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Convolution estimates for some degenerate curves

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1. Let $k \in \mathbf{R}$, $k \geqslant 3$. Consider the curve of class C^3 in $\mathbf{R}^3$ defined by

$$\gamma_k : t \to (t, t^2, t^k), \quad t \geqslant 0.$$

Let $\sigma_k(t) = t^{(k-3)/6}\,dt$ be the affine arclength measure on γ_k. Define the convolution operator $T\ (= T_k)$ by

$$\begin{aligned} Tf(x) &= (\sigma_k * f)(x) \\ &= \int_0^\infty f(x_1 - t, x_2 - t^2, x_3 - t^k)\, t^{(k-3)/6}\, dt. \end{aligned}$$

In [1] S. Drury proved the following $L^{\frac{3}{2}} \to L^2$ estimate for the operator T when $k \geqslant 4$:

THEOREM 1 (S. Drury). *Suppose $k \geqslant 4$. Then there exists a constant $C > 0$ such that*

$$\|Tf\|_{L^2(\mathbf{R}^3)} \leqslant C \|f\|_{L^{\frac{3}{2}}(\mathbf{R}^3)}, \tag{1}$$

for $f \in L^2(\mathbf{R}^3)$.

For $k = 3$, the measure σ_3 is the standard Lebesgue measure on the curve $t \to (t, t^2, t^3)$. In this case the curve is non-degenerate and the $L^{\frac{3}{2}} \to L^2$ estimate has been established by Oberlin [2]. We prove in this paper that the estimate holds if $k \in (3, 4)$, which was left open in [1].

2. The method of 'cut curves' was used by Oberlin to prove estimate (1) for non-degenerate curves. It was used again by Drury for degenerate curves. Using this technique, the problem is reduced to an $L^{\frac{3}{2}}(\mathbf{R}^2) \to L^3(\mathbf{R}^2)$ estimate for an operator S which is given by

$$Sg(x_1, x_2) = \int_1^\infty g(x_1 - t, x_2 - \varphi(t))\, (\varphi''(t))^{\frac{1}{3}}\, dt, \tag{2}$$

with $\varphi(t) = (t+1)^k - (t-1)^k$. To prove the desired $(\frac{3}{2}, 3)$ estimate for the operator S, one embeds S in an analytic family of operators $\{S_z\}$ where

$$S_z f = f * \mu_z \tag{3}$$

with

$$\mu_z(x_1, x_2) = \frac{\pi^{(z-1)/2}}{\Gamma(z/2)} \chi_{[a,b]}(x_1) \left(\frac{|x_2 - \varphi(x_1)|}{(\varphi''(x_1))^{\frac{1}{3}}} \right)^{z-1} dx_1\, dx_2. \tag{4}$$

Since $S \sim S_0$, by Stein's interpolation theorem [3], the $L^{\frac{3}{2}} \to L^3$ boundedness of S would follow if one could show that

$$\|S_{1+iy}\|_{L^1, L^\infty} \leqslant C(y), \tag{5}$$

$$\|S_{-\frac{1}{2}+iy}\|_{L^2, L^2} \leqslant C(y), \tag{6}$$

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where the constants $C(y)$ are independent of a, b and have at most exponential growth in $|y|$.

Equation (5) is true for any φ and the proof is straightforward. By Plancherel's theorem, (6) would follow if one could prove the following estimate on certain oscillatory integrals:

$$|I(\xi,\lambda)| \leqslant C(1+|s|)^{\frac{1}{2}}, \tag{7}$$

where

$$I(\xi,\lambda) = |\lambda|^{\frac{1}{2}} \int_a^b e^{i(\lambda\varphi(x)+\xi x)} (\varphi''(x))^{\frac{1}{2}+is}\, dx. \tag{8}$$

For more details, see [1].

It was proved in [1] that (7) holds if $k \geqslant 4$, but the argument used there breaks down if $k < 4$. Here we shall prove that (7) holds for all $k \in (3, \infty)$. Namely, we have

LEMMA 1. *Suppose $k > 3$. Then, there exists a constant $C = C(k)$ which is independent of a, b, s, λ and ξ such that*

$$|I(\xi,\lambda)| \leqslant C(1+|s|)^{\frac{1}{2}}, \tag{9}$$

for $s, \lambda, \xi \in \mathbf{R}$, and $a, b > 1$.

