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UNIVALENT FUNCTIONS AND RADIAL GROWTH

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Abstract. We address a question raised by Anderson, Hayman and Pommerenke relating to a classical result on univalent functions f in the unit disc due to Spencer, and involving the size of the set of $\theta \in [-\pi, \pi]$ for which we have $\log|f(re^{i\theta})| \neq o(\log(1/(1-r)))$ as $r \to 1$. An answer is given in terms of a certain generalized capacity, and also in terms of Hausdorff measure. Further results regarding the radial growth of univalent functions are also established, and some examples are constructed which relate to the sharpness of these results.

§1. *Introduction.* Let $\mathcal{S}$ denote the class of functions

$$f(z) = z + a_2 z^2 + \cdots$$

analytic and univalent in the unit disc $\Delta = \{z : |z| < 1\}$. A classical distortion theorem [**6**, p. 4] gives the sharp estimate

$$|f(z)| \leqslant \frac{r}{(1-r)^2}, \qquad |z| = r < 1,$$

and so, if $M(r, f) = \max_{|z|\leqslant r} |f(z)|$, then $\log M(r, f) = O(\log 1/(1-r))$ as $r \to 1$ for such functions. It has been shown by Spencer ([**10**], [**6**, p. 42]) that

$$\liminf_{r\to 1} \frac{\log|f(re^{i\theta})|}{\log(1/(1-r))} > 0 \tag{1.1}$$

for at most countably many values of θ, and N. Makarov raised the question whether the set of θ for which (1.1) holds with "lim inf" replaced by "lim sup" is also countable for functions in $\mathcal{S}$. Anderson, Hayman and Pommerenke [**1**] answered this question in the affirmative for starlike and close-to-convex functions, but gave a strongly negative answer for the full class $\mathcal{S}$ by proving that there exists a univalent function f for which the complement of the set

$$E_f = \left\{\theta \in [-\pi, \pi] : \limsup_{r\to 1} \frac{\log|f(re^{i\theta})|}{\log(1/(1-r))} > 0\right\} \tag{1.2}$$

is of Baire Category I, so that E_f is uncountably dense in every interval. An open question is raised in [**1**] regarding the size of the set E_f and whether it can be measured in terms of some generalized capacity, or Hausdorff measure.

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We provide an answer to this question here, and also establish some further results relating to the radial growth of univalent functions.

To describe our results we need the following classical notion of *capacity* ([4], [12, Vol. II, p. 194]). Let K be a *kernel*, that is, K is a non-negative, even and integrable function on $(-\pi, \pi)$, which is decreasing and unbounded on $(0, \pi)$ and is extended by periodicity to $\mathbf{R}$. A Borel set $E \subset [-\pi, \pi]$ is said to have positive K-capacity if there exists a positive measure μ of total mass 1 and supported on E for which

$$\sup_{\theta \in [-\pi, \pi]} \int_{-\pi}^{\pi} K(\theta - t)\, d\mu(t) < \infty.$$

Otherwise E is said to have zero K-capacity. We will be concerned here with the capacities connected with the particular kernels

$$K_\alpha(t) = \left(\log \frac{2}{|\sin(t/2)|} \right)^\alpha, \quad t \in \mathbf{R}, \quad \alpha > 0,$$

$$k_\beta(t) = \left(\log\log \frac{3}{|\sin(t/2)|} \right)^\beta, \quad t \in \mathbf{R}, \quad \beta > 0,$$

and the product kernel $\mathcal{K}_{\alpha,\beta} = K_\alpha k_\beta, \alpha > 0, \beta > 0$. We also write $\mathcal{K}_{0,\beta} = k_\beta$ for $\beta > 0$. When $K = K_\alpha, \alpha > 0$, we call the associated capacity "$\log_\alpha$-capacity", and note that when $\alpha = 1$ we obtain the classical logarithmic capacity. The capacity associated with the kernel k_β is called "$\log\log_\beta$-capacity", and we denote by "$\log_\alpha \log\log_\beta$-capacity" the capacity associated with $\mathcal{K}_{\alpha,\beta}$ when $\alpha > 0$. Observe that if a set E has $\log\log_\beta$-capacity zero for any $\beta > 0$, then it also has zero logarithmic capacity.

Before stating our first theorem we note that a familiar result of Beurling ([2], [8, p. 341]) for univalent functions implies that

$$f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$

exists as a finite limit outside a set of θ of logarithmic capacity zero, so E_f has logarithmic capacity zero for every $f \in \mathcal{S}$. On the other hand, it has been shown by Pommerenke [9, p. 211] that, given an F_σ set $E_0 \subset [-\pi, \pi]$ of zero logarithmic capacity, there exists a univalent function f such that $|f(re^{i\theta})| \to \infty$ as $r \to 1$ for every $\theta \in E_0$.

We prove the following result.

THEOREM 1.

(a) *If $f \in \mathcal{S}$, and E_f is defined as in (1.2), then E_f is a set of zero* $\log\log_\beta$*-capacity for every $\beta > 1$.*

(b) *If $f \in \mathcal{S}$ and $\log M(r, f) = O((\log(1/(1-r)))^\alpha), r \to 1$, where $\alpha \in (0, 1)$, then the set of points θ in $[-\pi, \pi]$ at which*

$$\limsup_{r \to 1} \frac{\log|f(re^{i\theta})|}{(\log(1/(1-r)))^\alpha} > 0 \tag{1.3}$$

has zero $\log_{1-\alpha}\log\log_\beta$*-capacity for every $\beta > 1$.*

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Remarks. (1) We can also estimate the size of the sets in the theorem in terms of Hausdorff measure. Let h be a continuous increasing function in $[0, 1)$ with $h(0) = 0$. The Hausdorff measure M_h of a set $E \subset [-\pi, \pi]$ with respect to h is then given by

$$M_h(E) = \liminf_{\varepsilon \to 0} \sum_{m=1}^{\infty} h(|I_m|), \quad E \subset \bigcup_{m=1}^{\infty} I_m, \quad |I_m| < \varepsilon,$$

where (I_m) is a countable collection of subintervals of $[-\pi, \pi]$ and $|I_m|$ denotes the length of I_m. It follows from a theorem of Carleson [4, Theorem I, § IV] that if E is a set of $\log \log_\beta$-capacity zero for every $\beta > 1$, then E has Hausdorff measure zero for every measure function h for which

$$\int_0^1 \left(\log \log \frac{3}{\rho} \right)^\beta dh(\rho) < \infty,$$

from which we deduce, by some straightforward calculations, that $M_h(E) = 0$ when $h(\rho) = (\log \log[3/\rho])^{-(1+\delta)}$ and $\delta > 0$. Hence, by part (a) of the theorem above, $M_h(E_f) = 0$ for such h. Analogous results hold for part (b).

