\lambda^3}$$
$$-\frac{2}{\lambda^3}\int_0^\infty dc_x \exp(-c_x^2-\delta/c_x)\frac{[4\delta c_x+\alpha(\delta^2-2\delta c_x+2c_x^2)]}{1-(1-\alpha)\exp(-2\delta/c_x)}\cosh(\lambda x/c_x), \tag{21}$$

$$\int_{-d}^{d} dx' K(|x-x'|)\cosh\left(\frac{x'}{\bar{\nu}}\right) = \bar{\nu}\int_{-\infty}^{\infty}\frac{dc_x\exp(-c_x^2)}{c_x+\bar{\nu}\lambda}\cosh\left(\frac{x}{\bar{\nu}}\right)$$
$$+\int_0^\infty \frac{dc_x\exp(-c_x^2-\delta/c_x)\cosh(\lambda x/c_x)}{1-(1-\alpha)\exp(-2\delta/c_x)}\left[\left(\frac{\bar{\nu}(1-\alpha)}{c_x+\bar{\nu}\lambda}+\frac{\bar{\nu}}{c_x-\bar{\nu}\lambda}\right)\exp(d/\bar{\nu})\right.$$
$$\left.-\left(\frac{\bar{\nu}(1-\alpha)}{c_x-\bar{\nu}\lambda}+\frac{\bar{\nu}}{c_x+\bar{\nu}\lambda}\right)\exp(-d/\bar{\nu})\right]. \tag{22}$$

We see therefore that a function of the form $\cosh(x/\bar{\nu})$ is 'replicated' by the integral operator in the sense that the function itself is reproduced plus a distribution of it over the range $0 \leqslant \bar{\nu} < \infty$. This behaviour confirms that the form of the solution given by (19) is the correct one.

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Inserting (19) into (17) and collecting up the coefficients of x^0, x^2, $\cosh(x/\bar{\nu})$ and $\cosh(x\lambda/\nu)$ we find that $B = \frac{1}{2}\kappa\lambda$ and $\bar{\nu}$ is given by the root of the equation

$$1 = \frac{\bar{\nu}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dc_x \exp(-c_x^2)}{c_x + \bar{\nu}},$$

which is clearly $\bar{\nu} = \infty$.

We also find that $\psi(\nu)$ is given by the following singular integral equation:

$$P\int_0^{\infty} \frac{\nu' \Phi(\nu')\, d\nu'}{\nu' - \nu} + p(\nu)\,\Phi(\nu) = f(\nu), \tag{23}$$

where

$$\left.\begin{aligned} \Phi(\nu) &\equiv e^{\delta'\nu}\{1-(1-\alpha)e^{-2\delta'\nu}\}\psi(\nu), \\ p(\nu) &= e^{\nu^2}\left\{\sqrt{\pi} + \nu\int_{-\infty}^{\infty} \frac{d\nu'\, e^{-\nu'^2}}{\nu' - \nu}\right\}, \end{aligned}\right\} \tag{24}$$

$$f(\nu) = \int_0^{\infty} \frac{\nu'\Phi(\nu')}{\nu' + \nu}\left\{\frac{1-\alpha-\exp(-2\delta/\nu')}{1-(1-\alpha)\exp(-2\delta/\nu')}\right\} d\nu'$$
$$- (1/\lambda)\,[2\alpha\lambda D - \kappa\alpha + 4\delta\nu\kappa + \alpha\kappa(\delta^2 - 2\delta\nu + 2\nu^2)], \tag{25}$$

and $D = A + C$.

The symbol P denotes that the principal value of the integral must be taken.

Equation (23) cannot be solved explicitly due to the presence of $\Phi(\nu)$ in $f(\nu)$. This term is, however, non-singular and under certain circumstances (to be discussed) will be small. If, therefore, we regard $f(\nu)$ as a known quantity, equation (23) is identical to an equation studied by Cercignani (12) in his proof of the completeness of the eigenfunctions of the Boltzmann equation in plane geometry.

Cercignani used the standard methods of Muskelishvili (17) to solve (23), and the important results of his analysis are

$$y\Phi(y) = N^+(y) - N^-(y) \tag{26}$$

where $N^{\pm}(y) = \lim_{\epsilon \to 0} N(y \pm i\epsilon)$ and

$$N(z) = \frac{1}{X(z)} \frac{1}{2\pi i} \int_0^{\infty} \frac{tX^-(t) f(t)\, dt}{(t-z)\,(p(t) - i\pi t)} \tag{27}$$

with

$$zX(z) = \exp\left[\frac{1}{2\pi i}\int_0^{\infty} \frac{dt}{t-z} \ln\left(\frac{p(t)+i\pi t}{p(t)-i\pi t}\right)\right]. \tag{28}$$

Equation (26) is the solution of (23) only if $f(t)$ satisfies the following additional condition:

$$\int_0^{\infty} \frac{tX^-(t) f(t)\, dt}{p(t) - i\pi t} = 0 \tag{29}$$

where, as before, $X^-(t) = \lim_{\epsilon \to 0} X(t - i\epsilon)$.

