TWO THEOREMS ON ENGEL GROUPS

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1. *Introduction.* Let G be a group and let us write $[x, y] = x^{-1}y^{-1}xy$. We shall say that G is an *Engel group*, or that it *satisfies the Engel condition*, if to any pair of elements x, y in G there can be assigned an integer k (which is allowed to depend on x, y) such that

$$[\ldots[[x, \underbrace{y], y], \ldots, y}_{k}] = 1.$$

If, in particular, there exists a fixed integer n such that

$$[\ldots[[x, \underbrace{y], y], \ldots, y}_{n}] = 1$$

for all pairs x, y in G, then G will be said to satisfy the *n-th Engel condition.*

Groups satisfying the second Engel condition were first investigated by Burnside (1). More recently, Levi (5) proved that such groups must be nilpotent of class at most 3, and that there are some of class equal to 3 (cf. Meier-Wunderli (6)).

The only other result on Engel groups, known so far, is that *every finite Engel group is nilpotent.* This theorem was discovered independently by Zorn (8) and Zassenhaus ((7), p. 5).

In this note we shall prove the following result:

THEOREM 1. *Every finitely generated soluble Engel group is nilpotent.*

As a corollary we shall deduce

THEOREM 2. *Every term of the lower central series of a finitely generated Engel group is itself finitely generated.*

The problem of deciding which Engel groups are nilpotent is, of course, suggested by the similar one in the theory of Lie rings. If L is a Lie ring (which may possibly admit as scalars the elements of some commutative ring) and the product of x, y in L is denoted by $[x, y]$, then we shall say that L satisfies the Engel condition if

$$[\ldots[[x, \underbrace{y], y], \ldots, y}_{k}] = 0$$

for each pair x, y in L and some integer $k = k(x, y)$. The question: does the Engel condition imply nilpotence? is still as open for Lie rings in general as it is for groups. The only Lie rings for which an affirmative answer is known are those which satisfy the maximal condition for subrings and in which the integer k can be taken independent of x (Zorn (9)). To these rings we can now add the finitely generated soluble Lie rings, since it turns out that our proof of Theorem 1 (and, as a matter of fact, that of Theorem 2) holds also in a Lie ring. This is explained in greater detail in § 4.

2. *Proof of Theorem* 1. Our proof centres round the basic sequences which arise in Philip Hall's commutator collecting process (3), and we begin with their definition.

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Let X be a finite subset of a given group. We consider a sequence

$$b_1, \quad b_2, \quad b_3, \quad \ldots,$$

in which (i) each b_k is either an element of X or of the form $[b_j, b_i]$. where $i, j < k$, and (ii) no two elements of the sequence are formally identical when written out as commutators in the elements of X (though, of course, they may be equal as elements of the given group). We define $k' = 0$, $k^* = 1$ if $b_k \in X$, and $k' = i$, $k^* = i^* + j^*$ if $b_k = [b_j, b_i]$. The given sequence is called a *basic sequence* (in the set X) if also (iii) $k < l$ whenever $k^* < l^*$, (iv) each element of X occurs once and only once, and (v) the commutator $[b_j, b_i]$ occurs if and only if $j' \leqslant i < j$. Such a basic sequence can, of course, always be constructed (cf. Marshall Hall (2)).

The importance of the basic sequences for our present purpose lies in the following result of Philip Hall (4):

THEOREM. *If G is a finitely generated group and $b_1, b_2, b_3, \ldots$ is a basic sequence in some finite set of generators of G, then G is nilpotent if and only if all but a finite number of the b's are equal to* 1.

To prove Theorem 1 we have therefore only to show that in an Engel group G, generated by a finite set X, all but a finite number of the elements of a basic sequence in X lie in a given term of the derived series of G. This is proved in the corollary to the following

LEMMA. *Let $b_1, b_2, b_3, \ldots$ be a basic sequence in d elements of an Engel group. If s is any given positive integer, then all but a finite number of the b_k's either are equal to* 1, *or have $k' \geqslant s$.*

Proof. When $s = 1$, all but the first d elements of our sequence are as required. We may therefore use induction on s. Suppose the lemma proved when $s = \sigma$. Then all but a finite number of the b's are equal to 1, or to b_k with $k' \geqslant \sigma$. We have to show that there are only a finite number of non-vanishing b_k's with $k' = \sigma$. Now if $k' = \sigma$, then, by (v) above, b_k can be written in the form

$$b_k = [\ldots[[b_l, b_\sigma], b_\sigma], \ldots, b_\sigma], \tag{1}$$

where $l' < \sigma$. By the induction hypothesis there are only a finite number of b_l's not equal to 1, and, by the Engel condition, for each b_l, there are only a finite number of b's of the form (1).