(2) The size of the set E_f when $f \in A^{-\infty}$, the Korenblum class of analytic functions f in U for which $|f(z)| \leqslant C/(1 - |z|)^k, z \in \Delta$, where C and k are constants, has recently been considered in [3] (see also [7]). The class $A^{-\infty}$ is significantly larger than its subclass $\mathcal{S}$ and this is reflected in the estimates for the size of E_f, $f \in A^{-\infty}$, obtained in [3].

(3) Some questions arise relating to the sharpness of the results in Theorem 1—whether, for instance, part (a) is true for some $\beta \in (0, 1]$, and whether the conclusions in part (b) hold for *all* functions in $\mathcal{S}$. These questions are open. With regard to Theorem 1(b), we note that it has been shown in [11] that, if f is close-to-convex, then the set of points θ for which (3.1) holds has zero $\log_{1-\alpha}$-capacity, and, further, that this result is sharp in the sense that, given any $\gamma \in (\alpha, 1)$, there is a set G of positive $\log_{1-\gamma}$-capacity and a starlike univalent function f such that (3.1) holds for all $\theta \in G$. We remark that no estimate is given of the rate of growth of $M(r, f)$ for the functions constructed by the author in [11] (or for the starlike examples of Pommerenke [9] referenced above). We address this question here, and strengthen the result obtained in [11] by proving the following theorem.

THEOREM 2. *Suppose that* $\alpha \in (0, 1)$ *and that* $\alpha < \gamma < 1$. *Then there is a set* $\mathcal{F} = \mathcal{F}(\alpha, \gamma) \subset [-\pi, \pi]$ *of positive* $\log_{1-\gamma}$*-capacity and a starlike function* h *for which*

$$\log M(r, h) = O\left(\left(\log \frac{1}{1 - r} \right)^\alpha \right), \quad r \to 1, \tag{1.4}$$

and such that

$$\log |h(re^{i\theta})| \neq o\left(\left(\log \frac{1}{1 - r} \right)^\alpha \right), \quad r \to 1, \tag{1.5}$$

for each $\theta \in \mathcal{F}$.

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Hence we cannot replace "(zero $\log_{1-\alpha}\log\log_{\beta}$-capacity)" in Theorem 1(b) by "(zero $\log_{1-\gamma}$ capacity)" for any $\gamma \in (\alpha, 1)$. Whether a stronger statement can be made for functions in $\mathcal{S}$ is an open question.

§2. *Some lemmas for Theorem 1.* We collect together in this section some results that we shall need for the proof of Theorem 1.

LEMMA 1. *Suppose that* $f \in \mathcal{S}$ *and set*

$$g(z) = \log \frac{f(z)}{z} = \sum_{n=1}^{\infty} c_n z^n, \quad z \in \Delta.$$

Suppose also that $\log M(r, f) = O((\log(1/(1-r)))^{\alpha}), r \to 1$, *where* $\alpha \in (0, 1]$. *Then*

$$\sum_{k=1}^{n} k|c_k|^2 = O((\log n)^{\alpha}), \quad n \to \infty. \tag{2.1}$$

(A well-known result due to Milin gives a sharp form of (2.1) for the case $\alpha = 1$; see [5, p. 151].)

Proof of Lemma 1. We note first that, for $f \in \mathcal{S}$,

$$I(r) = \iint_{\Delta(0,r)} \left| \frac{zf'(z)}{f(z)} \right|^2 dx\, dy = O(\log M(r, f)), \quad r \to 1, \tag{2.2}$$

where $\Delta(0, r) = \{z : |z| \leqslant r\}$ and $z = x + \mathrm{i}y$. This result follows from an obvious adaptation of the argument used to prove [6, Inequality (3.19), p. 74]. For the sake of completeness we show the details here, and begin by setting

$$E_0 = \{z : |f(z)| \leqslant 1\}, \qquad E_n = \{z : 2^{n-1} < |f(z)| \leqslant 2^n\} \quad (n \geqslant 1).$$

Since f is univalent, and $|f(z)| \leqslant 2^n$ for $z \in E_n$, we have

$$I_n = \iint_{E_n} |f'(z)|^2\, dx\, dy \leqslant \pi 2^{2n},$$

and so, for $n \geqslant 1$,

$$J_n = \iint_{E_n} \left| \frac{zf'(z)}{f(z)} \right|^2 dx\, dy \leqslant 2^{-2(n-1)} I_n \leqslant 4\pi.$$

For a fixed $r \in (0, 1)$, and $n \geqslant 1$, E_n does not intersect $\Delta(0, r)$ if

$$2^{n-1} > M(r, f), \quad \text{i.e. if } n > 1 + (\log M(r, f))/\log 2 = N(r),$$

and it follows that

$$I(r) \leqslant J_0 + \sum_{n=1}^{[N(r)]+1} J_n \leqslant J_0 + 4\pi[N(r) + 2].$$

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This gives (2.2). Next, $zg'(z) = zf'(z)/f(z) - 1$, and so, by (2.2), and the fact that $\log M(r, f) = O((\log(1/(1-r)))^{\alpha})$, we have

$$\pi \sum_{k=1}^{\infty} k|c_k|^2 r^{2k} \leqslant 2 \iint_{\Delta(0,r)} |zg'(z)|^2 \, dx\, dy = O\left(\left(\log \frac{1}{1-r}\right)^{\alpha}\right), \quad r \to 1.$$

The estimate (2.1) follows easily from this on noting that, with $r = 1 - 1/n$,

$$\sum_{k=1}^{n} k|c_k|^2 r^{2k} \geqslant r^{2n} \sum_{k=1}^{n} k|c_k|^2 = (\mathrm{e}^{-2} + o(1)) \sum_{k=1}^{n} k|c_k|^2,$$

and the lemma is proved. □

LEMMA 2. *If (2.1) holds, then*

$$\sum_{n=1}^{\infty} \frac{n|c_n|^2}{(\log 2n)^{\alpha} (\log\log 3n)^{\beta}} < \infty \tag{2.3}$$

for every $\beta > 1$.