Equation (29) serves to determine the unknown constant D. The solution of (23) may now be written

$$\Phi(\nu) = \frac{p(\nu) f(\nu)}{p(\nu)^2 + \pi^2\nu^2} - \frac{1}{X^-(\nu)\,[p(\nu) + i\pi\nu]} P\int_0^{\infty} \frac{tX^-(t) f(t)\, dt}{(t-\nu)\,[p(t) - i\pi t]}. \tag{30}$$

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Insertion of $f(\nu)$ into (29) leads to

$$\lambda D = -\tfrac{1}{2}\kappa\delta^2 + \tfrac{1}{2}\kappa + \kappa\delta\left(\frac{2}{\alpha}-1\right)Q_1 + \kappa Q_2 - \frac{\lambda}{2\alpha}\int_0^\infty \nu\Phi(\nu)g(\nu)X(-\nu)\,d\nu, \tag{31}$$

where

$$g(\nu) = \frac{1-\alpha-e^{-2\delta/\nu}}{1-(1-\alpha)e^{-2\delta/\nu}} \tag{32}$$

and

$$Q_n = \int_0^\infty \frac{t^{n+1}X^-(t)\,dt}{p(t)-i\pi t}. \tag{33}$$

Using some relations between the X-functions given by Cercignani, equation (30) simplifies to

$$\Phi(\nu) = \frac{\kappa\sqrt{\pi}}{\lambda}\frac{e^{\nu^2}X(-\nu)}{p^2+\pi^2\nu^2}[\delta(\alpha-2)+\alpha(Q_1-\nu)] + \frac{\sqrt{\pi}}{2}\frac{e^{\nu^2}X(-\nu)}{p^2+\pi^2\nu^2}\int_0^\infty \frac{\nu'\Phi(\nu')g(\nu')X(-\nu')\,d\nu'}{\nu'+\nu}. \tag{34}$$

It is also known that

$$Q_1 = -\zeta\lambda = -\sigma \tag{35}$$

and

$$Q_2 = -\tfrac{1}{2}(\sigma^2+\tfrac{3}{2}), \tag{36}$$

where ζ is the slip coefficient (12).

We may now write down the value of $q(x)$, with x in units of λ^{-1}, as

$$\lambda q(x) = \tfrac{1}{2}\kappa(x^2-\delta^2) + \kappa(Q_2+\tfrac{1}{2}) + \kappa\delta Q_1\{(2/\alpha)-1\} + \frac{\lambda}{2\alpha}\int_0^\infty \nu\Phi(\nu)g(\nu)X(-\nu)e^{-2\delta/\nu}\,d\nu + \lambda\int_0^\infty \Phi(\nu)\frac{e^{-\delta/\nu}\cosh(x/\nu)\,d\nu}{1-(1-\alpha)\exp(-2\delta/\nu)}. \tag{37}$$

In principle the problem is now solved. $\Phi(\nu)$ is obtained from the non-singular Fredholm equation (34) and inserted into (37). Certain limiting cases are, however, worth noting.

5. *Limiting cases.* When $\alpha = 1$, i.e. perfect accommodation, and δ is large, the integral term in (34) may be assumed small. Then we have an explicit expression for $\Phi(\nu)$ which may be inserted into (37); such a procedure, if repeated, would constitute an iterative solution of (34). Although this method would converge for sufficiently small $(1-\alpha)$ and large δ, the details of the convergence have not been investigated.

We note also that, for very large δ, $q(x)$ for x not too near $\pm d$, assumes the asymptotic form characteristic of the hydrodynamic equations, viz.

$$\lambda q(x) \sim \tfrac{1}{2}\kappa(x^2-\delta^2) + \lambda q_s, \tag{38}$$

where

$$\lambda q_s = \kappa(Q_2+\tfrac{1}{2}) + \kappa\delta Q_1\left(\frac{2}{\alpha}-1\right) - \frac{(1-\alpha)\lambda}{2\alpha}\int_0^\infty d\nu X(-\nu)\Phi(\nu)\,d\nu. \tag{39}$$

q_s may therefore be used as a slip boundary condition for the Navier–Stokes equations, thus accounting approximately for surface kinetic theory effects.