3. Proof of Lemma 1.

LEMMA 2. *Suppose $l > 0$. Then there exist $C_l, C_l' > 0$ such that*

$$C_l' x^{l-1} \leqslant (x+1)^l - (x-1)^l \leqslant C_l x^{l-1},$$

for all $x > 1$.

LEMMA 3. *Suppose $1 < t < x$, $l \geqslant 2$. There exists $C_l > 0$ which is independent of x and t such that*

$$[(x+1)^l - (x-1)^l] - [(t+1)^l - (t-1)^l] \geqslant C_l x^{l-2}(x-t).$$

The proof of the above results is elementary. We omit the details.

LEMMA 4 (van der Corput, [4]). *Suppose ϕ and ψ are smooth on $[a, b]$ and ϕ is real-valued. If $|\phi'(x)| \geqslant \lambda$, and ϕ' is monotone on $[a, b]$, then*

$$\left| \int_a^b e^{i\phi(x)} \psi(x)\, dx \right| \leqslant 4\lambda^{-1} \left(|\psi(b)| + \int_a^b |\psi'(x)|\, dx \right).$$

We now begin our proof of Lemma 1. It suffices to prove (8) for $\lambda > 0$, since $I(\xi, -\lambda) = \overline{I(-\xi, \lambda)}$. Let $\gamma = -\xi/\lambda$, $Q = [a, b]$. For $\varphi'(a) < \gamma < \varphi'(b)$, let $\tau = (\varphi')^{-1}(\gamma)$; for $\gamma \leqslant \varphi'(a)$, let $\tau = a$; for $\gamma \geqslant \varphi'(b)$, let $\tau = b$. Choose $\delta, \delta' > 0$ such that

$$\sqrt{\lambda}\delta(\tau+\delta)^{(k-3)/2} = \sqrt{\lambda}\delta'\tau^{(k-3)/2} = (1+|s|)^{\frac{1}{2}}.$$

Let $\eta = \max\{a, \tau-\delta'\}$ and $d = \min\{b, \tau+\delta\}$. We have

$$\begin{aligned} I(\xi,\lambda) &= \sqrt{\lambda} \int_a^\eta e^{i(\lambda\varphi(x)+\xi x)} (\varphi''(x))^{\frac{1}{2}+is}\, dx + \sqrt{\lambda} \int_\eta^\tau e^{i(\lambda\varphi(x)+\xi x)} (\varphi''(x))^{\frac{1}{2}+is}\, dx \\ &\quad + \sqrt{\lambda} \int_\tau^d e^{i(\lambda\varphi(x)+\xi x)} (\varphi''(x))^{\frac{1}{2}+is}\, dx + \sqrt{\lambda} \int_d^b e^{i(\lambda\varphi(x)+\xi x)} (\varphi''(x))^{\frac{1}{2}+is}\, dx \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

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By Lemma 2 we have

$$|I_2| \leqslant C\sqrt{\lambda}(\tau-\eta)\,\tau^{(k-3)/2}$$
$$\leqslant C\sqrt{\lambda}\delta'\tau^{(k-3)/2} = C(1+|s|)^{\frac{1}{2}} \tag{10}$$

and

$$|I_3| \leqslant C\sqrt{\lambda}(d-\tau)\,d^{(k-3)/2}$$
$$\leqslant C\sqrt{\lambda}\delta(\tau+\delta)^{(k-3)/2} = C(1+|s|)^{\frac{1}{2}}. \tag{11}$$

To treat I_4, we assume that $d = \tau+\delta$ (otherwise $d = b$ and $I_4 = 0$). Let $x\in[d,b]$, by Lemma 3 we have

$$\frac{d}{dx}(\lambda\varphi(x)+\xi x) \geqslant \lambda(\varphi'(x)-\varphi'(\tau)) \geqslant C\lambda\delta x^{k-3}.$$