Proof of Lemma 2. We prove the lemma by showing that if ψ is positive and increasing on $[1, \infty)$, and (2.1) holds, then

$$\sum_{n=1}^{\infty} \frac{1}{n(\log 2n)^{1-\alpha} \psi(n)} < \infty \Rightarrow \sum_{n=1}^{\infty} \frac{n|c_n|^2}{\psi(n)} < \infty, \tag{2.4}$$

and noting that (2.3) follows on taking $\psi(n) = (\log 2n)^{\alpha} (\log\log 3n)^{\beta}$ with $\beta > 1$. To prove (2.4) we note that, by (2.1) and a double use of partial summation, with $n \geqslant 2$,

$$\begin{aligned}
\sum_{k=1}^{n} \frac{k|c_k|^2}{\psi(k)} &= \sum_{k=1}^{n-1} \left(\frac{1}{\psi(k)} - \frac{1}{\psi(k+1)}\right) \sum_{s=1}^{k} s|c_s|^2 + \psi(n)^{-1} \sum_{s=1}^{n} s|c_s|^2 \\
&\leqslant C \left\{ \sum_{k=1}^{n-1} \left(\frac{1}{\psi(k)} - \frac{1}{\psi(k+1)}\right) (\log 2k)^{\alpha} + \psi(n)^{-1} (\log 2n)^{\alpha}) \right\}, \\
&\leqslant C \left\{ \sum_{k=2}^{n} \frac{[(\log 2k)^{\alpha} - (\log 2(k-1))^{\alpha}]}{\psi(k)} + \frac{(\log 2)^{\alpha}}{\psi(1)} \right\} \\
&\leqslant C \left\{ 2\alpha \sum_{k=2}^{n} \frac{1}{k(\log k)^{1-\alpha} \psi(k)} + \frac{(\log 2)^{\alpha}}{\psi(1)} \right\},
\end{aligned}$$

where $C = C(\alpha)$. Implication (2.4) follows and the proof of Lemma 2 is complete. □

Lemma 2 leads to the following representation formula for g.

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LEMMA 3. *Under the hypotheses of Lemma 1 we have*

$$g(z) = \int_{-\pi}^{\pi} \mathcal{H}_{\alpha,\beta}(z\mathrm{e}^{-\mathrm{i}t})F(t)\,dt, \quad z \in \Delta, \tag{2.5}$$

where $\alpha \in (0, 1]$, $\beta > 1$,

$$\mathcal{H}_{\alpha,\beta}(z) = (1-z)^{-1/2}\left(\log\frac{2}{1-z}\right)^{\alpha/2}\left(\log\log\frac{2e}{1-z}\right)^{\beta/2}, \quad (\alpha, \beta \in \mathbf{R}),$$

and the function F *(which depends on* α *and* β*) belongs to* $L^2(-\pi, \pi)$.

Proof of Lemma 3. We note first that if $\mathcal{H}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} A_n^{\alpha,\beta} z^n$, $z \in \Delta$, then

$$A_n^{\alpha,\beta} = (c + o(1))\, n^{-1/2}(\log 2n)^{\alpha/2}(\log\log 3n)^{\beta/2}, \quad n \to \infty, \tag{2.6}$$

for $\alpha \in (0, 1]$, $\beta \geqslant 0$, and where $c \neq 0$ is a constant. The special case $\beta = 0$ of this asymptotic estimate is established in [ZI, pp. 192–3] (where we are writing ZI for [**12**, Vol. I]) and the argument given there can be extended to yield (2.6). To show this, we first note that

$$L_2(\theta) = \log\log\frac{2e}{(1-\mathrm{e}^{\mathrm{i}\theta})} = U + \mathrm{i}V,$$

where, for $0 < \theta < \pi$,

$$U = \frac{1}{2}\log\left[\left(\log\frac{e}{\sin\theta/2}\right)^2 + \frac{(\pi-\theta)^2}{4}\right], \qquad V = \arctan\frac{\pi-\theta}{2\log(e/(\sin\theta/2))}.$$

Next, it is not difficult to establish, by some routine computations, that the function $|L_2(\theta)|^{\beta/2} = (U^2 + V^2)^{\beta/4}$ has, for every $\varepsilon > 0$, bounded variation in (ε, π), is *slowly varying* [ZI, p. 186] as $\theta \to 0^+$, and that $\arg L_2(\theta) = \arctan V/U$ has bounded variation on $[0, \pi]$. An examination of the argument used in [ZI] to obtain the case $\beta = 0$ of (2.6) shows that if these observations are added to the argument there, and [ZI, Theorem 2.24, p. 190] is applied to the Fourier series $\sum A_n^{\alpha,\beta}\cos n\theta$ of $\operatorname{Re}\mathcal{H}_{\alpha,\beta}(\mathrm{e}^{\mathrm{i}\theta}) = |\mathcal{H}_{\alpha,\beta}(\mathrm{e}^{\mathrm{i}\theta})|\arg\mathcal{H}_{\alpha,\beta}(\mathrm{e}^{\mathrm{i}\theta})$ in $(0, \pi)$, then (2.6) follows. (Note the remark at the top of page 194 in [ZI].)

Finally, by (2.3), $\sum(A_n^{\alpha,\beta})^{-2}|c_n|^2 < \infty$ when $\alpha \in (0, 1]$ and $\beta > 1$, and so, by the Riesz–Fischer theorem, there exists a function $F \in L^2(-\pi, \pi)$ having $((2\pi\, A_n^{\alpha,\beta})^{-1}c_n)_1^{\infty}$ for its Fourier coefficients. Then

$$\int_{-\pi}^{\pi} \mathcal{H}_{\alpha,\beta}(z\mathrm{e}^{-\mathrm{i}t})F(t)\,dt = \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} F(t)\mathrm{e}^{-\mathrm{i}nt}\,dt\right)A_n^{\alpha,\beta}z^n = \sum_{n=1}^{\infty} c_n z^n = g(z),$$

which is (2.5). This proves Lemma 3. □

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Our next lemma is a special case of [**11**, Lemma 3.1]. (In the referenced lemma take the increasing function to be $\|F\|_2^{-2}\int_{-\pi}^{x}|F(t)|^2$ for $x\in\mathbf{R}$, where $\|F\|_2$ is the L^2 norm of F on $[-\pi,\pi]$.)