COROLLARY. *Let G be the subgroup generated by $b_1, \ldots, b_d$ (i.e. by the d given elements of the lemma). For any given positive integer r, all but a finite number of the b's in our sequence lie in $G^{(r)}$, the r-th term of the derived series of G.*

Proof. We may use induction on r, since all b's from b_{d+1} onwards lie in G'. Suppose the result true for $r = \rho$. Then there exists an integer s such that all b's from b_s onwards lie in $G^{(\rho)}$. By the lemma we can find an integer t such that for all $k \geqslant t$ we have $b_k = 1$ or $k' \geqslant s$. Hence all b's from b_t onwards lie in $G^{(\rho+1)}$.

3. *Proof of Theorem* 2. Let G be any given Engel group, which can be generated by d of its elements. The first step is to show that the derived group G' is finitely generated.

If F is the free group on d generators, then $G \cong F/N$, where N is a suitable normal subgroup of F. Let us write $[N, F]$ for the subgroup of F generated by all the com-

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mutators $[a, b]$ with $a \in N, b \in F$. Then $E = F/[N, F]$ is an Engel group with d generators, say $x_1, \ldots, x_d$, and G is a homomorphic image of E. It will be sufficient to prove E' finitely generated. The reason for transferring our attention from G to E is that we require our group modulo its derived group to be *free* Abelian.

Write $y_i = x_i$ $(i = 1, \ldots, d)$, define $y_{2d-i+1} = x_i^{-1}$ $(i = 1, \ldots, d)$, and let

$$y_1, \cdots, y_{2d}, y_{2d+1}, \cdots$$

be a basic sequence in $y_1, \ldots, y_{2d}$. Any element u of E can be written in the form

$$u = x_{i_1}^{\epsilon_1} \ldots x_{i_r}^{\epsilon_r},$$

where each $\epsilon_\alpha = \pm 1$. Consequently

$$u = y_{j_1} \cdots y_{j_r},$$

and therefore, by the commutator collecting process (3),

$$u = y_1^{\lambda_1} \ldots y_{2d}^{\lambda_{2d}} c,$$

where c is a product of y_k's for which $0 < k' \leqslant 2d$. Since E/E' is the free Abelian group on $x_1E', \ldots, x_dE'$, $u \in E'$ if and only if $\lambda_i = \lambda_{2d-i+1}$ for $i = 1, \ldots, d$; i.e. $u \in E'$ if and only if $u = c$, and so E' is generated by those y_k's which have $0 < k' \leqslant 2d$. By the lemma, there are only a finite number of these*.

Thus E' is finitely generated. Hence G', and therefore *every* term of the derived series of G, is finitely generated.

Take now any term G_r, of the lower central series of G. Then G_r contains a member, $G^{(s)}$, say, of the derived series of G, and, by Theorem 1, $G^{(s)}$ contains some G_t. Since G/G_t is a finitely generated nilpotent group, so is G_r/G_t, and hence also $G_r/G^{(s)}$. But we have just proved that $G^{(s)}$ is finitely generated. Therefore G_r also has a finite set of generators.

4. *Lie rings.* Let L be a Lie ring in which the product of x, y is denoted by $[x, y]$.

THEOREM 1′. *Every finitely generated soluble Lie ring satisfying the Engel condition is nilpotent.*

We may define basic sequences in a finite subset of L exactly as in a group. Marshall Hall (2) has shown that, if a basic sequence in a finite subset X contains only a finite number of non-zero elements, then the Lie subring generated by X is nilpotent. The proof of Theorem 1′ can therefore proceed in the same way as that of Theorem 1. Since the lemma remains true when reinterpreted for Lie rings, Theorem 1′ is established.

THEOREM 2′. *Every term of the lower central series of a finitely generated Lie ring, which satisfies the Engel condition, is itself finitely generated.*

Let L be a Lie ring generated by d elements and let $b_1, b_2, \ldots$ be a basic sequence in these elements. We know (2) that the derived ring L' is spanned by $b_{d+1}, b_{d+2}, \ldots$, and hence L' is generated, as a Lie ring, by all those b_k's which have $0 < k' \leqslant d$. By the lemma there are only a finite number of such b's. Thus L' is finitely generated, and the proof may be completed in the same way as that of Theorem 2.

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* I am indebted to Mr P. Hall for this argument, which is considerably shorter than my original one.

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