As α decreases we note that the flow velocity becomes flatter and naturally larger. Indeed, for $\alpha \to 0$, we find $q(x) \to \text{constant}/\alpha$. This we expect, since for purely specular reflection the boundaries have no effect at all on the flow pattern.

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6. *Flow rate.* The average volumetric flow rate per unit thickness, which we define as

$$Q(d) = \frac{1}{2d}\int_{-d}^{d} q(x)\,dx, \tag{40}$$

is easily found to be given by

$$-\frac{\lambda}{\kappa}Q(d) = \tfrac{1}{4}(2\sigma^2+1) + \delta\sigma\left(\frac{2}{\alpha}-1\right) + \tfrac{1}{3}\delta^2 + \frac{\lambda}{2\alpha\kappa}\int_0^\infty \frac{\nu\Phi(\nu)}{1-(1-\alpha)\exp(-2\delta/\nu)} \times\left\{(1-\alpha)X(-\nu) - \frac{\alpha}{\delta} - \left(X(-\nu)-\frac{\alpha}{\delta}\right)\exp(-2\delta/\nu)\right\}d\nu. \tag{41}$$

For large δ, and α close to unity this expression reduces to

$$\begin{aligned} -(\lambda/\kappa)\,Q(d) \sim{}& \tfrac{1}{3}\delta^2 + \delta\{(2/\alpha)-1\}(\sigma - \tfrac{1}{2}(1-\alpha)\,I_2) \\ &+ \tfrac{3}{2}\sigma^2 - \tfrac{5}{4} + \tfrac{1}{2}\alpha(\tfrac{3}{2}-\sigma^2) - \tfrac{1}{2}(1-\alpha)(I_3+\sigma I_2) \\ &+ (\alpha/\delta)(\tfrac{1}{3}\sigma^3 - \tilde{I}) + O(\delta^{-2}), \end{aligned} \tag{42}$$

where

$$\tilde{I} = \frac{\sqrt{\pi}}{3}\int_0^\infty \frac{\nu^3 e^{\nu^2}\,d\nu}{p^2(\nu)+\pi^2\nu^2}$$

and

$$I_n = \sqrt{\pi}\int_0^\infty \frac{\nu^n e^{\nu^2} X^2(-\nu)\,d\nu}{p^2(\nu)+\pi^2\nu^2}.$$

For $\alpha = 1$, equation (42) reduces to

$$-(\lambda/\kappa)\,Q(d) \sim \tfrac{1}{3}\delta^2 + \sigma\delta + \sigma^2 - \tfrac{1}{2} + \frac{1}{\delta}(\tfrac{1}{3}\sigma^3 - \tilde{I}) + O(\delta^{-2}). \tag{43}$$

This is not the same as Cercignani's result (10) since there is an obvious error of sign in his equation (25).

Equation (42) has some practical value, since the asymptotic flow rate for very large δ has the form

$$-(\lambda/\kappa)\,Q(d) \sim \tfrac{1}{3}\delta^2 + \delta\{(2/\alpha)-1\}(\sigma - \tfrac{1}{2}(1-\alpha)\,I_2).$$

Thus, by plotting $-\lambda Q(d)/\kappa\delta$ versus δ, we may extrapolate to zero δ to find the slip coefficient ζ in terms of α.

7. *Summary and conclusions.* We have studied the flow of a rarefied gas between parallel plates with an arbitrary ratio of diffuse to specular reflection. An integral equation for the flow rate has been deduced and solved by employing a replication property of the kernel.

We have presented a slip boundary condition to be used with the hydrodynamic equations which includes a dependence on the accommodation coefficient and plate separation.

The main errors involved in the present analysis arise from the use of the B.G.K. model, and in particular the constant collision frequency approximation. This leads to an incorrect eigenvalue spectrum of the associated Boltzmann transport operator. A more detailed analysis of the eigenvalue spectrum (16) indicates that it consists of a number of discrete values below a limit point, given by $[V(c)/c]\min$, plus a continuous spectrum extending from this limit point to infinity. The B.G.K. model used

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in this paper preserves the discrete eigenvalues, corresponding to conservation of mass, momentum and energy, but it allows the continuous spectrum to extend from zero to infinity instead of from the limit point discussed above. One physical effect of this approximation is an over-emphasis of the importance of the non-hydrodynamic part of the solution. An investigation of these limitations is being carried out and will be reported at a later date.

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