LEMMA 4. *Suppose that K is a kernel and that $F\in L^2(-\pi,\pi)$, where F is assumed extended by periodicity to* $\mathbf{R}$. *If S_K denotes the set of $\theta\in[-\pi,\pi]$ for which*

$$\int_{-\pi}^{\pi}K(t-\theta)\,|F(t)|^2\,dt<\infty, \tag{2.7}$$

then $[-\pi,\pi]\setminus S_K$ has zero K-capacity.

Our penultimate lemma gives a simple growth result for functions in a Dirichlet-type class.

LEMMA 5. *Suppose that $h(z)=\sum_{n=0}^{\infty}b_nz^n$ is analytic in Δ with*

$$\sum_{n=1}^{\infty}n(\log 2n)^{1-2\alpha}\,|b_n|^2<\infty,$$

where $\alpha\in(0,1]$. Set $\lambda(r,h)=\sum_{n=0}^{\infty}|b_n|r^n$, $0\leqslant r<1$. Then

$$\lambda(r,h)=o\left(\left(\log\frac{1}{1-r}\right)^{\alpha}\right),\quad r\to 1. \tag{2.8}$$

Proof of Lemma 5. We begin by noting [**12**, Vol. I, p. 192] that

$$\left(\log\frac{2}{1-z}\right)^{2\alpha}=\sum_{n=0}^{\infty}A_n^{\alpha}z^n,\quad z\in\Delta, \tag{2.9}$$

where $A_n^{\alpha}=(2\alpha+o(1))\,n^{-1}(\log n)^{2\alpha-1}$ as $n\to\infty$. Hence $\sum_{n=0}^{\infty}(A_n^{\alpha})^{-1}|b_n|^2<\infty$, and so, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\sum_{n=N}^{\infty}|b_n|r^n&=\sum_{n=N}^{\infty}(A_n^{\alpha})^{-1/2}|b_n|r^{n/2}\cdot(A_n^{\alpha})^{1/2}r^{n/2}\\&\leqslant\left(\sum_{n=N}^{\infty}(A_n^{\alpha})^{-1}|b_n|^2\right)^{1/2}\left(\sum_{n=0}^{\infty}A_n^{\alpha}\,r^n\right)^{1/2}=\varepsilon_N\left(\log\frac{1}{1-r}\right)^{\alpha},\end{aligned}$$

where $\varepsilon_N\to 0$ as $N\to\infty$, and we have used (2.9). It is clear that (2.8) follows from this, and the proof of Lemma 5 is complete. □

Finally, we have the following technical lemma.

LEMMA 6. *Set*

$$H_{\gamma}(z)=(1-z)^{-1/2}\left(\log\frac{2}{1-z}\right)^{\gamma/2},\quad z=re^{i\theta}\in\overline{\Delta}\setminus\{1\},$$

with $\gamma\in[-1,1]$. Then

$$|H_{\gamma}(z)|\leqslant 6\,\mathrm{Re}\,[H_{\gamma}(z)],\quad z\in\Delta. \tag{2.10}$$

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Proof of Lemma 6. For $0 < \theta < \pi$ we have

$$\begin{aligned} 2 \arg H_\gamma(e^{i\theta}) &= \arg \frac{1}{1-e^{i\theta}} + \gamma \arg \log \frac{2}{1-e^{i\theta}} \\ &= \frac{\pi-\theta}{2} + \gamma \arctan \frac{(\pi-\theta)/2}{\log \csc(\theta/2)} \\ &= t + \gamma \arctan \frac{t}{\log \sec t}, \end{aligned}$$

where $t = (\pi - \theta)/2 \in (0, \pi/2)$, so

$$2|\arg H_\gamma(e^{i\theta})| \leqslant t + \arctan \frac{t}{\log \sec t} = \varphi(t), \tag{2.11}$$

say. Next, $\cos t < 1 - t^2/\pi$ for $0 < t < \pi/2$, so $\log \sec t > \log[(1-t^2/\pi)^{-1}] > t^2/\pi$, and it follows that

$$\varphi(t) < t + \arctan(\pi/t) < \pi/2 + \arctan(2) < 2.7, \quad 0 < t < \pi/2,$$

where we have used the fact that $t + \arctan(\pi/t)$ increases with t in $(0, \infty)$. It follows from (2.11), since $\overline{H_\gamma(e^{i\theta})} = H_\gamma(e^{-i\theta})$, that $\arg H_\gamma(e^{i\theta}) \in (-1.4, 1.4)$ for $0 < |\theta| < \pi$, and so $|\arg H_\gamma(re^{i\theta})| < 1.4$ for $0 \leqslant r < 1$. Consequently

$$\operatorname{Re} H_\gamma(z) = |H_\gamma(z)| \cos \arg H_\gamma(z) \geqslant |H_\gamma(z)| \cos 1.4, \quad z \in \Delta,$$

and (2.10) follows from this. The proof of Lemma 6 is complete. □

§3. *Proof of Theorem 1.* Suppose that $f \in \mathcal{S}$ and that $\log M(r, f) = O((\log(1/(1-r)))^\alpha)$ where $\alpha \in (0, 1]$. Note that when $\alpha = 1$ we are dealing with the full class $\mathcal{S}$. To prove Theorem 1 we must show that, with $g(z) = \log f(z)/z$, $z \in \Delta$, and $\alpha \in (0, 1]$,

$$\operatorname{Re} g(re^{i\theta}) = o\left(\left(\log \frac{1}{1-r}\right)^\alpha\right) \tag{3.1}$$

outside a set of θ of $\mathcal{K}_{1-\alpha,\beta}$-capacity zero for every $\beta > 1$. We note first that, by Lemma 3, for $\alpha \in (0, 1]$ and $\beta > 1$,

$$\begin{aligned} g(z) &= \int_{-\pi}^{\pi} \mathcal{H}_{\alpha,\beta}(ze^{-it}) F(t)\, dt \\ &= \int_{-\pi}^{\pi} H_{2\alpha-1}(ze^{-it})\, \mathcal{L}_{\alpha,\beta}(ze^{-it})\, F(t)\, dt, \end{aligned} \tag{3.2}$$

where H_γ is defined as in Lemma 6,

$$\mathcal{L}_{\alpha,\beta}(z) = \left(\log \frac{2}{1-z}\right)^{(1-\alpha)/2} \left(\log \log \frac{2e}{1-z}\right)^{\beta/2}, \quad z \in \Delta,$$

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and $F \in L^2(-\pi, \pi)$. We also note that for such F, by Lemma 4, the integral in (2.7), with K replaced by $\mathcal{K}_{1-\alpha,\beta}$, is finite for all θ outside a set of $\mathcal{K}_{1-\alpha,\beta}$-capacity zero. We prove Theorem 1 by showing that (3.1) holds for each such θ. We may assume, without loss of generality, that $\theta = 0$, so that

$$\int_{-\pi}^{\pi} \mathcal{K}_{1-\alpha,\beta}(t)\,|F(t)|^2\,dt = \int_{-\pi}^{\pi} K_{1-\alpha}(t)\,k_\beta(t)\,|F(t)|^2\,dt < \infty, \tag{3.3}$$

and it will be enough to prove that this implies (3.1) with $\theta = 0$.