Let $m \geqslant 0$. By van der Corput's lemma and Lemma 2 we have

$$\left|\sqrt{\lambda}\int_{[2^m d,\,2^{m+1}d]\cap Q} e^{i(\lambda\varphi(x)+\xi x)}\,(\varphi''(x))^{\frac{1}{2}+is}\,dx\right|$$
$$\leqslant \sqrt{\lambda}(\lambda\delta(2^m d)^{k-3})^{-1}\left[(2^m d)^{(k-3)/2} + (1+|s|)\int_{2^m d}^{2^{m+1}d} x^{-(k-3)/2}\,x^{k-4}\,dx\right]$$
$$\leqslant (1+|s|)\,\lambda^{-\frac{1}{2}}\delta^{-1}\,d^{-(k-3)/2}\,2^{-(k-3)\,m/2}$$
$$= C2^{-(k-3)\,m/2}(1+|s|)^{\frac{1}{2}}.$$

Therefore

$$|I_4| \leqslant C\Big(\sum_{m\geqslant 0} 2^{-(k-3)\,m/2}\Big)(1+|s|)^{\frac{1}{2}} = C(1+|s|)^{\frac{1}{2}}. \tag{12}$$

For I_1 the argument is even simpler. We assume that $\eta = \tau-\delta'$ (otherwise $\eta = a$ and $I_1 = 0$). Let $x\in[a,\eta]$; then

$$-\frac{d}{dx}(\lambda\varphi(x)+\xi x) \geqslant \lambda(\varphi'(\tau)-\varphi'(x)) \geqslant \lambda\delta'\tau^{k-3}.$$

Therefore we have

$$|I_1| \leqslant C\sqrt{\lambda}(\lambda\delta'\tau^{k-3})^{-1}\left[\tau^{(k-3)/2} + (1+|s|)\int_a^\tau x^{(k-3)/2-1}\,dx\right]$$
$$= C\lambda^{-\frac{1}{2}}\delta'^{-1}\tau^{-(k-3)/2}(1+|s|) = C(1+|s|)^{\frac{1}{2}}. \tag{13}$$

Combining (10)–(13), the lemma is proved.

4. By the argument preceding Lemma 1, we have the following result.

THEOREM 2. *Let S be given as in* (2), *$k \geqslant 3$. Then*

$$\|Sg\|_{L^3(\mathbf{R}^2)} \leqslant C\|g\|_{L^{\frac{3}{2}}(\mathbf{R}^2)}.$$

We are now able to show that T is bounded from $L^{\frac{3}{2}}(\mathbf{R}^3)$ to $L^2(\mathbf{R}^3)$ when $k \geqslant 3$.

THEOREM 3. *Suppose $k \geqslant 3$. Then there exists a constant $C > 0$ such that*

$$\|Tf\|_{L^2(\mathbf{R}^3)} \leqslant C\|f\|_{L^{\frac{3}{2}}(\mathbf{R}^3)}$$

for $f\in L^2(\mathbf{R}^3)$.

Theorem 3 extends Theorem 1 from $k \geqslant 4$ to $k \geqslant 3$. As pointed out in [1], there is only one step in Drury's proof of Theorem 1 that uses the assumption $k \geqslant 4$. The step

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in question uses the $L^{\frac{3}{2}} \to L^3$ boundedness of S, which was known previously for $k \geq 4$ only. Therefore, a proof of Theorem 3 can be obtained by combining Drury's argument with Theorem 2.

5. As an application of Theorem 1, Drury proved a restriction theorem for the Fourier transform on the curve γ_k when $k \geq 4$ ([**1**], p. 95). Theorem 3 enables us to extend Drury's restriction theorem to all $k \in [3, \infty)$.

THEOREM 4. *Suppose* $k \geq 3$, $1 \leq p < \frac{36}{31}$. *Let* $6/p + 1/q = 6$. *Then there exists* $C > 0$ *such that*

$$\|\hat{f}\|_{L^q(\gamma_k, d\sigma_k)} \leq C \|f\|_{L^p(\mathbf{R}^3, dx)},$$

for $f \in \mathcal{S}(\mathbf{R}^3)$.

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