We note next, since $|1 - re^{-it}| \geqslant |\sin t/2|$ if $r \geqslant 1/4$, that, for such r,

$$\left|\log \frac{2}{1 - re^{-it}}\right| \leqslant \log \frac{2}{|\sin(t/2)|} + \frac{\pi}{2} \leqslant 4 \log \frac{2}{|\sin(t/2)|},$$

and $|\log\log[2e/(1 - re^{-it})]| \leqslant A \log\log[3/|\sin t/2|]$. It follows that

$$|\mathcal{L}_{\alpha,\beta}(re^{-it})| \leqslant A(\alpha, \beta)\,K_{1-\alpha}(t)^{1/2} k_\beta(t)^{1/2} = G_{\alpha,\beta}(t), \tag{3.4}$$

say, where $A(\alpha, \beta)$ depends only on α and β, and $K_0(t) \equiv 1$. Hence we have, by (3.2), (3.4) and (2.10), for $r \geqslant 1/4$, that

$$\begin{aligned} |g(r)| &\leqslant \int_{-\pi}^{\pi} |H_{2\alpha-1}(re^{-it})|\; G_{\alpha,\beta}(t)\;|F(t)|\,dt \\ &\leqslant 6 \int_{-\pi}^{\pi} \operatorname{Re} H_{2\alpha-1}(re^{-it})\, G_{\alpha,\beta}(t)\,|F(t)|\,dt \\ &\leqslant 6 \left|\int_{-\pi}^{\pi} H_{2\alpha-1}(re^{-it})\, G_{\alpha,\beta}(t)\,|F(t)|\,dt\right|. \end{aligned} \tag{3.5}$$

Next, by [**12**, Vol. I, p. 192], $H_{2\alpha-1}(z) = \sum_{n=0}^{\infty} B_n^\alpha z^n$ where

$$B_n^\alpha = (c + o(1))\, n^{-1/2} (\log 2n)^{(2\alpha-1)/2}, \quad n \to \infty,$$

and, as the Fourier coefficients of $G_{\alpha,\beta}|F|$ are square summable, since $G_{\alpha,\beta}|F| \in L^2(-\pi, \pi)$ by (3.3), it is easy to see that if

$$h(z) = \int_{-\pi}^{\pi} H_{2\alpha-1}(ze^{-it})\, G_{\alpha,\beta}(t)\,|F(t)|\,dt = \sum_{n=0}^{\infty} b_n z^n,$$

then $\sum n(\log 2n)^{1-2\alpha}\,|b_n|^2 < \infty$. Hence, by (3.5) and (2.8),

$$|g(r)| \leqslant 6\,|h(r)| \leqslant 6\,\lambda(r, h) = o\left(\left(\log \frac{1}{1-r}\right)^{\alpha}\right), \quad r \to 1.$$

This gives gives (3.1) with $\theta = 0$, so (3.1) holds for all θ outside a set of zero $\mathcal{K}_{1-\alpha,\beta}$-capacity when $\alpha \in (0, 1]$ and $\beta > 1$. Since this gives both parts of the theorem, the proof of Theorem 1 is complete.

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§4. *Proof of Theorem 2.* The examples we construct to prove Theorem 2 are based on the examples used by the author in [**11**].

Suppose that α is fixed in $(0, 1)$ and that $\gamma \in (\alpha, 1)$. We define a sequence (r_n) in $(0, 1)$ such that

$$2^n = n^2\left(\log \frac{1}{1-r_n}\right)^{1-\gamma}, \quad n \geqslant 1, \tag{4.1}$$

and set

$$\delta_n = (1-r_n)\,2^n, \quad n \geqslant n_0, \tag{4.2}$$

where $n_0 = [9/(\gamma-\alpha)]$, the integer part of $9/(\gamma-\alpha)$. Then δ_n decreases for $n \geqslant n_0$, and we assume that the definition of δ_n is completed in such a way that $\delta_0 = 2\pi$ and $(\delta_n)_{n=0}^{\infty}$ is decreasing. We note here also that it follows from (4.1) and (4.2) that

$$\delta_n < \delta_{n-1}/2, \quad n \geqslant n_0. \tag{4.3}$$

We next use the sequence $(\delta_n)_{n=0}^{\infty}$ to construct a standard Cantor-type set, and begin by setting $F_0 = [0, 2\pi]$ and $F_1 = [0, \delta_1/2] \cup [2\pi - \delta_1/2, 2\pi]$. For $n \geqslant 2$, let F_n denote the union of the 2^n closed subintervals obtained by removing from each of the 2^{n-1} intervals of F_{n-1} a concentric open subinterval in such a way that 2^n closed subintervals of equal length $\delta_n/2^n$ remain. We define

$$\mathcal{F} = \bigcap_{n=0}^{\infty} F_n,$$

and note, by a result of Carleson [**4**, p. 31], that $\mathcal{F}$ has positive $\log_{1-\gamma}$-capacity since, by (4.1) and (4.2),

$$\sum_{n=1}^{\infty} \frac{(\log(2^n/\delta_n))^{1-\gamma}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

We next write

$$z_{mn} = r_n \exp(\mathrm{i}\,\theta_{mn}), \quad 1 \leqslant m \leqslant 2^{n+1}, \quad n \geqslant n_0,$$

where (θ_{mn}), $1 \leqslant m \leqslant 2^{n+1}$, denotes the ordered sequence of endpoints of the intervals of F_n. We now define the starlike function h by

$$h(z) = z \prod_{n=n_0}^{\infty} \prod_{m=1}^{2^{n+1}} (1 - \overline{z}_{mn}\, z)^{-2\mu_n}, \quad z \in U,$$

where

$$\mu_n = c\left(\log \frac{1}{1-r_n}\right)^{-1+\alpha},$$

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with $c = c(\alpha, \gamma)$ chosen so that

$$\sum_{n=n_0}^{\infty} 2^{n+1}\,\mu_n = 2c \sum_{n=n_0}^{\infty} n^{2(1-\alpha)/(1-\gamma)} 2^{-n(\gamma-\alpha)/(1-\gamma)} = 1.$$

Note that

$$\log 4\left|\frac{h(z)}{z}\right| = 2\sum_{n=n_0}^{\infty} \mu_n \sum_{m=1}^{2^{n+1}} \log \frac{2}{|1-\overline{z}_{mn}\, z|}, \quad z \in U, \tag{4.4}$$

from which we obtain, for $p \geqslant n_0$ and $1 \leqslant m \leqslant 2^{p+1}$,

$$\log \frac{4\,|h(z_{mp})|}{r_p} \geqslant 2\mu_p \log \frac{1}{1-r_p} = 2c\left(\log \frac{1}{1-r_p}\right)^{\alpha}. \tag{4.5}$$

Suppose next that $\theta \in \mathcal{F}$. Then $\theta \in F_p$ for each $p \geqslant n_0$ and so there is an integer $m \in [1, 2^{p+1}]$ such that $|\theta - \theta_{mp}| \leqslant \delta_p/2^p = 1 - r_p$. Since, by a familiar estimate for univalent functions [**6**, p. 5], $|zh'(z)/h(z)| \leqslant (1+r)/(1-r)$, we have

$$\begin{aligned}
&\log\left(\frac{|h(z_{mp})|}{r_p}\right) - \log\left(\frac{|h(r_p e^{i\theta})|}{r_p}\right) \\
&= \operatorname{Re} \int_{\theta}^{\theta_{mp}} \left(\frac{h'(\zeta)}{h(\zeta)} - \frac{1}{\zeta}\right) i r_p e^{it}\, dt \quad (\zeta = r_p e^{it}) \\
&= O(1), \quad p \to \infty,
\end{aligned}$$

(cf. [**11**, p. 240]), and thus by (4.5) we have

$$\limsup_{p\to\infty} \frac{\log |h(r_p e^{i\theta})|}{(\log (1/(1-r_p)))^{\alpha}} > 0$$

for every $\theta \in \mathcal{F}$. This gives (1.5).

To complete the proof of Theorem 2 we must establish that (1.4) holds, and for this purpose we require two further lemmas.

LEMMA 7. *Let the sequences (r_n), (θ_{mn}), and (z_{mn}) be defined as above. Then, if $|z| = r$, $3/4 \leqslant r < 1$, and $n \geqslant n_0$,*

$$\sum_{m=1}^{2^{n+1}} \log \frac{2}{|1-\overline{z}_{mn}\, z|} \leqslant 4 \max_{j\in[1,2^{n+1}]} \sum_{m=1}^{2^{n+1}} \log \frac{C_0^2}{(1-r_n r)^2 + (\theta_{jn} - \theta_{mn})^2}, \tag{4.6}$$

where $C_0^2 = 1 + 4\pi^2$.

Proof of Lemma 7. We begin by proving that

$$\sum_{m=1}^{2^{n+1}} \log \frac{2}{|1-\overline{z}_{mn}\, z|} \leqslant 2 \sum_{m=1}^{2^{n+1}} \log \frac{C_0^2}{(1-r_n r)^2 + (\theta_{mn} - \theta)^2}, \tag{4.7}$$

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where $\theta \in [0, \pi]$, $z = re^{i\theta}$, and $r \in [3/4, 1)$. Note first that if $|\theta_{mn} - \theta| \leqslant \pi$ then

$$|1 - \bar{z}_{mn} z|^2 = (1 - r_n r)^2 + 4 r_n r \sin^2 \frac{\theta_{mn} - \theta}{2}$$
$$\geqslant \frac{1}{4}[(1 - r_n r)^2 + |\theta_{mn} - \theta|^2],$$

where we have used the simple inequality $\sin x \geqslant 2x/\pi$ for $0 \leqslant x \leqslant \pi/2$, and the fact that $4r_n r/\pi^2 \geqslant 1/4$ when $r \geqslant 3/4$ and $n \geqslant n_0$. Next, if $|\theta_{mn} - \theta| > \pi$ with $\theta \in [0, \pi]$, then $\theta_{mn} \in (\pi, 2\pi]$ and $\theta_{mn} - \theta > \pi$, so that $2\pi - (\theta_{mn} - \theta) < \pi$. Note that, by the symmetry of F_n, $\theta'_{mn} = 2\pi - \theta_{mn}$ is an endpoint of one of the subintervals of F_n contained in $[0, \pi)$, and we have

$$\sin \frac{\theta_{mn} - \theta}{2} = \sin\left[\pi - \frac{\theta_{mn} - \theta}{2}\right]$$
$$= \sin\left(\frac{\theta'_{mn}}{2} + \frac{\theta}{2}\right) \geqslant \sin \frac{|\theta'_{mn} - \theta|}{2} \geqslant \frac{|\theta'_{mn} - \theta|}{\pi},$$

since $|\theta'_{mn} - \theta| \leqslant \theta'_{mn} + \theta < \pi$. This gives

$$|1 - \bar{z}_{mn} z|^2 \geqslant \tfrac{1}{4}[(1 - r_n r)^2 + |\theta'_{mn} - \theta|^2], \quad r \geqslant 3/4, \quad n \geqslant n_0.$$

Combining these results we obtain (4.7) for $\theta \in [0, \pi]$.

Next, for $\theta \in [0, \pi]$ and fixed $n \geqslant n_0$, there exists $k \in [1, 2^{n+1} - 1]$ for which $\theta_{kn} \leqslant \theta \leqslant \theta_{(k+1)n}$. Then, for $1 \leqslant m \leqslant k$ we have $|\theta - \theta_{mn}| \geqslant |\theta_{kn} - \theta_{mn}|$, while for $k + 1 \leqslant m \leqslant 2^{n+1}$ we have $|\theta - \theta_{mn}| \geqslant |\theta_{(k+1)n} - \theta_{mn}|$. Inequality (4.6) now follows from (4.7) for $z = re^{i\theta}$ when $\theta \in [0, \pi]$.

We complete the proof of Lemma 7 by deducing that (4.6) holds also for all $\theta \in [\pi, 2\pi]$. To this end, we note first that, with n fixed, $\{\theta_{mn}\} = \{\theta'_{mn}\}$, where $\theta'_{mn} = 2\pi - \theta_{mn}$ for $1 \leqslant m \leqslant 2^{n+1}$, and we are using the symmetry of F_n again. Next, since

$$|1 - \bar{z}_{mn} z| = |1 - r_n r e^{i(\theta_{mn} - \theta)}| = |1 - r_n r e^{i(2\pi - \theta - \theta'_{mn})}|,$$

we clearly have $L_n(re^{i(2\pi - \theta)}) = L_n(re^{i\theta})$, where $L_n(re^{i\theta})$ denotes the first sum in (4.6). Inequality (4.6) follows for $\theta \in [\pi, 2\pi]$ with $z = re^{i\theta}$, $r \geqslant 3/4$, and the proof of Lemma 7 is complete. □

LEMMA 8. *Let the sequences (r_n), (θ_{mn}), and (μ_n) be defined as above, and suppose that $p \geqslant n_0 + 3$. Then*

$$\sum_{n=n_0}^{\infty} \mu_n \max_{j \in [1, 2^{n+1}]} \sum_{m=1}^{2^{n+1}} \log \frac{C_0^2}{(1 - r_n r_p)^2 + (\theta_{jn} - \theta_{mn})^2} \leqslant A \left(\log \frac{1}{1 - r_p}\right)^{\alpha}. \tag{4.8}$$

(Here, and below, A denotes a positive quantity which may depend on α or γ, or both, but is independent of p. A does not necessarily denote the same quantity for different occurrences.)

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Proof. We begin by taking $n \geqslant n_0 + 3$ and fixing $j \in [1, 2^{n+1}]$. Next we set $S_n = \{\theta_{mn} : 1 \leqslant m \leqslant 2^{n+1}\}$, and define disjoint sets E_q, $1 \leqslant q \leqslant n - n_0 + 1$, such that $S_n = \bigcup E_q$. First, E_1 is the set consisting of the two endpoints of the subinterval in F_n that contains θ_{jn}. Next, E_2 is the subset of $S_n \setminus E_1$ contained in the same subinterval of F_{n-1} as E_1. Then E_2 contains two elements each at a distance from θ_{jn} of at least

$$\frac{\delta_{n-1}}{2^{n-1}} - 2\frac{\delta_n}{2^n} \geqslant \frac{\delta_{n-1}}{2^n},$$

where we have used (4.3). Next, E_3 contains the elements of S_n in the same subinterval of F_{n-2} as E_2, but disjoint from $E_2 \cup E_1$. Then E_3 contains four elements, each at a distance of at least $\delta_{n-2}/2^{n-1}$ from θ_{jn}. Continuing in this way we define, for $2 \leqslant q \leqslant n - n_0$, a subset E_q of S_n which is disjoint from $\bigcup_{k=1}^{q-1} E_k$, is contained in the same subinterval of F_{n-q+1} as E_{q-1}, and contains 2^{q-1} elements each at a distance of at least $\delta_{n-q+1}/2^{n-q+2}$ from θ_{jn}. The remaining points of S_n, numbering $2^{n+1} - 2^{n-n_0}$, each at a distance of at least $\delta_{n_0}/2^{n_0+1}$ from θ_{jn}, constitute E_q with $q = n - n_0 + 1$.

Next we define

$$T_n(r_p, j) = \sum_{m=1}^{2^{n+1}} \log \frac{C_0^2}{(1 - r_n r_p)^2 + (\theta_{jn} - \theta_{mn})^2}, \quad 0 \leqslant r < 1, \quad n \geqslant n_0,$$

and use the decomposition of S_n obtained above for $n \geqslant n_0 + 3$ to show that

$$\sum_{n=p+1}^{\infty} \mu_n \max_{j \in [1, 2^{n+1}]} T_n(r_p, j) \leqslant A\left(\log \frac{1}{1 - r_p}\right)^{\alpha}. \tag{4.9}$$

To this end, we write, for $n \geqslant p + 1$,

$$T_n(r_p, j) = \sum_{q=1}^{n-p} \sum_{E_q} + \sum_{q=n-p+1}^{n-n_0+1} \sum_{E_q} = \sigma_1 + \sigma_2,$$

say, and note that, since each term in $T_n(r_p, j)$ is at most $2 \log[C_0/(1 - r_p)]$, and $1/(1 - r_p) > C_0$ for $p > n_0$,

$$\begin{aligned} \sigma_1 &\leqslant 2\left(2 + \sum_{q=2}^{n-p} 2^{q-1}\right) \log \frac{C_0}{1 - r_p} \leqslant 2^{n-p+2} \log \frac{1}{1 - r_p} \\ &= 2^{n+2} p^{-2} \left(\log \frac{1}{1 - r_p}\right)^{\gamma}, \end{aligned}$$

where we have used (4.1). For σ_2 we note first, since $|\theta_{jn} - \theta_{mn}| \geqslant \delta_{n-q+1}/2^{n-q+2}$ for each $\theta_{mn} \in E_q$ when $n - p + 1 \leqslant q \leqslant n - n_0$, that

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$$\sigma_2 \leqslant 2 \sum_{q=n-p+1}^{n-n_0} 2^{q-1} \log \frac{C_0\, 2^{n-q+2}}{\delta_{n-q+1}} + 2[2^{n+1} - 2^{n-n_0}] \log \frac{C_0\, 2^{n_0+1}}{\delta_{n_0}}$$
$$\leqslant 2^{n+1} \sum_{s=n_0+1}^{p} 2^{-s} \log \frac{C_0 2^{s+1}}{\delta_s} + 2^{n+2} \log \frac{C_0 2^{n_0+1}}{\delta_{n_0}}.$$

Next, using (4.2) and (4.1),

$$\sum_{s=n_0+1}^{p} 2^{-s} \log \frac{C_0 2^{s+1}}{\delta_s} \leqslant 3 \sum_{s=n_0+1}^{p} 2^{-s} \log \frac{1}{1-r_s}$$
$$= 3 \sum_{s=n_0+1}^{p} 2^{\gamma s/(1-\gamma)} s^{-2/(1-\gamma)}$$
$$\leqslant A \int_{n_0}^{p} \frac{2^{\gamma t/(1-\gamma)}}{t^{2/(1-\gamma)}}\, dt = A\, I_\gamma,$$

say, and by integration by parts, temporarily setting $a = \gamma/(1-\gamma), b = 2/(1-\gamma)$,

$$I_\gamma = \int_{n_0}^{p} \frac{2^{at}}{t^b} dt \leqslant \frac{1}{a \log 2} \left(\frac{2^{ap}}{p^b} + b \int_{n_0}^{p} \frac{2^{at}}{t^{b+1}} dt \right) \leqslant \frac{2}{a} \left(\frac{2^{ap}}{p^b} + \frac{b}{n_0} I_\gamma \right),$$

so that

$$\left(1 - \frac{2b}{an_0}\right) I_\gamma \leqslant \frac{2^{ap+1}}{ap^b}.$$

Thus, noting that $1 - 2b/an_0 = 1 - 4/n_o\gamma \geqslant 1/2$, since $n_0 > 8/\gamma$, we have

$$\frac{1}{2} I_\gamma \leqslant (2/\gamma) \frac{2^{\gamma p/(1-\gamma)}}{p^{2/(1-\gamma)}} = A\, p^{-2} \left(\log \frac{1}{1-r_p} \right)^{\gamma}.$$

Combining the above estimates we obtain, for $n \geqslant p+1$,

$$\max_{j \in [1, 2^{n+1}]} T_n(r_p, j) \leqslant 2^n \left(A\, p^{-2} \left(\log \frac{1}{1-r_p} \right)^{\gamma} + 4 \log \frac{C_0 2^{n_0+1}}{\delta_{n_0}} \right). \quad (4.10)$$

It follows from (4.1), since $\mu_n = c/(\log(1/(1-r_n)))^{(1-\alpha)}$, that

$$\sum_{n=p+1}^{\infty} \mu_n 2^n = c \sum_{n=p+1}^{\infty} n^{2(1-\alpha)/(1-\gamma)} 2^{-[(\gamma-\alpha)/(1-\gamma)]n}$$
$$\leqslant A \int_{p}^{\infty} t^{2(1-\alpha)/(1-\gamma)} 2^{-[(\gamma-\alpha)/(1-\gamma)]t} dt$$
$$\leqslant A\, p^{2(1-\alpha)/(1-\gamma)} 2^{-[(\gamma-\alpha)/(1-\gamma)]p} = A\, p^2 \left(\log \frac{1}{1-r_p} \right)^{-(\gamma-\alpha)},$$

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where we have used the inequality $p > n_0 = [9/(\gamma - \alpha)]$. Noting finally that $p \leqslant A(\log\log(1/(1-r_p)))$, it is clear that this yields (4.9) when combined with (4.10).

We complete the proof of Lemma 8 by showing that

$$\sum_{n=n_0}^{p} \mu_n \max_{j\in[1,2^{n+1}]} T_n(r_p, j) \leqslant A\left(\log\frac{1}{1-r_p}\right)^{\alpha}. \tag{4.11}$$

First, by arguments similar to those used above, for $n \geqslant n_0 + 3$,

$$T_n(r_p, j) \leqslant 4\sum_{q=1}^{n-n_0} 2^{q-1}\log\frac{C_0 2^{n-q+2}}{\delta_{n-q+1}} + [2^{n+1} - 2^{n-n_0}]\log\frac{C_0 2^{n_0+1}}{\delta_{n_0}}$$

and

$$\begin{aligned}\sum_{q=1}^{n-n_0} 2^{q-1}\log\frac{C_0 2^{n-q+2}}{\delta_{n-q+1}} &\leqslant 2^n\, 3\sum_{s=n_0+1}^{n} 2^{-s}\log\frac{1}{1-r_s}\\ &\leqslant A\,2^n\sum_{s=n_0+1}^{n} 2^{s\gamma/(1-\gamma)}s^{-2/(1-\gamma)}\\ &\leqslant A\,2^{n[1+\gamma/(1-\gamma)]}n^{-2/(1-\gamma)} = A\log\frac{1}{1-r_n}.\end{aligned}$$

Hence, by the definition of μ_n and the fact that $T_n(r_p, j) \leqslant 2^{n_0+3}\log(1/(1-r_n))$ for $n_0 \leqslant n \leqslant n_0 + 2$, we have

$$\begin{aligned}\sum_{n=n_0}^{p} \mu_n \max_{j\in[1,2^{n+1}]} T_n(r_p, j) &\leqslant A\sum_{n=n_0}^{p}\mu_n\left\{\log\frac{1}{1-r_n} + 2^n\right\}\\ &\leqslant A\left[c\sum_{n=n_0}^{p}\left(\log\frac{1}{1-r_n}\right)^{\alpha} + 1/2\right],\end{aligned}$$

since $\sum \mu_n 2^n = 1/2$. Finally,

$$\begin{aligned}\sum_{n=n_0}^{p}\left(\log\frac{1}{1-r_n}\right)^{\alpha} &= \sum_{n=n_0}^{p} 2^{n\alpha/(1-\gamma)}n^{-2\alpha/(1-\gamma)}\\ &\leqslant A\,2^{p\alpha/(1-\gamma)}p^{-2\alpha/(1-\gamma)} = A\left(\log\frac{1}{1-r_p}\right)^{\alpha}.\end{aligned}$$

Hence (4.11) holds. Combining (4.9) and (4.11) we obtain (4.8) and Lemma 8 is proved. □

As a consequence of Lemmas 7 and 8, and (4.4), we now have the growth estimates $\log M(r_p, h) \leqslant A(\log(1/(1-r_p)))^{\alpha}$ for every $p > n_0$. It is easy to see

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that (1.4) follows from these estimates, for if $r_{n_0+1} \leqslant r < 1$, so that $r_p \leqslant r \leqslant r_{p+1}$ for some $p > n_0$, then

$$\log M(r,h) \leqslant \log M(r_{p+1},h) \leqslant A\left(\log \frac{1}{1-r_{p+1}}\right)^{\alpha}$$
$$\leqslant 2^{3\alpha/(1-\gamma)} A\left(\log \frac{1}{1-r_p}\right)^{\alpha} \leqslant A\left(\log \frac{1}{1-r}\right)^{\alpha},$$

which gives (1.4). This completes the proof of Theorem 2.

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